TD

# Midterm 2 — Dupuy — Math 121 — Fall 2016

**Instructions** Remember to show all your work to get full credit. Please leave answers in their exact form. This is a closed book test. You may not use a calculator. If you need extra paper let me know.

Name: KEY

Section:

Deductions

- For writing things unrelated to the problem.

Missing "="

| Problem | Possible | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 15 | |
| EC1 | 10 | |
| EC2 | 10 | |
| Total | 85 | |
| Percentage | | |

1. (15 points) Compute partial derivatives of the indicated functions

(a) $\frac{\partial}{\partial y}[x^2 + y^2 + z^2]$ $= 2y$

(+5)

(b) $\frac{\partial^2}{\partial y \partial x}[e^{xy}]$ $= \frac{\partial}{\partial y}[ye^{xy}] = e^{xy} + xye^{xy}$

(+5)

(c) $\frac{\partial}{\partial w}[\ln(wx) + 1]$ $= \dfrac{\frac{\partial}{\partial w}[wx]}{wx} = \dfrac{x}{wx} = \dfrac{1}{w}.$

(+5)

2. (10 points) Find the plane tangent to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$= 3 + 4(x-1) + 2(y-1)$$

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+10 correct plane

partial: +5 for plane equation.

3. (10 points) Determine if the limit exists or not. If the limit exists state its value. If not, explain why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}.$$

Approach along two paths & show they are not the same.

$$\frac{x^2}{x^2+y^2} = \frac{r^2\cos(\theta)^2}{r^2} = \cos(\theta)^2$$

angle dependent, so the limit doesn't exist.

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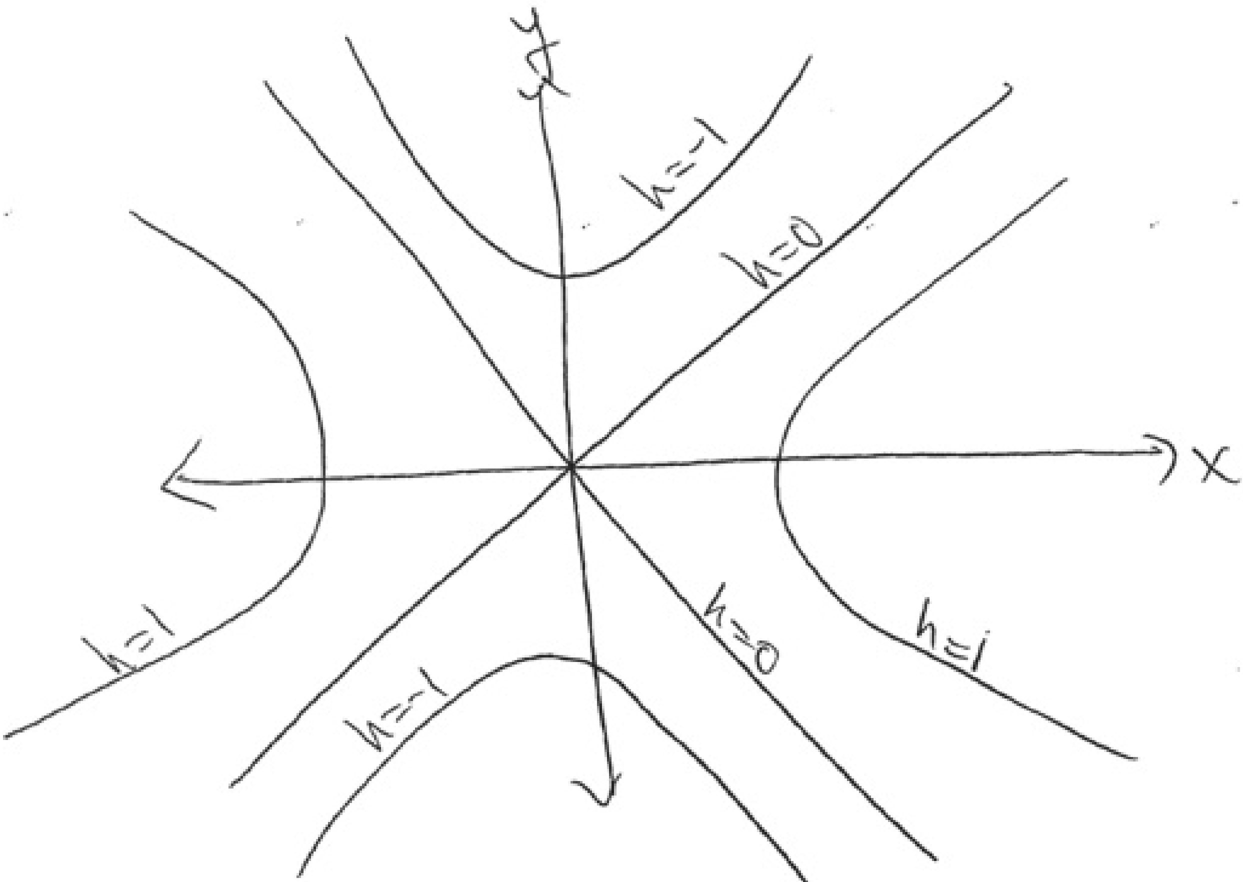
(+5) correct answer

(+5) correct argument

4. (10 points) Create a level set plot (contour plot) of

$$h(x,y) = x^2 - y^2.$$

make sure to label the level sets for $h(x,y) = -1, h(x,y) = 1$ and $h(x,y) = 0$.




$x^2 - y^2 = 1,$

$x^2 - y^2 = -1,$

$\Leftrightarrow y^2 - x^2 = 1$

(+5) for any level set plot

(+5) for correct level set plot w/ labels

5. (10 points) Let $f(x,y,z) = e^{xy}z$.

(a) In what direction does $f$ have the maximal rate of change at the point $(0,0,0)$?

(b) What is the value of the maximal rate of change?

$$\nabla f = (yze^{xy}, xze^{xy}, e^{xy})$$

$$\nabla f(0,0,0) = (0,0,1)$$

(a) Direction of max'l rate of change is the direction of $\dfrac{\nabla f(0,0,0)}{|\nabla f(0,0,0)|} = \dfrac{(0,0,1)}{1} = (0,0,1)$.

(b) The value of the max'l rate of change is

$$|\nabla f(0,0,0)| = |(0,0,1)| = 1.$$

+5 part (a)

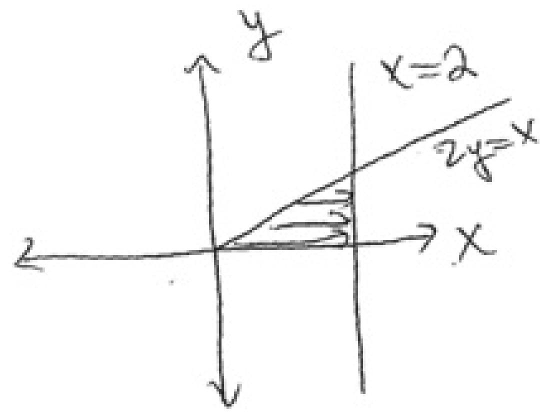
+5 part (b)

<u>partial credit:</u>

+3 for knowing $\nabla f$ is the direction of max'l change.

6. (15 points) Evaluate the iterated integral by reversing the order of integration

$$\int_0^1 \int_{2y}^2 e^{x^2}\,dx\,dy.$$




~~Usual~~

Given Order: $2y \le x \le 2$
$0 \le y \le 1.$

Changed Order: $y \in [0, \frac{x}{2}]$
$x \in [0,2]$

+10 [correct switch order

$$\int_0^1 \int_{2y}^2 e^{x^2}\,dxdy = \int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx$$

$$= \int_0^2 e^{x^2}\,\frac{x}{2}\,d\cancel{y}$$

$$= \frac{1}{2}\int_0^2 \frac{d}{dx}\left[\frac{e^{x^2}}{2}\right] dx$$

$$= \frac{1}{4}\; e^{x^2}\Big|_0^2 = \frac{1}{4}\left(e^4-1\right). \;//$$

+5 [correct answer]

7. (15 points) Use Lagrange multipliers to maximize the function $f(x,y) = 2x^2 - y^2$ subject to the constraint $2x^2 + y^2 = 2$.

$\nabla f = (4x, -2y)$, $\nabla g = (4x, 2y)$ [SETUP] +5

Note: $2x^2+y^2=$ is a compact set so extreme values are achieved by extreme value theorem.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 2 \end{cases} \iff \begin{cases} 4x = \lambda 4x & (1) \\ -2y = \lambda 2y & (2) \\ 2x^2+y^2 = 2. & (3) \end{cases}$$

so

$(1) \Rightarrow x - \lambda x = 0 \iff x(1-\lambda) = 0$

$\lambda = 1$ or $x = 0$.

if $\lambda = 1$: $-2y = 2y \to 4y = 0 \Rightarrow y = 0$.

From constraint, $2x^2 + 0^2 = 2 \Rightarrow x = \pm 1$

$\leadsto$ crit points $(+1, 0)$, $(-1, 0)$.

if $x = 0$: From constraint, $2(0)^2 + y^2 = 2$

$\Rightarrow y = \pm\sqrt{2}$

$\leadsto$ crit points $(0, +\sqrt{2})$, $(0, -\sqrt{2})$

$f(\pm 1, 0) = 2(\pm 1)^2 - 0^2 = 2.$ ← max

$f(0, \pm\sqrt{2}) = 2(0)^2 - (\pm\sqrt{2})^2 = -2$ (← min)

+5 [ANSWER]

[No mistakes] +5

**EC1** (10 points) Let $c$ be a constant. Suppose that a level set $\{(x,y) : f(x,y) = c\}$ is parametrized by $\mathbf{r}(t)$, that is

$$f(\mathbf{r}(t)) = c.$$

Show that for every time $t$, $\mathbf{r}'(t)$ and $\nabla f(\mathbf{r}(t))$ are perpendicular. Make sure to use English sentences if appropriate (I am going to be very strict when grading extra credit so please be precise and explain your steps).

$$\frac{d}{dt}\left[f(\vec{r}(t))\right] = \frac{d}{dt}[c] = 0$$

$$\Rightarrow \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 \quad \text{(By Chain Rule)}$$

The dot product of two vectors being zero is the same thing as them being perpendicular, so we are done.

+10 [if complete & correct (which means no extra nonsense on the page)]

**EC2** (10 points) A bell curve with mean $\mu$ and standard deviation $\sigma$ (both constants) is given by

$$g(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Show that

$$\int_{-\infty}^{\infty} g(x)dx = 1.$$

(Half credit will be given if you assume the value of $\int_{-\infty}^{\infty} e^{-x^2}dx$, which you can deduce from a result in Problem 2 of the written part of Homework 07. If you derive the value of $\int_{-\infty}^{\infty} e^{-x^2}dx$ and then use it in your computation you will get full credit.)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ by "Polar Coordinates trick" (see HW07, Problem 02)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du$$

$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\infty} e^{-u^2} du\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(\sqrt{\pi}\right) = 1.$$

$\boxed{\text{Chg of Vars}}$

$u = \frac{x-\mu}{\sqrt{2\sigma^2}}$, $du = \frac{dx}{\sqrt{2\sigma^2}}$

$x = \infty \Rightarrow u = \infty$

$x = -\infty \Rightarrow u = -\infty$

+10 [for being complete & correct]