1. Determine whether the following operations make sense. In what follows $\mathbf{F} = \mathbf{F}(x, y, z)$ and $\mathbf{G} = \mathbf{G}(x, y, z)$ are vector fields and $f = f(x, y, z)$ a scalar function. (simply say Yes or No).

   (a) $\mathrm{div}(\mathbf{F} \times \mathbf{G})$

   Yes

   (b) $\mathrm{curl}(\mathrm{curl}(\mathbf{F} \times \mathbf{G}))$

   Yes

   (c) $\mathrm{curl}(\mathrm{div}(\mathbf{F}))$.

   No

   (d) $\mathrm{div}(f)$

   No

   (e) $\mathrm{curl}(\nabla f)$

   Yes

2. Consider the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 0\mathbf{k}$

(a) Compute curl($\mathbf{F}$).

(b) Compute div($\mathbf{F}$).

(a) $\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

$= (2x - 2y)\vec{k}$

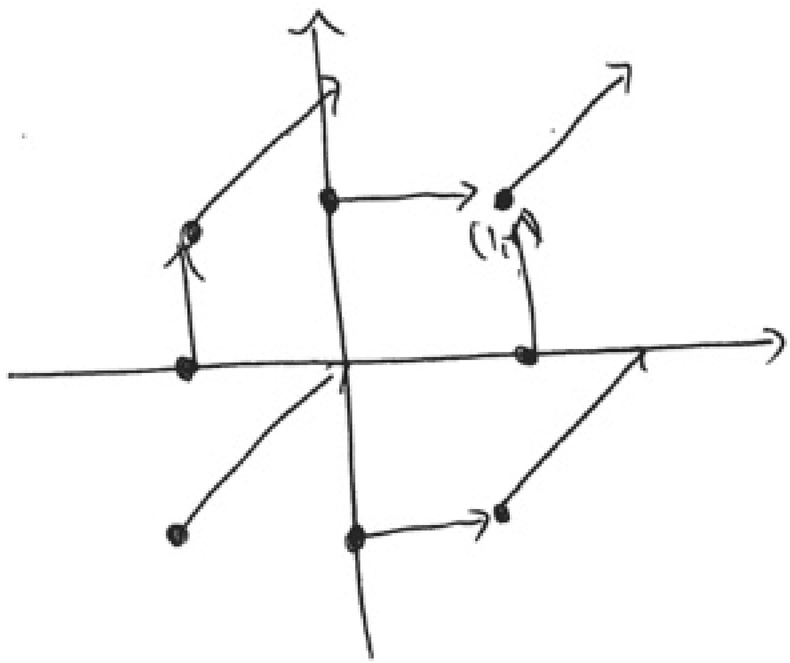
(b) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(x^2) = 0.$

3. Let $\vec{F} = (y^2, x^2)$.

(a) Sketch $\vec{F}$.

(b) Determine if the following vector field has a potential. If it does, find the potential.

(a)



(b) $\text{curl}(\vec{F}) = (2x - 2y)\hat{k} \neq 0$

so there is no potential.

4. Using a triple integral compute the volume of a ball of radius R. I don't care what coordinates you use. Show all of your work.

$$\mathrm{vol}(B) = \iiint_B dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin(\phi)\, d\phi\right)\left(\int_0^{R} \rho^2\, d\rho\right)$$

$$= (2\pi)\left(-\cos(\phi)\Big|_0^{\pi}\right)\left(\frac{R^3}{3}\right)$$

$$= (2\pi)(2)\left(\frac{R^3}{3}\right)$$

$$= \frac{4\pi R^3}{3}.$$

close but bad setup: 12/15

5. Compute

$$\int_C 2x^2 dx + 2y^2 dy,$$

where C is a the oriented curve parametrized by

$$\vec{r}(t) = (\cos(t)^9(1 - 5\sin(t)^5), \sin(t)^{20} + 2\sin(t)^4\cos(t))$$

for $t \in [0, \pi]$. (Hint: $\vec{r}(0) = (1, 0)$ and $\vec{r}(\pi) = (-1, 0)$)

$\vec{F} = 2x^2\hat{i} + 2y^2\hat{j}$ has

$$f(x,y) = \frac{2x^3}{3} + \frac{2y^3}{3}$$

as a potential.

$$\Rightarrow \int_C 2x^2 dx + 2y^2 dy = f(-1,0) - f(1,0)$$

$$= \frac{-2}{3} - \frac{2}{3} = \frac{-4}{3}. //$$

wrong pot: 10/15

curve only: 5/15

green's: 0/15

$D = \left(\text{disc of radius } R\right)$

6. Compute the following contour integral:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy$$

where C is a full counterclockwise circle of radius R.

$$\frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dA$$

$$= \frac{1}{2}\iint_D 2\,dA$$

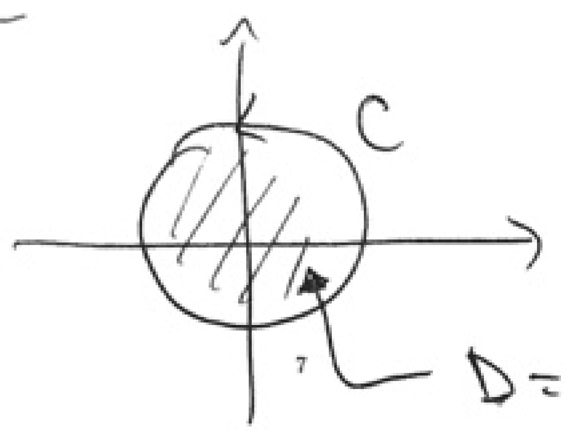
$$= \iint_D dA$$

$$= \pi R^2$$

"hybrid method": 0/15

wrong area of disc: 10/15

missing $1/2$: 14/15

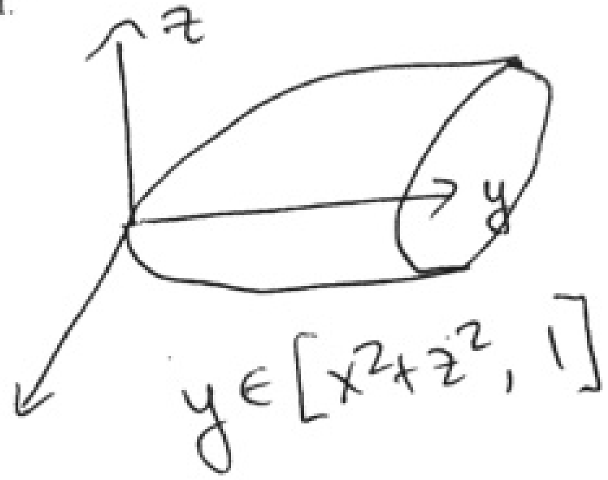


D = interior.

(x, y, z) · Setup: 12/15

drawing only? 5/15

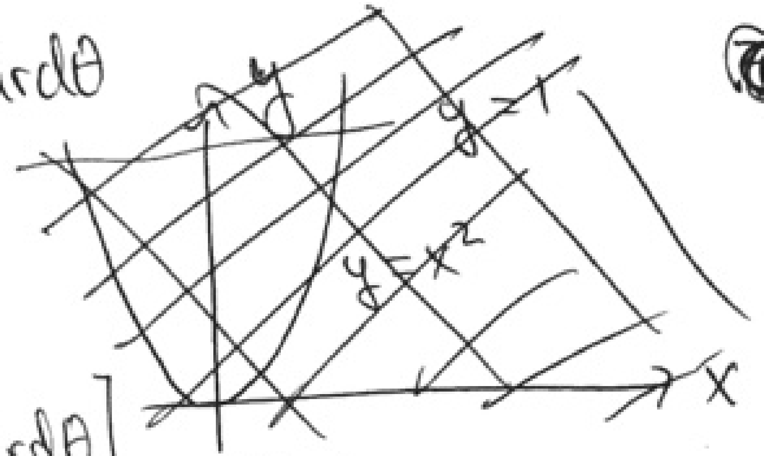
7. Evaluate $\int_T f(x,y,z)dV$ where $f(x,y,z) = 7y$ and T is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 1$.



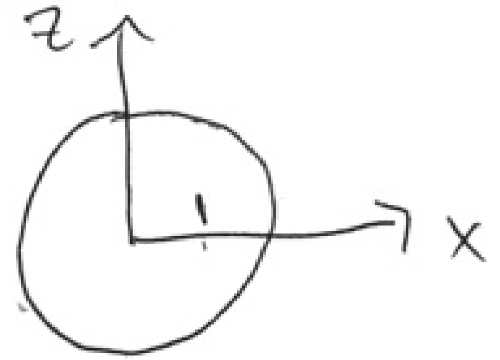
$$\iiint_T 7y\,dV$$

$y \in [x^2+z^2, 1]$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 7y\,dy\,r\,dr\,d\theta$$



$$= 7\left[\int_0^{2\pi}\int_0^1 \frac{y^2}{2}\Big|_{y=r^2}^{1} dr\,d\theta\right]$$



$$= 7\left[\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \left(\frac{1}{2}-\frac{r^4}{2}\right) r\,dr\right)\right]$$

$$= 7(2\pi)\left(\frac{r^2}{4} - \frac{r^6}{12}\Big|_{r=0}^{r=1}\right)$$

circle of radius 1.

$$= 7(2\pi)\left(\frac{1}{2}-\frac{1}{6}\right)$$

$\Rightarrow$ use cylindrical coordinates. $\begin{cases} x = r\cos(\theta) \\ z = r\sin(\theta) \end{cases}$

$$= 7\pi\left(\frac{3}{6}-\frac{1}{6}\right) = \frac{7\pi}{3}.$$

8. State the fundamental theorem of line integrals. (Be precise).

If $\vec{F} = \nabla f$ then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{B}) - f(\vec{A})$$

where $\vec{B}$ is the end point of C & $\vec{A}$ is the starting point of C.

EC1 Prove the fundamental theorem of line integrals. (Be precise).

Suppose $\vec{F} = \nabla f$ & C is parametrized by $\vec{r}(t)$ for $t \in [a,b]$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)). //$$

EC2 State an prove Poincare's Theorem on the existence of a potential in two dimensions. (To get credit you need to be precise. You may use things we proved in class.)

- If $\vec{F}$ has a potential then $\nabla \times \vec{F} = 0$.
- Conversely, if $\nabla \times \vec{F} = 0$ on a simply connected open region then $\vec{F}$ has a potential.

[illegible]

<u>proof.</u>

- $\nabla f = f_x \hat{i} + f_y \hat{j}$. One checks $\nabla \times \nabla f = 0$ directly (see homework __) on back of prev page.
- The second part uses that the following are equivalent
  1) Path independence
  2) Integrals around closed loops are zero. ☆
  3) $\nabla \times \vec{F} = 0$.

Suppose ☆

One defines

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F}\cdot d\vec{r} = \int_{(x_0,y_0)}^{(x,y)} P\,dx + Q\,dy.$$

by path independence.

Since we are on an open simply connected region, we can find an open box completely contained in our region and break up our path as drawn:

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

so we will have,

$$f_x = \frac{\partial}{\partial x}\int_{(x_0,y_0)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

$$= \frac{\partial}{\partial x}\Big[\underbrace{\int_{C_1} \vec{F}\cdot d\vec{r}}_{\text{const}} + \underbrace{\int_{C_2} \vec{F}\cdot d\vec{r}}_{\text{only depends on } y} + \int_{C_3} \vec{F}\cdot d\vec{r}\Big]$$

$$= 0 + 0 + \frac{\partial}{\partial x}\Big[\int_{x_1}^{x} P(t,y)\,dt\Big] = P(x,y).$$

A computation similar to this shows $\frac{\partial f}{\partial y} = Q(x,y)$.

(**)

$$\nabla \times \nabla f = \left(\frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x\right)\vec{k}$$

$$= 0$$

by equality of mixed partials.