Elliptic Functions — For Math 331

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The notes here are from Whittaker and Watson and Knapp.

1 Overview of elliptic functions

As an application of Mittag-Leffler, we gave the following example:

Example 1.1. For τ in the upper-half complex plane H we defined the **Weierstrass $\wp$ function**:

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-n\tau-m)^2} - \frac{1}{(n\tau+m)^2} \right].$$

We remarked that we needed to subtract off the 0th order Taylor approximations of $\frac{1}{(z-n\tau-m)^2}$ at $z=0$ to get this series to converge as the sum $\sum_{(n,m)\neq(0,0)} \frac{1}{(z-n\tau+m)^2}$ was comparable to the integral $\int_{\mathbf{R}^2} |\vec{x}|^{-2} d\vec{x}$ which didn't converge.[1]

I drew a picture of the "lattice of points" we were summing over.

The above example is an example of an "elliptic functions". These were an attempt to generalize trig functions. Let $\wp(z) = \wp(z;\tau)$. It will turn out that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

for some g_2 and g_3 which depend on $\tau \in H$.[2] If we let

$$\begin{aligned} x &= \wp(z) \\ y &= \wp'(z) \end{aligned}$$

then for each $z \in \mathbf{C} \setminus \Lambda_\tau$ where $\Lambda_\tau = \{m\tau + n : m, n \in \mathbf{Z}\}$ we have

$$y^2 = 4x^3 - g_2 x - g_3. \tag{1.2}$$

When g_2 and g_3 are real, and we let x and y be real variables, the curve E defined by the equations looks like this:

[1] The p-test in higher dimensions has $1/|\vec{x}|^{n+\varepsilon}$ integrable in $\mathbf{R}^n$

[2] These equations have been super influential in Mathematics. For example, the study of the so-called Painleve equations were motivated by generalizing these equations.

** I draw a picture here. **

Such curves are called elliptic curves. We will see more of them later when we talk about Riemann Surfaces.

Anyway, the functions $\wp(z)$ and $\wp'(z)$ were developed as an attempt to generalize trig functions. Note the similarities:

$$\begin{aligned} x &= \sin(z) \\ y &= \sin'(z) \end{aligned}$$

Then for each $z \in \mathbf{C}$ we have

$$y^2 = 1 - x^2.$$

Alternatively, we could have defined trig functions implicitly via

$$z = \int_0^{\sin(z)} (1-t^2)^{1/2} dt.$$

For elliptic curves, one could do something similar for elliptic integrals: we define P in terms of z so that

$$z = \int_P^{\infty} (4t^3 - g_2 t - g_2)^{-1/2} dt \tag{1.3}$$

where the integral is over any curve which does not pass through the zeros of $4t^3 - g_2 t - g_3$. One can then do implicit differentiation to find

$$1 = -(4 - P^3 - g_2 P - g_3)^{-1/2} \frac{dP}{dz}$$

which implies

$$(\frac{dP}{dz})^2 = 4P^4 - g_2 P - g_3.$$

One can see that $P = P(z) = \wp(x+\alpha)$ for some constant α would give a solution of (1.3). The integrals of the type in (1.3) are called *elliptic integrals*. They appear for example when one tries to compute the surface area or the volume of a ellipsoid in three dimensions.

2 Elliptic functions

Definition 2.1. A **lattice** $\Lambda \subset \mathbf{R}^n$ is a free $\mathbf{Z}$-module of rank n.

In practice this means that

$$\Lambda = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_n$$

where $v_1, \ldots, v_n$ are linearly independent vectors over $\mathbf{R}$.

Definition 2.2. Let $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2 \subset \mathbf{C}$ be a lattice. An **elliptic function with period lattice** Λ is a function $f \in \mathrm{Mer}(\mathbf{C})$ such that

$$f(z+\lambda) = f(z) \text{ for all } \lambda \in \Lambda.$$

We will denote the collection of elliptic functions with period lattice Λ by $\mathrm{Mer}_\Lambda(\mathbf{C})$.

Note that $\wp(z;\tau) = \wp_{\Lambda_\tau}(z)$ where $\Lambda_\tau = \mathbf{Z} + \mathbf{Z}\tau$.

Example 2.3. Concretely, if $\Lambda = \mathbf{Z} + \mathbf{Z}i$ then

$$\wp_{\Lambda_i}(z) = \wp(z;i) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-m-ni)^2} - \frac{1}{(m+ni)^2}\right].$$

Remark 2.4. The set of elliptic functions for a period lattive, $\mathrm{Mer}_\Lambda(\mathbf{C})$ is a field. [3]

Exercise 2.5. Let $\Lambda \subset \mathbf{C}$ be a lattice. Let $\Lambda^* = \Lambda \setminus \{0\}$. We define the **Weierstrass $\wp$-function** for Λ to be

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right].$$

Show that $\wp_\Lambda(z) \in \mathrm{Mer}_\Lambda(\mathbf{C})$.

Remark 2.6. Exercise 2.5 is not entirely trivial. If $\omega \in \Lambda$ we recommend looking at the two formulas for translates of the input:

$$\wp_\Lambda(z+\omega) = \frac{1}{(z+\omega)^2} + \sum_{\lambda\in\Lambda^*} \left[\frac{1}{(z+\omega-\lambda)^2} - \frac{1}{\lambda^2}\right]$$

$$\wp_\Lambda(z+\omega) = \frac{1}{(z+\omega)^2} + \sum_{\lambda\in(\Lambda^*-\omega)} \left[\frac{1}{(z+\lambda)^2} - \frac{1}{(\lambda+\omega)^2}\right]$$

then rearranging terms (in a convergent way) to show $2\wp_\Lambda(z+\omega) = 2\wp_\Lambda(z)$.

Example 2.7. The function

$$\wp'_\Lambda(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}$$

is an elliptic function with period lattice Λ. To see this note that for $\omega \in \Lambda$ we have

$$\begin{aligned}
\wp'_\Lambda(z+\omega) &= -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda-\omega)^3} \\
&= -2\sum_{\lambda\in\Lambda+\omega} \frac{1}{(z-\lambda)^3} \\
&= -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3} = \wp'_\Lambda(z).
\end{aligned}$$

[3] It turns out that this will be the "field of functions" of an "elliptic curve". These are the Riemann surfaces we alluded to in the previous section.

The last line we just used that Λ was a group and hence that $\omega + \Lambda = \Lambda$ for all $\omega \in \Lambda$.

3 Group actions and lattices

Definition 3.1. Let Γ be a group. Let X be a topological space. A (left) **group action** will be a map

$$\rho : \Gamma \times X \to X$$

satifying certain axioms. Let's use the notation $\gamma(x) := \rho(\gamma, x)$. The axioms are

1. The maps $X \to X$ defined by

$$x \mapsto \gamma(x)$$

   are continuous for each fixed γ and

2. $(\gamma_1(\gamma_2(x))) = (\gamma_1\gamma_2)(x)$,

3. $1_\Gamma(x) = x$.

Remark 3.2. The definition above isn't standard everywhere. We are modifying it for our purposes.

1. The last two axioms are just the axioms of a group action. The first axiom imposes continuity.

2. We could have defined a **group action** to be a group homomorphism $\Gamma \to \mathrm{Aut}(X)$.

3. One can consider Γ as a topological group (meaning that Γ is a topological space and all of its group operations are continuous with respect to this topology). In this case we take $\Gamma \times X \to X$ to be a continous map.

Exercise 3.3. Let $\rho : \Gamma \times X \to X$ be a group action in the sense of Definition 3.1. Show that if we give Γ the discrete topology then ρ is continuous in the sense of Remark 3.2, Item 3.

Definition 3.4. An action $\rho : \Gamma \times X \to X$ is **discretely** if for all $x \in X$ there exists some open subset $U \ni x$ such that for all $\gamma \neq 1_G$ we have

$$\gamma(U) \cap U = \emptyset.$$

Definition 3.5. An action $\rho : \Gamma \times X \to X$ is **free** provided $\mathrm{Stab}_\Gamma(x)$ is trivial for all $x \in X$.

Example 3.6. Let $X = \mathbf{C}$ and $\Gamma = \Lambda$ be a lattice in $\mathbf{C}$. The action $\rho : \Lambda \times \mathbf{C} \to \mathbf{C}$ given by $\rho(\lambda, z) = \lambda + z$ is a discrete, free group action.

Definition 3.7. Let a, b, c, d be elements of $\mathbf{C}$ such that $ad - bc \neq 0$. Tranformations $\mathbf{C} \to \mathbf{C}$ given by

$$z \mapsto \frac{az + b}{cz + d}$$

are called **Möbius transformations**.[4].

Exercise 3.8. Show that $\mathrm{SL}_2(\mathbf{R})$ acts on $H = \{\tau \in \mathbf{C} : \mathrm{Im}\,\tau > 0\}$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Definition 3.9. Let $\rho : \Gamma \times X \to X$ be a group action. A **true fundamental domain** for the action is a connected subset $\Omega \subset X$ such that for all $\gamma \in \Gamma$ not the identity be have

1. $\gamma(\Omega) \cap \Omega = \emptyset$
2. $X = \coprod_{\gamma \in \Gamma} \gamma(\Omega)$.

In class, I had called "true fundamental domains" fundamental domains, but because we will want to consider some domains with stablizers we will modify this.

Definition 3.10. Let $\rho : \Gamma \times X \to X$ be a group action. A **fundamental domain** for the action is a connected closed subset $\Omega = \overline{\Omega} \subset X$ such that for all $\gamma \in \Gamma$ not the identity be have

1. $\gamma(\Omega) \cap \Omega$ has empty interior
2. $X = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$.

Example 3.11. Consider the action of a lattice $\Lambda = \mathbf{Z} + i\mathbf{Z}$ on $\mathbf{C}$. The set

$$\Omega = \{x + iy : 0 \leq x < 1 \text{ and } 0 \leq y < 1\}$$

is a fundamental domain for the action of Λ on $\mathbf{C}$.

Exercise 3.12.
1. Show that $\mathrm{SL}_2(\mathbf{Z})$ acts via Möbius transformations on the upper half plane.
2. Show that a fundamental domain for this action is the complement of the unit disc in a vertical strip of length 1 centered around zero in the upper half plane. In other words
$$\Omega = A \cup B$$
where $A = \{z : |z| > 1 \text{ and } -1/2 < \mathrm{Re}(z) < 1/2\}$ and $B = \{z \in \partial A : \mathrm{Re}(z) \leq 0\}$ is a fundamental domain for this action.

[4]The domain and range of these transformations are naturally a compactified version of the complex numbers, so we will omit discussion on the domain and range of these maps for a later times

4 Properties of Elliptic Functions

Let $\Lambda \subset \mathbf{C}$ be a lattice. Note that an elliptic function $f \in \mathrm{Mer}_\Lambda(C)$ is completely determined by its restriction to a fundamental domain Ω.

Definition 4.1. Let $f \in \mathrm{Mer}_\Lambda(\mathbf{C})$. A fundamental domain $\Omega \subset \mathbf{C}$ for Λ will be called **good** for f provided

$$\mathrm{Poles}(f) \cap \partial\Omega = \emptyset.$$

Example 4.2. The fundamental domain $\Omega = \{x + iy : 0 \leq x < 1 \text{ and } 0 \leq y < 1\}$ is not good for $\wp(z; i)$ since $\wp(z; i)$ has a pole at 0 which is contained in Ω.

One can check that by perturbing a given fundamental domain by some small $\varepsilon \in \mathbf{C}$ that there exists a good fundamental domain for any given elliptic function f.

Lemma 4.3. *Let $\Lambda \subset \mathbf{C}$ be a lattice. Let $f \in \mathrm{Mer}_\Lambda(\mathbf{C})$. Let Ω be a good fundamental domain for f.*

1. *The number of poles of f in Ω is finite.*
2. *The number of zeros of f in Ω is fiite.*
3. *The sum of the residues of f over a good fundamental domain is zero.*
4. *Any elliptic function without poles is constant.*

Proof.

- By meromorphicity, the poles and zeros of f are isolated.
- The last claim follows from Liouville's theorem.
- The third claim requires a small amount of work. Let $\Lambda = \omega_1\mathbf{Z} + \omega_2\mathbf{Z}$ and consider a good fundamental domain with corners $\alpha, \alpha + \omega_1, \alpha + \omega_1 + \omega_2, \alpha + \omega_2$.

  *** draw a picture ***

  We observe that $\partial(\alpha + \Omega)$ is the sum of the edges of the parallelogram. If we let $\gamma_j : [0, 1] \to \mathbf{C}$ parametrize these edges we have

  $$\gamma_1(t) + \omega_2 = \gamma_3(1 - t),$$

  and hence

  $$\begin{aligned} \int_{-\gamma_3} f(z)dz &= \int_{\gamma_2 + \omega_2} f(z)dz \\ &= \int_0^1 f(\gamma_1(t) + \omega_2)\gamma'(t)dt \\ &= \int_0^1 f(\gamma_1(t))\gamma_1'(t)dt \\ &= \int_{\gamma_1} f(z)dz \end{aligned}$$

Similarly,

$$\int_{\gamma_2} f(z)dz = -\int_{\gamma_4} f(z)dz.$$

This implies

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)dz = 0.$$

□

Using these properties we can finally derive our differential equation.

Exercise 4.4. In this exercise we will show

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_2(\Lambda) \tag{4.5}$$

for some constants g_2 and g_3.

1. For a lattice $\Lambda \subset \mathbf{C}$ and $m \geq 3$ define

$$G_m = G_m(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-m}.$$

[5] Show that

$$\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k.$$

2. Conclude that

$$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2 = O(z^2),$$

as $z \to 0$, which shows that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2$ is analytic at the origin of $\mathbf{C}$.

3. Conclude that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2$ is constant.[6]

4. Show the constant in the previous number is zero.

5 Characterization of Elliptic Functions

We will not prove every elliptic function is a combination of $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$. That is

$$\mathrm{Mer}_\Lambda(\mathbf{C}) = \mathbf{C}(\wp_\Lambda, \wp'_\Lambda).$$

The approximate strategy is to interpolate a function with the number of zeros and poles.

[5] $G_{2m}(\tau) := G_{2m}(\Lambda_\tau)$ is

[6] Hint: use 4.3

Lemma 5.1. *If $f(z) = f(-z)$ is a meromorphic, then the order of* $\mathrm{ord}_{z=0}(f(z)))$ *is even.*

Proof. Write

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n.$$

Writing out the equation $f(z) = f(-z)$ and using the fact that $\{z^n : n \in \mathbf{Z}\}$ is a basis for the vector space of Laurent series gives the result. □

Lemma 5.2 (Main Lemma)**.** *Let $\Lambda \subset \mathbf{C}$ be a lattice. Let $f(z) \in \mathrm{Mer}_\Lambda(\mathbf{C})$. If $f(z) = f(-z)$ then $f(z) \in \mathbf{C}(\wp_\Lambda(z))$.*

We first prove a series of reductions.

Lemma 5.3. *Let $f(-z) = f(z)$. Let $f(z) \in \mathrm{Mer}_\Lambda(\mathbf{C})$ for some lattice $\Lambda \subset \mathbf{C}$. In 5.2 we can assume without loss of generality that $f(z)$ has no zeros or poles on Λ.*

Proof. We first show that we can assume there are no poles. If $f(z)$ has a pole in Λ by translation we can reduce to examinine the case where the pole is at zero and has order $2m$. If $f(z)$ has a pole of order 2 then these exist some c such that

$$\widetilde{f}(z) := f(z) - c\wp(z)$$

no longer has a pole.

Inductive step: if the pole is order $2m$ for $m > 1$ then there exists some c such that

$$\widetilde{f}(z) = f(z) - c\wp(z)^m$$

has a pole of order $2m - 2$ and we can reduce to the previous case.

Suppose now $f(z)$ has no poles in Λ but has some zeros. If we take

$$\widetilde{f}(z) = f(z) - c$$

where c is non-zero then $\widetilde{f}$ no longer has a zero in Λ and we are done. □

We now show how to reduce the number of zeros and poles we need to consider for the interpolation prolem. Let $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ and let Ω_0 a fundamental parallelogram containing 0. we define $* : \Omega_0 \to \Omega_0$ by

$$a^* = \begin{cases} \omega_1 + \omega_2 - a, & a \text{ not on boundary} \\ \omega_1 - a, & a \text{ on } \omega_1\text{-line} \\ \omega_2 - a, & a \text{ on } \omega_2\text{-line} \end{cases}.$$

Identifying any other fundamental domain with Ω_0 allows us to define a similar operation on any fundamental domain (in fact, this operation is really defined on $\mathbf{C}/\Lambda$.)

Exercise 5.4. The zeros of $\wp(z) - c$ are simple with precisely double zeros at the points congruent to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$. (Hint: one need to check the zeros of $\wp'(z)$.)

Remark 5.5. THe double zeros are precisely the points fixed under $a \mapsto a^*$.

Lemma 5.6. *Let $f \in \mathrm{Mer}_\Lambda(\mathbf{C})$. If $f(a) = 0$ then $f(a^*) = 0$.*

Proof. If $f(a) = f(-a) = f(\omega_1 + \omega_2 - a) = 0$. The proof is similar for the boundary cases. $\square$

The above lemma and exercise imply that $\wp(z) - \wp(a)$ has roots at a and a^* with multiplicity one except in the special cases.

We can now prove Lemma 5.2

Proof of Lemma 5.2. The $\mathrm{Zeros}(f) \cap \Omega$ consists of points $a_1, \ldots, a_n$ and points congruent to $-a_1, \ldots, -a_n$. The set $\mathrm{Poles}(f) \cap \Omega$ consist of points $b_1, \ldots, b_n$ and points congruent to $-b_1, \ldots, -b_n$.

Forming the product with the appropriate multiplicities we have

$$\frac{1}{f(z)} \prod_{j=1}^{n} \frac{(\wp(z) - \wp(a_j))^{m_j}}{\prod_{j=1}^{l} (\wp(z) - \wp(b_j))^{n_j}} \in \mathrm{Mer}_\Lambda(\mathbf{C}).$$

Since this function has no poles it must be constant which proves our result. $\square$

Theorem 5.7. *Let $\Lambda \subset \mathbf{C}$ be a lattice. We have*

$$\mathrm{Mer}_\Lambda(\mathbf{C}) = \mathbf{C}(\wp_\Lambda(z), \wp'_\Lambda(z)).$$

Proof. Suppose $f(z) \in \mathrm{Mer}_\Lambda(\mathbf{C})$. In this proof we will use the shorthand $\wp_\Lambda(z) = \wp(z)$ and $\wp'_\Lambda(z) = \wp'(z)$. The functions

$$f(z) + f(-z), \frac{f(z) - f(-z)}{\wp'(z)}$$

are even functions in $\mathrm{Mer}_\Lambda(\mathbf{C})$. Hence, by Lemma 5.2, there exists rational functions $A(z), B(z) \in \mathbf{C}(z)$ such that

$$\begin{aligned} A(\wp(z)) &= f(z) + f(-z) \\ B(\wp(z)) &= \frac{f(z) - f(-z)}{\wp'(z)} \end{aligned}$$

This implies that

$$f(z) = A(\wp(z)) + B(\wp(z))\wp'(z).$$

$\square$