Hadamard's Theorem and Entire Functions of Finite Order — For Math 331

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1 Entire functions of finite order

Definition 1.1. An entire function f is **finite order** if and only if $\exists \rho_0, \exists R_0$ such that

$$|f(z)| < \exp(|z|^{\rho_0}) \text{ whenever } |z| \geq R_0.$$

The infimum of such ρ_0 is called the **order** of f and is denoted by $\rho = \rho(f)$.

Lemma 1.2. *Let f be a entire function of finite order.*

$$\rho(f) = \lim_{R\to\infty} \sup_{r\geq R} \frac{\log\log M(f,r)}{\log(r)},$$

were $M(f,r) = \max_{|z|=r} |f(z)|$.

Proof. If f is finite order,

$$\begin{aligned} & M(f,r) \leq \exp(r^{\rho_0}) \\ \implies & \log M(f,r) \leq r^{\rho_0} \\ \implies & \log\log M(f,r) \leq \rho_0 \log(r) \end{aligned}$$

So we have

$$\lim_{R\to\infty} \lim_{r\geq R} \frac{\log\log M(f,r)}{\log r} \leq \rho_0$$

$\square$

Example 1.3. Here are some functions and their orders:

1. e^z, $\rho = 1$
2. $\sin(z), \rho = 1$
3. $\cos(\sqrt{z}), \rho = 1/2$
4. $e^{e^z}, \rho = \infty$
5. $e^{z^2}, \rho = 2$

Here is the converse of the Weierstrass product formula. This is the main goal of these notes.

Theorem 1.4 (Hadamard)**.** *If f be an entire function of finite order $\rho \geq 0$ then f can be written as*

$$f(z) = e^{g(z)} z^m \prod_{j=1}^{\infty} E_d(z/a_j) \tag{1.5}$$

where

- $a_1, a_2, \ldots$ *are the zeros of* $f(z)$ *repeating according to their multiplicity.*
- $m = \operatorname{ord}_{z=0} f(z)$
- $g(z) \in \mathbf{C}[z]$

Furthermore we have $\deg(g), d \leq \rho$.

The proof in Ahlfors is a little incomplete and the proof in McMullen uses hyperbolic geometry. The proof in Schlag isn't finished. The proof in Green and Krantz is essentially the same al Ahlfors. We are going to follow McMullen's proof supplementing it with computations from Ahlfors and replacing his hyperbolic geometry arguments with a more down-to-earth estimate.

Strategy of Proof.

1. Prove this for entire function $f(z)$ without zeros: by existence of logarithms we know $f(z) = e^{g(z)}$. We show $g(z)$ is a polynomial (This involves the so-called Borel-Cartheodory inequality)[1]
2. By Weierstrass any entire function can be written as $f(z) = e^{g(z)}P(z)$ where $P(z)$ is a canonical product and $g(z)$ is some function.
   (a) Study the order of $P(z)$ (this uses Jensen's Formula — a formula that relates the size of the zeros of a function to its absolute value)
   (b) Get bounds on $1/P(z)$
   (c) Conclude that $f(a)/P(z)$ is entire of finite order as in the first case.

□

2 Case: functions without zeros

Exercise 2.1. If $f : D_R(0) \to D_R(0)$ is a conformal map with $f(0) = 0$ then $|f(z)| \leq |z|$. (Hint: consider $g(z) = g(z/R)/R$ and apply Schwarz.)

[1]This replaces a hyperbolic geometry argument that McMullen uses — an actually uses the Schwarz Lemma, an incarnation of hyperbolic geometry

Theorem 2.2 (Borel-Caratheodory). *Let f be holomorphic on a region containing $\overline{D_R(0)}$. For all $r < R$,*

$$M(f,r) \leq \frac{2r}{R-r} M(\operatorname{Re} f, R) + \frac{R+r}{R-r}|f(0)|.$$

Proof. Let $A = M(\operatorname{Re} f, R)$ (since $\operatorname{Re} f$ is harmonic this is positive). We break the proof into two cases: $f(0) = 0$ and $f(0) \neq 0$.

- Suppose $f(0) = 0$. The map f takes $D_R(0)$ to the region $\{z : \operatorname{Re} f(z) < A\}$. The map in figure 1 explains how the map

  $$g(z) = \frac{Rf(z)}{f(z) - 2A}$$

  is a conformal map $g : D_R(0) \to D_R(0)$ with $g(0) = 0$. By Schwarz's Lemma (see Exercise 2.1) we have $|g(z)| \leq |z|$. This gives

  $$\frac{R|f(z)|}{|f(z) - 2A|} \leq |z|.$$

  If $|z| < r$ we have

  $$R|f(z)| \leq r|f(z) - 2A| \leq r|f(z)| + 2Ar$$

  which implies

  $$|f(z)| \leq \frac{2r}{R-r} M(\operatorname{Re} f, R).$$

- Suppose $f(0) \neq 0$. Consider $g(z) = f(z) - f(0)$ and apply the previous case (you will find the string of inequalities below)

  $$\begin{aligned} |f(z)| - |f(0)| &\leq |f(z) - f(0)| \\ &\leq \frac{2r}{R-r} \max_{|w| \leq R} \operatorname{Re}(f(w) - f(0)) \\ &\leq \frac{2r}{R-r} (\max_{|w| \leq R} \operatorname{Re} f(w)) + |f(0)|. \end{aligned}$$

  This implies

  $$|f(z)| \frac{2r}{R-r} \max_{|w| \leq R} \operatorname{Re} f(w) + \frac{R+r}{R-r}|f(0)|,$$

  here we used $\frac{2r}{R-r} + 1 = \frac{r+R}{R-r}$.

$\square$

Theorem 2.3. *If $f(z)$ is an entire function of finite order ρ without zeros then $f(z) = e^{g(z)}$ where $g(z)$ is a polynomial of degree ρ (in particular $\rho \in \mathbf{N}$).*

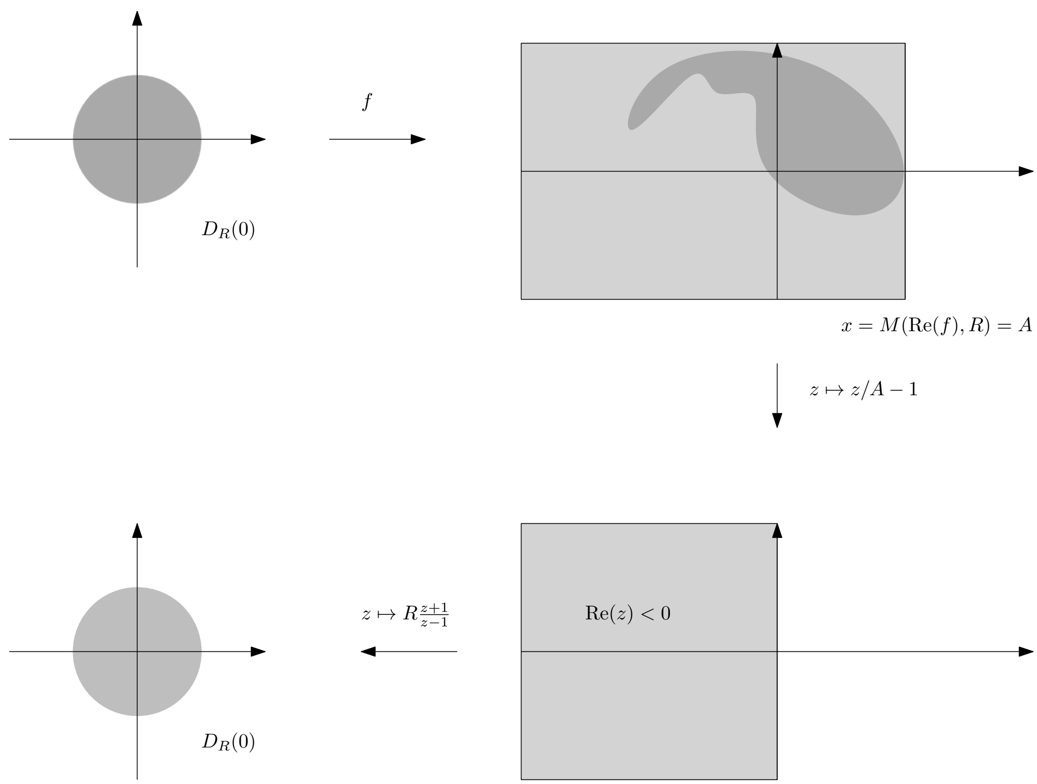

Figure 1: The composition of conformal maps used to imply the Borel-Caratheodory theorem.

Remark 2.4. Theorem 2.3 is equivalent to saying the following: if for all $\varepsilon > 0$ there exists some $r_n \to \infty$ such that

$$\operatorname{Re} g(z) < r_n^{\rho+\varepsilon}$$

whenever $|z| = r_n$ then $g(z)$ is a polynomial of degree at most ρ.

The sequence $r_n \to \infty$ is necessary sometimes to avoid weird behavior at particular radii in applications.

Proof. By the finite order hypothesis, there exists some R_0 such that $|f(z)| = e^{\operatorname{Re} g(z)} \leq e^{|z|^{\rho+\varepsilon}}$ for $|z| \geq R_0$. This implies $\operatorname{Re} g(z) \leq |z|^{\rho+\varepsilon}$. By Borel-Caratheodory, taking $R = 2r$ we have

$$\begin{aligned} M(g,r) &\leq \frac{2r}{R-r} M(\operatorname{Re} g, R) + \frac{R+r}{R-r}|g(0)| \\ &= 2M(\operatorname{Re} g, 2r) + 3|g(0)| \\ &\leq 2(2r)^{\rho+\varepsilon} + 3|g(0)| \\ &= O(r^{\rho+\varepsilon}). \end{aligned}$$

By Liouville's Theorem (the souped-up version) $g(z)$ must be a polynomial of degree less than or equal to ρ. [2] ∎

3 Jensen's formula

To move prove Hadamard's theorem where the entire function $f(z)$ has zeros we need to know something about the growth of the zeros. This is provided by Jensen's Formula:

Theorem 3.1 (Jensen's Formula)**.** *Let $f(z)$ be analytic in a region containing $D_R(0)$, $f(0) \neq 0$ and $f(z)$ has no zeros with $|z| = R$. We have*

$$\log|f(0)| \leq \sum_{j=1}^{n} \log|\frac{a_j}{R}| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta, \tag{3.2}$$

here $a_1, a_2, \ldots, a_n$ are the zeros of $f(z)$ insider the regions $D_R(0)$.

When looking at this at first, think of the case when $f(0) = 1$ so the left hand term vanishes. What this formula does is relate the absolute values of the roots to the absolute value of $f(z)$ on the circle $|z| = R$.

Proof. Suppose $f(z)$ has no zeros. Then $\log|f(z)|$ is harmonic and

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta$$

[2] Observe that in the application of Liouville's theorem we just need bounds on $M(f,r)$ for particular radii approaching infinity — so the version in the remark holds

by the mean value theorem.

Suppose now $f(z)$ has zeros inside $D_R(0)$. Let $a_1, \dots, a_n$ be the zeros of $f(z)$ in $D_R(0)$ where we repeat zeros according to multiplicity. Define

$$g(z) = f(z) \prod_{j=1}^{n} \frac{R^2 - a_j z}{R(z - a_j)}.$$

This function is analytic, free from zeros inside $D_R(0)$ (see Exercise 3.3). Consequently, $\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta$. Since, $g(0) = f(0)\prod_{j=1}^{n} \frac{R}{-a_j}$, we have

$$\log|g(0)| = \log|f(0)| + \sum_{j=1}^{n} \log|\frac{R}{a_j}|$$

and hence

$$\log|f(0)| + \sum_{j=1}^{R} \log|\frac{R}{a_j}| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

□

Exercise 3.3. The factors $B_{R/a}(z) = \frac{R^2 - az}{R(z-a)}$ appearing in the product are called **Blashke factors**. Here $|a| < R$.

1. Show that $||B_{R/a}(z)| = 1$ when $|z| = R$
2. Show that the zeros of $B_{R/a}(z)$ lie outside $D_R(0)$.

Exercise 3.4. This exercise extends Jensen's formula to the case when $f(z)$ has zeros on the boundary.

1. Show that $\int_0^{2\pi} \log|1 - e^{i\theta}|d\theta = 0$ (first show that this integral makes sense).[3]
2. Consider $g(z) = f(z)/\prod_{j=1}^{r}(z - Re^{i\theta_j})$ where $Re^{i\theta_1}, Re^{i\theta_2}, \dots, Re^{i\theta_r}$ are the zeros of $f(z)$ on the boundary. Show that by applying Jensen's Formula to $g(z)$ we recover Jensen's formular for $f(z)$.

4 Case: canonical products

We want to know when canonical products of zeros of an entire function converge. In this section we keep in mind the following criterion:

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|^{h+1}} < \infty \implies \prod_{j=1}^{\infty} E_h\left(\frac{z}{a_j}\right) \text{ converges },$$

where $E_h(z) = (1-z)\exp(z + z^2/2 + \cdots + z^h/h)$. We will control the left hand side using Jensen's formula.

[3] See Ahlfors, Chapter 5, Section 6, Exercise 4 — I think,

Definition 4.1. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of non-zero complex number ordered by size (meaning $|a_j| \leq |a_{j+1}|$). The **critical exponent** of the sequence is

$$\alpha = \inf\{\beta : \sum_{j=1}^{\infty} \frac{1}{|a_j|^{\beta}} < \infty\}.$$

Definition 4.2. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of non-zero complex number ordered by size. The **counting function** of the sequence is

$$N(r) = \#\{n : |a_n| \leq r\}.$$

A basic philosophy is that knowledge about the $N(r)$ is the same as knowledge of $r_n = |a_n|$.

Lemma 4.3. *For all $\beta > 0$ we have*

$$\sum_{n=1}^{\infty} |a_n|^{-\beta} = \int_0^{\infty} N(r)\beta r^{-\beta-1} dr. \tag{4.4}$$

Proof. Let $|a_j| r_j$ and define $r_0 = 0$. We have

$$\begin{aligned}
\int_0^{\infty} N(r)\beta r^{-\beta-1} dr &= \sum_{j=1}^{\infty} N(r_j) \int_{r_j}^{r_{j+1}} \beta r^{-\beta-1} dr \\
&= \sum_{j=1}^{\infty} N(r_j)(r_j^{-\beta} - r_{j+1}^{-\beta}) \\
&= \sum_{j=1}^{\infty} (N(r_j) - N(r_{j-1}) r_j^{-\beta} \\
&= \sum_{j=1}^{\infty} |a_j|^{-\beta}.
\end{aligned}$$

The last line follows from the fact that $N(r_j) - N(r_{j-1}) = \#\{a_k : |a_k| = r_j\}$ □

Lemma 4.5. *Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of non-zero complex numbers ordered by size. Let $N(r) = \#\{j : |a_j| < r\}$. We have*

$$\alpha = \lim_{R\to\infty} \sup_{r>R} \frac{\log N(r)}{\log r}. \tag{4.6}$$

I couldn't make a proof of this work using the p-test.

Proof. Let $\alpha_0 = \lim_{R\to\infty} \sup_{r>R} \frac{\log N(r)}{\log r}$. Let α be the critical exponent. We need to show $\alpha \leq \alpha_0$ and $\alpha_0 \leq \alpha$.

- We first show $\alpha_0 \geq \alpha$. We show convergence of $\sum |a_j|^{-\beta}$ whenever $\beta > \alpha_0$. By the definition of lim sup for all $\varepsilon > 0$ there exist some R such that for all $r > R$ we have $\log(N(r))/\log(r) \leq \alpha_0 + \varepsilon$. This implies

$$N(r) > r^{\alpha_0+\varepsilon} \text{ when } r \geq R \ .$$

Since

$$\int_0^\infty N(r) r^{-\beta-1} dr \text{ converges } \iff \int_R^\infty N(r) r^{-\beta-1} dr \text{ converges}$$

and

$$\int_R^\infty N(r) r^{-\beta-1} dr \leq \int_R^\infty r^{\alpha_0+\varepsilon-\beta+1} dr$$

we have convergence when $\alpha_0 + \varepsilon - \beta - 1 < -1$ or $\alpha_0 + \varepsilon < \beta$. Since $\varepsilon > 0$ was arbitrary this condition turns says

$$\{\beta : \beta > \alpha_0\} \subset \{\beta : \sum |a_j|^{-\beta} < \infty\},$$

and hence

$$\alpha_0 = \inf\{\beta : \beta > \alpha_0\} \geq \inf\{\beta : \sum |a_j|^{-\beta} < \infty\} = \alpha.$$

- [4] [5] [6]

We show now $\alpha_0 \geq \alpha$. We divergence of $\sum |a_j|^{-\beta}$ whenever $\beta < \alpha_0$.

We would like to use a similar divergence argument as in the pervious case but we don't get clear lower bounds because of the lim sup in the definition $\limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \alpha_0$ we have: $\forall \varepsilon > 0, \exists R \geq 0$ and a sequence $r_n \to \infty$ with $r_n \geq R$ such that

$$N(r_n) > r_n^{\alpha_0-\varepsilon}.$$

Because r_n could be spread out in some weird way, we don't get a simple lower bound on our integral.

First, observe that

$$\sum_{u \leq |a_j| \leq 2u} \frac{1}{|a_j|^\beta} \geq \sum_{u \leq |a_j| \leq 2u} \frac{1}{(2u)^\beta} = \frac{N(2r) - N(r)}{(2u)^\beta}.$$

[4] In [McM] he shows divergence of $\sum |a_j|^{-\beta}$ when we have $N(r) > r^{\alpha_0-\varepsilon}$ and $\beta < \alpha_0$.

[5] We do the opposite of `http://math.stackexchange.com/questions/275417/on-the-convergence-exponent-of-zeros-of-entire-functions`. Although the person answering is technically giving the wrong direction, this is a good example of where a technically wrong response gives the correct tool to solve a problem.

[6] Another post `http://math.stackexchange.com/questions/275070/order-of-growth-of-a-counting-function` obnoxiously asserts α_0 is the quantity we should start with, asserts that it is equal to $\alpha_0 = \alpha_1 = \limsup_{n\to\infty} \frac{\log(n)}{\log |a_n|}$ without proof (which is done is say [War] or [Lev96]) and proves $\alpha_1 = \alpha$ making a funny convention for ε.

The treatment of Hadamard's Theorem in [BY14] proves $\alpha_1 = \alpha$ and doesn't use α_0 at all.

This means

$$\begin{aligned}
\sum_{|a_j|<2^{m+1}u} \frac{1}{|a_j|^\beta} &= \sum_{|a_j|\le u} \frac{1}{|a_j|^\beta} + \sum_{j=0}^{m} \sum_{2^j u<|a_j|\le 2^{j+1}u} \frac{1}{|a_j|^\beta} \\
&\ge \sum_{j=1}^{m} \frac{N(2^{j+1}u)-N(2^j u)}{(2^j u)^\beta} \\
&\ge \frac{1}{(2^{m+1}u)^\beta} \sum_{j=1}^{m} N(2^{j+1}u)-N(2^j u) \\
&= \frac{N(2^{j+1}u)-N(u)}{(2^{m+1}u)^\beta}
\end{aligned}$$

Now we have a lower bound! Define m_n and u_n from the sequence r_n by

$$\begin{aligned}
\lfloor \log_2(r_n) \rfloor &= m_n + 1, \\
\frac{r_n}{2^{m_n+1}} = u_n,
\end{aligned}$$

we get $r_n = 2^{m_n+1}u_n$ with $u_n < 2$ and

$$\begin{aligned}
\sum_{|a_j|\le r_n} |a_j|^{-\beta} &\ge \frac{N(r_n)-N(u_n)}{r_n^\beta} \\
&\ge \frac{N(r_n)-N(2)}{r_n^\beta} \sim \frac{N(r_n)}{r_n^\beta} \ge \frac{r_n^{\alpha_0-\varepsilon}}{r_n^\beta} = r_n^{\alpha_0-\varepsilon-\beta}.
\end{aligned}$$

Hence the series diverges if $\alpha_0 - \varepsilon > \beta$. Since this is done for arbitrary $\varepsilon > 0$ we have the series diverges whenever $\alpha_0 > \beta$. Hence

$$\{\beta : \beta < \alpha_0\} \subset \{\beta : \sum |a_j|^{-\beta} \text{ diverges }\}$$

which implies

$$\alpha_0 = \sup\{\beta : \beta < \alpha_0\} \le \sup\{\beta : \sum |a_j|^{-\beta} \text{ diverges }\} = \alpha.$$

□

Exercise 4.7. Show that

$$\alpha = \limsup_{n\to\infty} \frac{\log n}{\log |a_n|}.$$

Proof. Let $\alpha_1 = \limsup_{n\to\infty} \frac{\log n}{\log |a_n|}$ and let $\alpha = \inf\{\beta : \sum_j |a_j|^{-\beta} < \infty\}$. We show that $\alpha_1 \le \alpha$ and $\alpha \le \alpha_1$.

- Suppose $\beta > \alpha_1$. This implies $\beta > \alpha_1 + \varepsilon$ for some $\varepsilon > 0$. By the definition of lim sup: $\forall \varepsilon > 0, \exists n_0 \geq 0, \forall n \in \mathbf{N}$:

  $$n \geq n_0 \implies \frac{\log n}{\log |a_n|} < \alpha_1 + \varepsilon.$$

  Fix such an n_0. This condition is the same as $n^{1/(\alpha_1+\varepsilon)} < |a_n|$. Hence for $n \geq n_0$ we have

  $$\frac{1}{|a_n|^\beta} \leq \frac{1}{n^{\frac{\beta}{\alpha_1+\varepsilon}}}.$$

  Since $\beta > \alpha_1 + \varepsilon$ the series $\sum_{n\geq 0} |a_n|^{-\beta}$ converges.

- Suppose $\beta < \alpha_1$. This implies $\beta < \alpha_1 - \varepsilon$ for some $\varepsilon > 0$. By the definition of lim sup: $\forall \varepsilon > 0, \exists n_0 \geq 0$ and a sequence $n_j \to \infty$ as $j \to \infty$ with $n_j \geq n_0$ such that for all j

  $$\frac{\log n_j}{\log |a_{n_j}|} > \alpha_1 - \varepsilon.$$

  Fix such a sequence n_j. This condition is equivalent to $|a_{n_j}| < n_j^{1/(\alpha_1-\varepsilon)}$. Hence for n_j we have

  $$\sum_{n=1}^{n_j} \frac{1}{|a_n|^\beta} \geq \frac{n_j}{|a_{n_j}|^\beta} > \frac{n_j}{n_j^{\frac{\beta}{\alpha_1-\varepsilon}}} = n_j^{1-\frac{\beta}{\alpha_1-\varepsilon}}.$$

  Since $\beta < \alpha_1 - \varepsilon$ we have $\beta/(\alpha_1 - \varepsilon) < 1$ and the series diverges.

$\square$

Lemma 4.8. *Let f be an entire function of finite order. The critical exponent of the non-zero zeros of $f(z)$ is less than the order of f:*

$$\alpha(f) \leq \rho(f).$$

Proof.

$$\begin{aligned}
& \log(2)N(r) \leq \log M(2r) \\
\implies \quad & \log N(r) \leq \log^2 M(2r) - \log^2(2) \\
\implies \quad & \frac{\log N(r)}{\log(r)} \leq \frac{\log^2 M(2r) - \log^2(2)}{\log(2r) - \log(2)} = \frac{\log^2(M(2r))/\log(2r) - \log^2(2)/\log(2r)}{1 - \log(2)/\log(2r)} \\
\implies \quad & \alpha = \limsup_{r\to\infty} \frac{\log N(r)}{\log(r)} \leq \limsup_{r\to\infty} \frac{\log^2 M(2r)}{\log(2r)} = \rho.
\end{aligned}$$

$\square$

Remark 4.9. We actually have $\alpha(f) = \rho(f)$.

Lemma 4.10. *Let f be an entire function of finite order ρ. Let $a_1, a_2, \dots$ be its non-zero zeros. The number $d = \lfloor \rho \rfloor$ is the smallest integer making $\prod_{j=1}^{\infty} E_d\left(\frac{z}{a_j}\right)$ converge.*

Proof. We have $\lfloor \rho \rfloor \le \rho \le \lfloor \rho \rfloor + 1$. Since $\alpha < \rho$ we have that $\alpha < \lfloor \rho \rfloor + 1$. This implies $\sum_{j=1}^{\infty} |a_j|^{-\lfloor \rho \rfloor - 1}$ converges and hence the desired canonical product converges. □

If $f(z)$ is an entire function of finite order ρ and zeros $a_1, a_2, \dots$ we now know that $f(z)/\left(z^m \prod_{j=1}^{\infty} E_{\lfloor \rho \rfloor}(z/a_j)\right)$ is an entire function of finite order without zero. It remains to get a bound on the order of this function. To do this, we need a lower bound on $E_{\lfloor \rho \rfloor}(z/a_j)$.

McMullen and Ahlfors say some things about lower bounds and it looks like McMullen is following Stein and Shakarchi. Here we follow the material before Stein and Shakarchi, Lemma 5.3 to get the bound.

Lemma 4.11. *1. There exists some C_1 such that*

$$\log |E_h(z)| \ge -C_1 |z|^{h+1}$$

when $|z| \le 1/2$.

2. There exists some C_2 such that

$$\log |E_h(z)| \ge -C_2 |z|^h$$

when $|z| \ge 1/2$.

Proof. We follow [SS03].

1. Suppose $|z| \le 1/2$. This implies $\log(1-z) = -\sum_{j \ge 1} z^j/j$. We have

$$E_h(z) = \exp\left(\log(1-z) + \sum_{j=1}^{h} z^j/j\right) = \exp\left(-\sum_{j=h+1} z^j/j!\right) = \exp(w).$$

   Since $|e^w| \ge e^{-|w|}$ and $|w| \le C|z|^{h+1}$ we have

$$|E_h(z)| \ge \exp\left(-|\sum_{h+1}^{\infty} z^j/j|\right) \ge e^{-C|z|^{h+1}}.$$

2. Suppose $|z| \ge 1/$. We have

$$|E_h(z)| = |1-z| \cdot |e^{z+z^2/2+\cdots+z^h/h}|.$$

   The proof follows from

$$|e^{z+z^2/2+\cdots+z^h/h}| \ge e^{-|z+z^2/2+\cdots z^h/h|} \ge e^{-C|z|^h}$$

   for some $C \ge 0$.

$\square$

Lemma 4.12. *Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of complex numbers with critical exponent α. Let $h = \lfloor \alpha \rfloor$. For all z sufficiently large and outside $\bigcup_{j\geq 1} D_{1/r_j^{\alpha+\varepsilon}}(a_j)$ there exists a constant $B = B_\varepsilon$ such that*

$$|\prod_{j=1}^{\infty} E_h(z/a_j)| \geq \exp(-B|z|^{\alpha+\varepsilon}).$$

(Hence there exists a sequence $z_j \to \infty$ such that this inequality holds.)

Proof.

$$\prod_{j=1}^{\infty} |E_h(z/a_j)| = \underbrace{\prod_{|z/a_j|<1/2}^{\infty} |E_h(z/a_j)|}_{\text{case I}} \cdot \underbrace{\prod_{|z/a_j|\geq 1/2} |E_h(z/a_j)|}_{\text{case II}}$$

- Case I: $|z/a_j| < 1/2$ (there are infinitely many such a_j since $a_j \to \infty$).

  By the bounding lemma, $\log |E_h(z/a_j)| \geq -c_1|z/a_j|^{h+1}$

  Observe that

  $$|a_j|^{-h-1} = |a_j|^{-\alpha-\varepsilon}|a_j|^{\alpha+\varepsilon-h-1} \leq |a_j||2z|^{\alpha+\varepsilon-h-1}.$$

  Here we used $\alpha + \varepsilon - \lfloor \alpha \rfloor - 1 < 0$ and $|2z| < |a_j|$.

  These together give

  $$\begin{aligned} \prod_{|z/a_j|<1/2} |E_h(z/a_j)| &\geq \exp\left(-c_1 \sum_{|z/a_j|<1/2} |z/a_j|^{h+1}\right) \\ &\geq \exp\left(-c_1|z|^{h+1} \sum_{|z/a_j|<1/2} |a_j|^{-\alpha-\varepsilon}|2z|^{\alpha+\varepsilon-h-1}\right) \\ &= \exp\left(\frac{-c_1|2z|^{\alpha+\varepsilon}}{2^{h+1}} \sum_{|z/a_j|<1/2} |a_j|^{-\alpha-\varepsilon}\right) \\ &= e^{-B_1|z|^{\alpha+\varepsilon}} \end{aligned}$$

- Case II: $|z/a_j| \geq 1/2$ (there are finitely many such a_j). We have

  $$\prod_{|z/a_j|\geq 1/2} |E_h(z/a_j)| \geq \underbrace{\prod_{|z/a_j|\geq 1/2} |1-z|}_{\text{case II.2}} \cdot \underbrace{\prod_{|z/a_j|\geq 1/2} e^{-c_2|z/a_j|^h}}_{\text{case II.2}}$$

– Case II.1: We have

$$\prod_{|z/a_j|\geq 1/2} e^{-c_2|z/a_j|^h} = \exp\left(-c_2 \sum_{|z/a_j|\geq 1/2} |z/a_j|^h\right).$$

We will now bound the exponent from below:

$$\begin{aligned}
-c_2 \sum_{|z/a_j|\geq 1/2} |z/a_j|^h &= -c_2 \sum_{|z/a_j|\geq 1/2} |z/a_j|^h \\
&\geq -c_2|z|^{\alpha+\varepsilon} \sum_{|z/a_j|>1/2} \frac{(|a_j|/2)^{h-\alpha-\varepsilon}}{|a_j|^h} \qquad (\ |z|>|a_j|/2 \text{ and } h-\alpha-\varepsilon<0\ ) \\
&= -c_2|z|^{\alpha+\varepsilon} \sum_{|z/a_j|\geq 1/2} \frac{1}{2^{h-\alpha-\varepsilon}} \frac{1}{|a_j|^{\alpha+\varepsilon}} \\
&= -b_2|z|^{\alpha+\varepsilon}.
\end{aligned}$$

– Case II.2: We look at $\prod_{|z/a_j|\geq 1/2} |1-\frac{z}{a_j}|$.
Since $|z/a_j|>1/2$ we have $2|z|>|a_j$ which implies there are exactly $N(2|z|)$ terms.
From the hypotheses $|z-a_j|>1/|a_j|^{\alpha+\varepsilon}$ we get

$$\begin{aligned}
|1-\frac{z}{a_j}| &= |z-a_j|\frac{1}{|a_j|} \\
&> \frac{1}{|a_j|^{\alpha+\varepsilon}}\frac{1}{|a_j|} \\
&= \frac{1}{|a_j|^{\alpha+1+\varepsilon}}
\end{aligned}$$

This implies

$$\begin{aligned}
\log(\prod_{j=1}^{N(2|z|)} |1-z/a_j|) &= \sum_{j=1}^{N(2|z|)} \log|1-z/a_j| \\
&> \sum_{j=1}^{N(2|z|)} (\alpha+1+\varepsilon)\log|a_j| \\
&> -\sum_{j=1}^{N(2|z|)} (\alpha+1+\varepsilon)\log(2|z|) \\
&= -(\alpha+1+\varepsilon)\log(2|z|)N(2|z|). \\
&\geq -(\alpha+1+\varepsilon)\log(2|z|)(2|z|)^{\alpha} \quad \text{by formula for } \alpha \\
&\geq -b_1'|z|^{\alpha+\varepsilon}
\end{aligned}$$

Combining cases I, II.1 and II.2 show that for every $\varepsilon > 0$ there exists a constant such that for all z not in $\bigcup_j D_{1/r_j^{\alpha+\varepsilon}}(a_j)$ we have

$$|\prod_{j=1}^{\infty} E_h(z/a_j)| \geq \exp(-B|z|^{\alpha+\varepsilon})$$

this is equivalent to

$\square$

5 Proof of Hadamard's Factorization Theorem

Proof of Hadmard's Theorem. Consider now

$$f(z)/z^m P(z) = e^{g(z)}$$

where $P(z) = \prod_{j\geq 1} E_{\lfloor\rho\rfloor}(z/a_j)$. For $z \in \mathbf{C} \setminus \bigcup_{j\geq 1} D_{1/r_n^{\alpha}}(a_j)$ and $|z| = r$ sufficiently large we have

$$\begin{aligned} \frac{|f(z)|}{|z^m P(z)|} &\leq \frac{e^{r^{\rho+\varepsilon}}}{r^m e^{-Br^{\alpha+\varepsilon}}} \\ &= \exp(r^{\rho+\varepsilon} + Br^{\alpha+\varepsilon} - m\log(r)) \\ &\leq e^{(1+B)r^{\rho+\varepsilon}}. \end{aligned}$$

This shows that $\operatorname{Re} g(z) = O(r^{\rho+\varepsilon})$ (along particular radii where the lower bound applies) and hence we are in the hypotheses of Theorem 2.3 and Remark 2.4. This proves that $g(z)$ is a polynomial of degree at most ρ. $\square$

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