#1) $\zeta_n = e^{2\pi i/n}$,

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satisfies, $z^n - 1 = 0$

$$z^n - 1 = (z-1)(1 + z + z^2 + \cdots + z^{n-1})$$

$$\Rightarrow \frac{\zeta_n^n - 1}{\zeta_n - 1} = 1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = 0.$$

#2) • Let $z \in \mathcal{H}$ so $y > 0$.

$$w = \frac{z-i}{z+i}$$

$$\left|\frac{z-i}{z+i}\right|^2 = \frac{x^2+(y-1)^2}{x^2+(y+1)^2} < 1 \quad \text{when } y>0$$

(since $y+1 > y-1$)

this shows $f: \mathcal{H} \to \mathbb{D}$.

• We show the map is injective & surjective. We solve for the inverse:

$$w = \frac{z-i}{z+i} \Rightarrow w(z+i) = z-i$$

$$\Rightarrow (w-1)z + i(w+1) = 0$$

$$\Rightarrow z = -\frac{i(w+1)}{w-1}$$

thus $g(w) = -i\frac{w+1}{w-1}$ is the inverse.

One can actually check directly this is an inverse,

$$f(g(w)) = \frac{g(w)-i}{g(w)+i}$$

$$= \frac{-i\left(\frac{w+1}{w-1}\right)-i}{-i\left(\frac{w+1}{w-1}\right)+i}$$

$$= \frac{-i\left((w+1)+(w-1)\right)}{i\left(-(w+1)+(w-1)\right)}$$

$$= \frac{-(2w)}{-2} = w.$$

This show the map is surjective (for every w, $g(w) \in \mathcal{H}$ maps to w under the map f).

$$\frac{z_1-i}{z_1+i} = \frac{z_2-i}{z_2+i}$$

$$\Rightarrow (z_1-i)(z_2+i) = (z_2-i)(z_1+i)$$

$$\Rightarrow \cancel{z_1z_2} - iz_2 + iz_1 + \cancel{1} = \cancel{z_2z_1} - iz_1 + iz_2 + \cancel{1}$$

$$\Rightarrow 2iz_1 = 2iz_2 \Rightarrow z_1 = z_2.$$

So the map is injective.

#3)

$$\int_0^{2\pi} \cos(\theta)^{2n}\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^{2n} d\theta$$

$$= \int_0^{2\pi} \left[\sum_{j=0}^{2n} \binom{2n}{j} \frac{1}{2^{2n}} \left(e^{i\theta}\right)^{j} \left(e^{-i\theta}\right)^{2n-j}\right] d\theta$$

$$= \sum_{j=0}^{2n} \binom{2n}{j} \frac{1}{2^{2n}} \int_0^{2\pi} e^{i\theta(j-(2n-j))}\, d\theta$$

$$= \sum_{j=0}^{2n} \binom{2n}{j} \frac{1}{2^{n}} \int_0^{2\pi} e^{i\theta(2j-2n)}\, d\theta$$

$$= \binom{2n}{n} \frac{1}{2^n}\, 2\pi \quad //$$

NOTE: for $\ell \in \mathbb{Z}$

$$\int_0^{2\pi} e^{i\ell\theta}\, d\theta = \begin{cases} 0, & \ell \neq 0 \\ 2\pi, & \ell = 0 \end{cases}$$

#4)

[I made a typo in the original post
Most people write

$$\frac{1}{R} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{1/m}$$ ← this is the version good for the proofs.

this is the same as

$$R = \lim_{n\to\infty} \inf_{m\geq n} |a_m|^{-1/m}.$$]

$$A_n - \frac{1}{R} = \varepsilon_n,$$

$$|a_m|^{1/m} \leq A_n, \quad m \geq n$$

$$\left|\sum_{n=0}^{N} a_n z^n\right| \leq \sum_{n=0}^{N} |a_n| |z|^n$$

$$\leq \sum_{n=0}^{N} \left(|a_n|^{1/n}\right)^n |z|^n$$

$$\leq \sum_{n=0}^{N} \left(\sup_{m\geq n} |a_m|^{1/m}\right)^n |z|^n$$

$$\leq \underbrace{\sum_{n=0}^{m_1} \left(\sup_{m\geq n} |a_m|^{1/m}\right)^n |z|^n}_{\text{Bounded}} + \sum_{n=m_1}^{N} \left(\frac{1}{R-\varepsilon}\right)^n |z|^n$$

For a given $\varepsilon > 0$ we may choose m_1 large enough (and depending on ε) so that

$$\sup_{m\geq n} |a_m|^{1/m} < \frac{1}{R-\varepsilon},$$

Take $|z| < r_1 < R$, let ε be such that $r_1 < R-\varepsilon$. We have,

$$\sum_{n=m_1}^{N} a_n z^n \leq \sum_{n=m_1}^{N} \left(\frac{1}{R-\varepsilon}\right)^n |z|^n \leq \sum_{n=m_1}^{N} \frac{1}{(R-\varepsilon)} r_1^n$$

which implies that our series converges absolutely and uniformly as $N\to\infty$ by the Weierstrass M-test.

#5)

Suppose that $|z| < r < R$ where

$$R = \liminf_{n\to\infty} \inf_{m\geq n} |a_m|^{-1/m}.$$

(Use the same trick! $\frac{1}{R} = \lim_{n\to\infty} \sup_{m\geq n} |a_m|^{1/m}$

$$\left|\sum_{n\geq 0}^{N} n a_n z^{n-1}\right| \leq \sum_{n=0}^{N} n|a_n|\,|z|^{n-1}$$

$$= \sum_{n=0}^{N} n\left(|a_n|^{1/n}\right)^n r^{n-1}$$

$$= \underbrace{\sum_{n=0}^{M} n\left(|a_n|^{1/n}\right)^n r^{n-1}}_{\text{bounded}} + \underbrace{\sum_{n=M+1}^{N} n\left(|a_n|^{1/n}\right)^n r^{n-1}}_{\text{stuff we need to check converges.}}$$

We pick M large enough so that for $n \geq M+1$,

$$\sup_{n\geq M+1} |a_n|^{1/n} \leq \frac{1}{R-\varepsilon}$$

and $r < R-\varepsilon$,

$$\Rightarrow \sum_{n=M+1}^{N} n\left(|a_n|^{1/n}\right)^n r^{n-1}$$

$$= \sum_{n=M+1}^{N} n\left(\frac{1}{R-\varepsilon}\right)^n r^{n-1}$$

$$= \frac{1}{r}\sum_{n=M+1}^{N} n\left(\frac{r}{R-\varepsilon}\right)^n$$

This series converges so by the Weierstrass M-test, the series

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

converges absolutely & uniformly on sets $D_r(0)$ when $r < R$.

· For the 2nd part we use the following lemma.

Lemma: Suppose $u_n(z) \to u(z)$ uniformly on a set S. If the limit $u_n'(z)$ converges uniformly on S & $u'(z)$ exists then $u_n'(z) \to u'(z)$ on S.

This is the situation we have.

#6

$$\sum_{n\geq 1}\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \sum_{n\geq 1}\frac{z^{n-1}}{(1-z)(1+z+\cdots+z^{n-1})(1-z)(1+z+\cdots+z^n)}$$

$$= \frac{1}{z(1-z)^2}\sum_{n\geq 1}\frac{z^n}{(1+z+\cdots+z^{n-1})(1+z+\cdots+z^n)}$$

$$= \frac{1}{z(1-z)^2}\sum_{n\geq 1}\left(\frac{1}{1+z+\cdots+z^{n-1}} - \frac{1}{1+z+\cdots+z^n}\right)$$

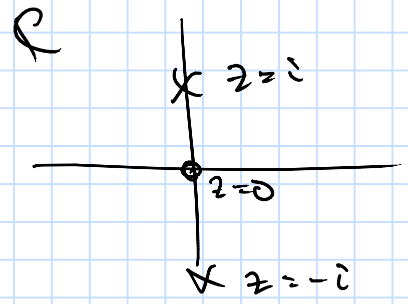
$$= \frac{1}{z(1-z)^2}\left(1 - \lim_{n\to\infty}\frac{1}{\sum_{k=0}^{\infty} z^k}\right)$$

$$= \begin{cases} \dfrac{1}{(1-z)^2}, & |z|<1 \\[2ex] \dfrac{1}{z(1-z)^2}, & |z|>1. \end{cases}$$

#7)

(a) $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (z^2)^j = \sum_{n=0}^{\infty} z^{2j}.$$

the radius of convergence is actually the distance to the nearest pole, so in this case $R=1$



(b) $$\sum_{n=0}^{\infty} n!\, z^n$$

$$\frac{a_n}{a_{n+1}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$$ as $n \to \infty$ so it has a radius of convergence zero.

(c) $$\frac{z^2+2z-4}{z} = f(z)$$

has a pole a $z=0$, so the radius of convergence is 1.

Let's rework the numerator:

$$z^2+2z-4 = h(z)$$

$$h(z) = h(1) + h'(1)(z-1) + \frac{h''(1)}{2}(z-1)^2 = 4(z-1) + (z-1)^2.$$

$$h(1) = 0$$

$$h'(1) = 2(1)+2 = 4$$

$$h''(1) = 2$$

Let's rework the denominator:

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Put them together,

$$\frac{h(z)}{z} = \left[4(z-1) + (z-1)^2\right]\left(\sum_{n=0}^{\infty} (-1)^n (z-1)^n\right)$$

$$= 4\sum_{n=0}^{\infty} (-1)^n (z-1)^{n+1} + \sum_{n=0}^{\infty} (-1)^n (z-1)^{n+2}$$

$$= 4(z-1) + \sum_{n=2}^{\infty} \underbrace{\left(4(-1)^{n-1} + (-1)^n\right)}_{(-1)^{n-1}(4-1)} (z-1)^n$$

$$= 4(z-1) + \sum_{n=2}^{\infty} (-1)^{n+1}\, 3\, (z-1)^n.$$

#3)

Proof using Cauchy's Formula: (1st Approach)

$$F'(z) = \frac{1}{2\pi i}\int_\alpha \frac{F(\xi)}{(\xi - z)^2}\,d\xi, \qquad \alpha^+ \ni z$$

$$= \frac{1}{2\pi i}\int_\alpha \frac{1}{(\xi-z)^2}\int_\gamma f(\xi, s)\,ds\,d\xi$$

Fubini

$$= \int_\gamma \frac{1}{2\pi i}\int_\alpha \frac{f(\xi,s)}{(\xi-z)^2}\,d\xi\,ds$$

$$= \int_\gamma \frac{\partial f}{\partial z}(z,s)\,ds$$

You can check that Fubini works by breaking this into real integrals.

I give two other approaches, one which works & the other I had trouble finishing ...

Remark.

This is pretty unsatisfying though because in class we wanted to use the fact that

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z}\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z)^2}\,d\xi$$

to show that holomorphic functions were ∞-ly differentiable.

Here is another way to give a self contained explanation of the formulas

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z)^{n+1}}\,d\xi.$$

Procedure:

1) We show that holomorphic fns are analytic by using the Cauchy Integral formula & uniform convergence of power series to get a power series formula for $f(z)$.
2) We then use that power series are infinitely differentiable as in exercise 3 of HW01.
3) We then use Taylor's Theorem to conclude the derivative formulas. i.e. If $f(z) = \sum_{n=0}^\infty a_n (z-z_0)^n$ then

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

From the expansion

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{(\xi - z_0)}\,\frac{1}{1 - \frac{z-z_0}{\xi - z_0}} = \sum_{n=0}^\infty \left(\frac{z-z_0}{\xi-z_0}\right)^n \frac{1}{\xi - z_0}$$

like this

we get that

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi$$

and hence that

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi.$$

4) After we have established these formulas we conclude.

2nd Approach

Lemma 1

$$g(z,\xi,h) := \begin{cases} \dfrac{f(z+h,\xi) - f(z,\xi)}{h}, & h \neq 0 \\[2mm] \dfrac{\partial f}{\partial z}(z,\xi), & h = 0 \end{cases}$$

is a continuous function on $A \times B \times D_r(0)$ & hence uniformly continuous.

Given the Lemma, if $h_n = \frac{1}{n}$ then

$$g(z,\xi,\tfrac{1}{n}) := g_n(z,\xi)$$

converges uniformly to

$$\frac{\partial f}{\partial z}(z,\xi)$$

for $(z,\xi) \in A \times B$.

Lemma 2 If $f_n(z)$ converges uniformly to $f(z)$ on some domain D then for all $\gamma \subseteq D$ C^1-arcs,

$$\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz.$$

Using this lemma & the uniform convergence, for fixed z we have

$$\int_\gamma g_n(z,\xi)\,d\xi \longrightarrow \int_\gamma \frac{\partial f}{\partial z}(z,\xi)\,d\xi.$$

On the other hand,

$$\int_\gamma g_n(z,\xi)\,d\xi := \int_\gamma \left(\frac{f(z+\frac{1}{n},\xi) - f(z,\xi)}{1/n}\right)d\xi \to \frac{\partial}{\partial z}\int_\gamma f(z,\xi)\,d\xi$$

By the definition of the derivative. This shows,

$$\frac{\partial}{\partial z}\int_\gamma f(z,\xi)\,d\xi = \int_\gamma \frac{\partial f}{\partial z}(z,\xi)\,d\xi.$$

It remains to prove the Lemmas.

proof of Lemma 2

We resort to real variables. Let $f_n(z) = u_n(z) + i v_n(z)$. Let $f(z) = u(z) + iv(z)$ & $\gamma(t) = x(t) + iy(t)$, for $t \in [a,b]$.

By the defn of the norm

$$f_n \to f$$

uniformly $\Rightarrow$ $u_n \to u$ & $v_n \to v$ uniformly.

$$\int_\gamma f_n(z)\,dz = \int_a^b \Big[\big(u_n(t)x'(t) - v_n(t)y'(t)\big) + i\big(u_n(t)y'(t) + v_n(t)x'(t)\big)\Big]$$

where $u(t) = u(\gamma(t))$ & $v(t) = v(\gamma(t))$

Each of these integrals is real & uses the measures

$$x'(t)\,dt, \quad y'(t)\,dt.$$

We also have $u_n(t) \to u(t)$ & $v_n(t) \to v(t)$ uniformly (we are using that the pullback of uniformly convergent sequences of fns by continuous maps is uniformly — we are pulling back by $\gamma(t)$). This means that the limits of the integrals are the integrals of the limit & we are done.

Some side work where I was thinking about sequences: diff'ble

$$g(z_n,\xi_n,h_n) \to g(z_n,\xi_n,h_0)$$

cont of $g(z,\xi,h_n)$

cont of $g(z,\xi,0)$

$h_n \neq 0$

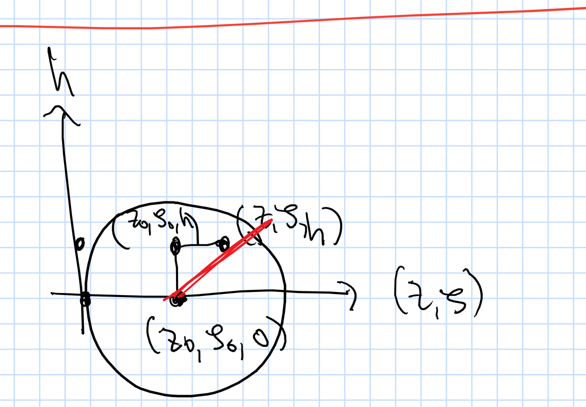
cont

diff'ble

$$g(z_0,\xi_0,h_n) \longrightarrow g(z_0,\xi_0,h_0)$$

$h_0 = 0$

$h_0 \neq 0$ just continuity.



#9) [Extra Credit]

[Sudbery, Theorem 1, page 9]

$$\frac{df}{dq} = \frac{\partial f}{\partial t} = -i\frac{\partial f}{\partial x} = -j\frac{\partial f}{\partial y} = -k\frac{\partial f}{\partial z}$$

Write,

$$f(q) = g(v,w) + j\,h(v,w)$$

where

$$q = v + iw$$
$$v = t + ix,$$
$$w = y - iz,$$

with, $x,y,z,t \in \mathbb{R}$. The C.R. eqns then become

$$\frac{\partial g}{\partial t} = i\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} = i\frac{\partial h}{\partial z},$$
$$\frac{\partial h}{\partial t} = i\frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y} = i\frac{\partial g}{\partial z}.$$

In terms of complex derivatives, we have

$$\begin{cases} \frac{\partial g}{\partial \bar v} = \frac{\partial h}{\partial \bar w} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial w} = 0, & (1) \\ \frac{\partial g}{\partial v} = \frac{\partial h}{\partial w}, & (2) \\ \frac{\partial h}{\partial \bar v} = -\frac{\partial g}{\partial \bar w}. & (3) \end{cases}$$

Eqn (1) gives:

- $g = g(v, \bar w)$ $\mathbb{C}$-analytic (1.2)
- $h = h(\bar v, w)$ $\mathbb{C}$-analytic. (1.3)

Eqn (2) gives:

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{\partial h}{\partial w}\right] = 0 \quad \text{By (1.3).}$$

$$\frac{\partial^2 h}{\partial w^2} = \frac{\partial}{\partial w}\left[\frac{\partial g}{\partial v}\right] = 0 \quad \text{By (1.2)}$$

Eqn (3) gives:

$$\frac{\partial^2 h}{\partial \bar v^2} = \frac{\partial}{\partial \bar v}\left[-\frac{\partial g}{\partial \bar w}\right] = 0 \quad \text{By (1.2)}$$

$$\frac{\partial^2 g}{\partial \bar w^2} = \frac{\partial}{\partial \bar w}\left[\frac{\partial h}{\partial \bar v}\right] = 0 \quad \text{By (1.3).}$$

So, h is linear in w & $\bar v$ & g is linear in v & $\bar w$.

$$\Rightarrow \begin{cases} g = \alpha + \beta v + \gamma \bar w + \delta v \bar w \\ h = \varepsilon + \xi \bar v + \eta w + \theta \bar v w \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \xi, \eta, \theta \in \mathbb{C}$.

Using equations (2) & (3) one more time we get

$$\begin{cases} \beta = \eta \\ \xi = -\gamma \\ \delta = \theta = 0 \end{cases}$$

$\therefore$

$$f(q) = g + jh$$
$$= \alpha + j\varepsilon + (v + jw)(\beta - j\gamma)$$
$$= a + qb$$

where

$$a = \alpha + j\varepsilon$$
$$b = \beta - j\gamma. \quad //$$

#10) $\Phi(x,y) = \phi(u(x,y), v(x,y))$

You just use the chain rule & C.R. eqns:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}$$

$$\Rightarrow \Phi_{xx} = u_x(\phi_{uu}u_x + \phi_{uv}v_x) + \underline{u_{xx}\phi_u}$$
$$+ v_x(\phi_{vv}v_x + \phi_{uv}u_x) + \underline{v_{xx}\phi_v}$$

$$\Phi_{yy} = u_y(\phi_{uu}u_y + \phi_{uv}v_y) + \underline{u_{yy}\phi_u}$$
$$+ v_y(\phi_{vv}v_y + \phi_{uv}u_y) + \underline{v_{yy}\phi_v}$$

$$\Phi_{xx} + \Phi_{yy} =$$

$$\phi_{uu}(u_x^2 + u_y^2) + \phi_{uv}(u_x(-u_y)) + \phi_{uv}(u_x(-u_y))$$
$$+ \phi_{vv}(u_x^2 + u_y^2) + \phi_{uv}(u_y(u_x)) + \phi_{uv}(u_y(u_x))$$

$$= (\phi_{uu} + \phi_{vv})(u_x^2 + u_y^2)$$

$$= 0.$$