Dupuy — Complex Analysis — Spring 2017 — Homework 01

1. Let ζ_n be a primative nth root of unity. Show that $\sum_{j=0}^{n-1} \zeta_n^j = 0$.

2. (New Mexico, Summer 1999) Let $\mathcal{H}$ be the upper half complex plane $\mathcal{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Let D be the unit disc $D = \{w : |w| < 1\}$. Show that the map $f(z) = w = (z-i)/(z+i)$ defines a bijection $\mathcal{H} \to D$.

3. (Wallis' Formula) Using the complex representation of cosine, find a formula for

$$\int_0^{2\pi} \cos(\theta)^{2n} d\theta.$$

4. Prove Hadamard's formula for the radius of convergence of a series $\sum_{n=0}^{\infty} a_n z^n$.

$$R = \lim_{n\to\infty} \inf_{m \geq n} |a_m|^{-1/m}.$$

Also show that the series converges absolutely and uniformly, (the differentiability thing is the next problem).

5. Suppose that $f(z) = \sum_{n\geq 0} a_n z^n$ has a radius of convergence R.

   (a) Show $\sum_{n\geq 0} n a_n z^{n-1}$ converges with the same radius of convergence R.

   (b) Show $\frac{d}{dz}\left[\sum_{n=0}^{\infty} a_n z^n\right] = \sum_{n\geq 0} \frac{d}{dz}[a_n z^n]$ on the disc of convergence.

   (Warning: it is not true that for general $u_n(z) \to u(z)$ uniformly that $u_n'(z) \to u_n(z)$ uniformly! This is a special fact about power series.)

6. (Extra Credit, see WW page 59) Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

converges to $\frac{1}{(z-1)^2}$ when $|z| < 1$ and $\frac{1}{z(z-1)^2}$ when $|z| > 1$

7. If the series converges do some analysis to determine the radius of convergence at the boundary.

   (a) Expand $\frac{1}{1+z^2}$ in a power series around $z = 0$, find the radius of convergence.

   (b) Find the radius of convergence of $\sum_{n\geq 0} n! z^n$.

   (c) (New Mexico, Jan 1998) Expand $\frac{z^2+2z-4}{z}$ in a power series around $z = 1$ and find its radius of convergence.

8. (a) Let $B \times A \subset \mathbf{C} \times \mathbf{C}$ be an open region with compact closure. Let $f : B \times A \to \mathbf{C}$ be a function. Let $\gamma \subset A$ be a C^1-curve (so it has finite length). Define $F : B \to \mathbf{C}$ by

$$F(z) = \int_\gamma f(z,s) ds.$$

   Assuming $\frac{\partial f}{\partial z}(z,s)$ exists and is continuous for all $s \in \gamma$ and all $z \in B$ show that

$$\frac{d}{dz}[F(z)] = \int_\gamma \frac{\partial f}{\partial z}(z,s) ds.$$

(b) Let $\Omega \subset \mathbf{C}$ be an open set. Let $\gamma : [0,1] \to \Omega$ be an C^1 curve. Let $f \in \text{hol}(\Omega)$ and $g \in L^2(\Omega)$. Show that

$$F(z) := \int_\gamma f(\zeta - z) g(\zeta) d\zeta$$

is holomorphic on Ω.

9. (Extra Credit, This exercise show how nice the complex numbers are and how if one tries to develop a notion of holomorphic function in higher dimensions it doesn't work.)

   The quaternions are the Division algebra (non commutative field) over the reals defined by

$$\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k, (\cong \mathbf{R}^4 \text{ as a vector space})$$

   where i,j and k satisfy

$$ijk = -1 \text{ and } i^2 = j^2 = k^2 = -1.$$

   The norm on the quaternions is defined as

$$|a + bi + cj + dk|^2 = a^2 + b^2 + c^3 + d^2,$$

   here $a, b, c, d \in \mathbf{R}$.

   For $U \subset \mathbf{H}$ open, we say a function $f : U \to \mathbf{H}$ is **holomorphic** if

$$f(q) = \lim_{h \to 0} \left((f(q + h) - f(q))h^{-1} \right).$$

   Show that the only quaternionic holomorphic functions are of the form

$$f(q) = \alpha q + \beta.$$

   where $\alpha, \beta \in \mathbf{H}$.

10. (New Mexico, Summer 2000) Show that the pullback of a harmonic function by an holomorphic map is harmonic (what these words means is explained below). Assume that $w = f(z) = u(z) + iv(z)$ is holomorphic map $f : D \to D' \subset \mathbf{C}$. We consider D in the z-plane to a domain D' in the w-plane. If ϕ is harmonic on D', show that

$$\Phi(x, y) := \phi(u(x, y), v(x, y))$$

   is harmonic in D.

   (The function Φ is called the pullback of ϕ by f. Sometimes in the literature these you will see the notation $f^*\phi$ for Φ.)