Dupuy — Complex Analysis — Spring 2016 — Homework 02

1. (CUNY, Fall 2005) Let D be the closed unit disc. Let g_n be a sequence of analytic functions converging uniformly to f on D.

   (a) Show that g_n' converges.

   Solution We have

   $$g_n'(z) = \frac{1}{2\pi i}\oint \frac{g_n(\zeta)}{(\zeta - z)^2} d\zeta \to \frac{1}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z)^2}.$$

   since limit of the integral is the integral of the limit for uniformly convergent functions. $\blacksquare$

   (b) Conclude that f is analytic.

   Solution If $g_n \to f$ and $g_n' \to g$ with f continuous, then $\lim_{n\to\infty} g_n' = f'$. $\blacksquare$

   2nd Solution By Morera's theorem, it is enough to show that $\oint_\gamma f(z)dz = 0$ for all closed circles in the domain. But $\oint_\gamma f(z)dz = \lim_{n\to\infty} \oint_\gamma g_n(z)dz = 0$. $\blacksquare$

2. Here is a first example of an analytic continuation "from the wild".

   (a) Show that the Riemann Zeta function

   $$\zeta(z) := \sum_{n\geq 1} \frac{1}{n^z}$$

   converges for $\operatorname{Re} z > 1$ and is analytic on this domain. (You need to use the "analytic convergence theorem", which states that a uniform limit of analytic functions is analytic. This is just a slight generalization of the previous problem.)

   Solution Let $D \subset H = \{z \in \mathbf{C} : \operatorname{Re} z > 1\}$ be compact. Since this is bounded away from the line $\operatorname{Re} z = 1$, all $z \in D$ have $\operatorname{Re} z > \sigma > 1$. This means $|n^{-z}| = n^{-x} < n^{-\sigma}$. The series $\sum_{n=1}^\infty n^{-\sigma}$ converges by the p-test, hence by the M-test $\sum_{n=1}^\infty n^{-s}$ converges absolutely and uniformly on all such regions D. Since the uniform limit of analytic functions is analytic we know that $\zeta(s)$ is analytic.

   (b) (Whittaker and Watson, 2.8, problem 10)

      i. Show that when $\operatorname{Re} s > 1$,

      $$\sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^\infty \left[\frac{1}{n^s} + \frac{1}{s-1}\left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right)\right]$$

      Solution It is enough to show that

      $$\lim_{N\to\infty} \sum_{n=1}^N \frac{1}{s-1}\left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right) = \frac{-1}{(s-1)}$$

      when $\operatorname{Re} s > 1$.
      We observe that

      $$\sum_{n=1}^N \left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right) = \frac{1}{(N+1)^s} - 1 \to -1 \text{ as } N \to \infty,$$

      since $\operatorname{Re} s > 1$. $\blacksquare$

ii. Show that the series on the right converges when $0 < \mathrm{Re}\, s < 1$. (This means the series above gives us access to the interesting part of the Riemann-Zeta function. Hint: $\int_n^{n+1} x^{-s}dx = \frac{(n+1)^{-s+1}}{1-s} - \frac{n^{-s+1}}{1-s}$)

Solution By the hint

$$\sum_{n=1}^{N}\left[\frac{1}{n^s} + \frac{1}{s-1}\left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right)\right]\sum_{n=1}^{N}\int_n^{n+1}(n^{-s} - x^{-s})dx$$

so it is enought to show the series on the right converges as $N \to \infty$ when $0 < \mathrm{Re}\, s < 1$. For $x \in [n, n+1]$ we have

$$|n^{-s} - x^{-s}| = |s\int_n^x y^{-1-s}dy| \leq |s|n^{-1-\mathrm{Re}\, s},$$

so the series converges by the p-test. It, of course, converges absolutes and uniformly on compact subsets and hence is analytic on this region. ∎

3. (New Mexico, not sure which year) Let f be analytic on $\mathbf{C}$. Assume that $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of valued r going to infinity. Show that f is a polynomial of degree less than or equal to n.

Solution By Cauchy's integral formula for derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{\gamma_R}\frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta.$$

Let $R > 2|z|$ so that

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi}\frac{M(f,R)}{(R-|z|)^{n+1}}2\pi R \\ &\leq n!\frac{M(f,R)}{(R/2)^{n+1}}R \to 0 \text{ as } R \to \infty \end{aligned}$$

where the vanishing follows from the hypotheses on $M(f,R)$. Since $f^{(n)}(z) = 0$ for all z, $f(z)$ must be a polynomial of degree n. ∎

4. (a) Prove the Riemann Extension Theorem: Let $U \subset \mathbf{C}$ be a region containing a point z_0. Let $f \in \mathrm{hol}(U \setminus \{z_0\})$. If f is bounded on U show that there exists a unique $\widetilde{f} \in \mathrm{hol}(U)$ such that $\widetilde{f}|_{U\setminus\{z_0\}} = f \in \mathrm{hol}(U \setminus \{z_0\})$.

Solution By analytic continuation, it is enough to extend the function in a small neighborhood around z_0. Let $r > 0$ be such that the closure of $D := D_r(z_0)$ is contained in U. Consider

$$\widetilde{f}(z) := \frac{1}{2\pi i}\oint_{\partial D} f(\zeta)/(\zeta - z)d\zeta.$$

The integral is defined by boundedness of f, is holomorphic in all D since we can switch integration and summation. For $z \neq z_0$ we have

$$\begin{aligned} \frac{1}{2\pi i}\oint_{\partial D} f(\zeta)/(\zeta - z)d\zeta &= \mathrm{Res}(\frac{f(\zeta)}{\zeta - z}; \zeta = z) + \mathrm{Res}(\frac{f(\zeta)}{\zeta - z}; \zeta = z_0) \\ &= f(z) + \frac{1}{z_0 - z}\mathrm{Res}(f(\zeta); \zeta = z_0) \\ &= f(z) + \frac{a_{-1}(f; z_0)}{z_0 - z}. \end{aligned}$$

We can show the residue vanishes: since f is bounded in a neighborhood of z_0 we have

$$|a_{-1}(f; z_0)| = |\frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)d\zeta| \leq \sup_{|\zeta - z_0| = r} r \to 0 \text{ as } r \to 0 .$$

Hence,

$$\widetilde{f}(z) = \frac{1}{2\pi i}\oint_{\partial D} f(\zeta)/(\zeta - z)d\zeta = f(z) \text{ for } z \neq z_0$$

and $\widetilde{f}(z)$ is defined and analytic at $z = z_0$, so we are done. ∎

(b) Recall that a morphism of topological spaces $f : X \to Y$ is "proper" if and only if the inverse image of every compact set is compact. Show that an analytic map $f : \mathbf{C} \to \mathbf{C}$ is proper if and only if for all $z_j \to \infty$ we have $f(z_j) \to \infty$.

Solution We will show the contrapositive: There exists some sequence $z_j \to \infty$ with $f(z_j)$ not going to infinity, if and only if there exists a compact subset $K \subset \mathbf{C}$ such that $f^{-1}(K)$ is not compact.

- Suppose that we have a $z_j \to \infty$ with $f(z_j)$ bounded. There exists a convergent subsequence $f(z_{j_k}) \to w_0$. For K a compact subset around w_0 we know that $f^{-1}(K)$ is not compact as $z_{j_k} \in f^{-1}(K)$ and $z_{j_k} \to \infty$.
- Conversely, suppose there exists some compact subset K such that $f^{-1}(K)$ is not compact. Since $f^{-1}(K)$ is not compact it is not bounded (the inverse image of a closed set is always closed). Let z_j be an unbounded sequence of points in $f^{-1}(K)$. We can arrange so that $z_j \to \infty$ by going to a subsequence if necessary. We know that $f(z_j)$ is bounded which provides the example we wanted.

∎

(c) Show that the only proper maps $f : \mathbf{C} \to \mathbf{C}$ are polynomials. (see page 27 of McMullen, you need to consider the function $g(z) = 1/f(1/z)$ and show that $g(z) = z^n g_0(z)$ where $g_0(z)$ is analytic and non-zero. This will allows you to conclude $|g(z)| > c|z|^n$ for some n which will allows you to conclude behavious about the growth of $f(z)$ as $z \to \infty$.)

Solution Let $f : \mathbf{C} \to \mathbf{C}$ be propert. Consider the function $g(z) = 1/f(1/z)$. Note that $g \in \text{hol}(\mathbf{C} \setminus \{0\})$. By properness, if $z_j \to 0$, $g(z_j) \to 0$. This means $g(z)$ is bounded in a neighborhood of zero so by the Riemann extension theorem $g(z)$ is actually entire with a finite order zero at $z = 0$. By the local structure of analytic functions we can write $g(z) = z^n g_0(z)$ where $g_0(0) \neq 0$. We conclude $|g(z)| \geq c|z|^n$ near $z = 0$ where $c = c_r = \inf_{|z|<r} |g_0(z)|$ for some r sufficiently small. Now we have

$$f(z) = 1/g(1/z)$$

So for $|z| > 1/r$ we have

$$|f(z)| < 1/|g(1/z)| < |z|^n/c,$$

hence, by the modified versions of Liouville's Theorem for polynomial growth (Problem 3), we are done. ∎

5. (New Mexico, not sure which year) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume that g is not identically zero. Show that f/g is rational.

Solution Since $g(z)$ is analytic, its zeros are isolated. Let $\{z_1, \ldots, z_n\} = \{z \in \mathbf{C} : |z| < 100 \text{ and } g(z) = 0\}$. Let $m_j = \text{ord}_{z_j}(g)$. Let

$$p(z) := \prod_{j=1}^{n}(z - z_j)^m.$$

By construction $p(z)/g(z)$ has no poles when $|z| < 100$. Similarly, since $f(z)/g(z)$ has not poles where $|z| > 1$ since $|f(z)| \leq |g(z)|$. This means that $p(z)f(z)/g(z)$ is entire and $|p(z)f(z)/g(z)| \leq |p(z)|$ for $|z| > 100$. By the modified version of Liouville's theorem for polynomial growth (Problems 3) the function $p(z)f(z)/g(z) = q(z) \in \mathbf{C}[z]$. This implies $f(z)/g(z) = q(z)/p(z)$ and we are done. ∎

6. Prove Goursat's theorem. Let γ be a simple contour. If $f : \overline{\gamma^+} \to \mathbf{C}$ is holomorphic (but whose derivative is not necessarily continuous) then

$$\int_\gamma f(\zeta) d\zeta = 0.$$

Solution See the appendix of Greene and Krantz.

7. Suppose that $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ has a finite radius of convergence. Let $g(z) = a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots$. Show that $g(z)$ has the same radius of convergence as $f(z)$ at z_0. (Hint: don't think about this too much)

Solution Note that

$$a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots = (a_0+\cdots+a_{n-1}(z-z_0)^n)+(z-z_0)^n(a_n+a_{n+1}(z-z_0)+a_{n+2}(z-z_0)^2+\cdots).$$

This shows that

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

converges if and only if

$$a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots$$

converges. ∎

8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let R be the radius of convergence (which is possibly infinite). Let $S_N(f)(z) = \sum_{n=0}^{N} a_n z^n$. Show that for all $r < R$ and all $z \in \mathbf{C}$ with $|z| < r$ we have

$$|f(z) - S_N(f)(z)| \leq \frac{M(f,r)}{r - |z|} \frac{|z|^{N+1}}{r^N}$$

where $M(f,r) = \max_{|z|=r} |f(z)|$.

Solution By expanding Cauchy's formula in a geometric series and truncating we get

$$\begin{aligned}
S_N(z) &= \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta} \left(\sum_{n=0}^{N} \left(\frac{z}{\zeta}\right)^n \right) d\zeta \\
&= \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \left(1 - \left(\frac{z}{\zeta}\right)^{N+1} \right) d\zeta \\
&= \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \left(\frac{z}{\zeta}\right)^{N+1} d\zeta \\
&= f(z) + E_N(z)
\end{aligned}$$

where

$$E_N(z) := \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \left(\frac{z}{\zeta}\right)^{N+1} d\zeta$$

is the error term in the truncated power series approximation. Estimating $E_N(z)$ using a circle of radius r gives

$$|E_N(z)| \leq \frac{1}{2\pi} \frac{M(f,r)}{r - |z|} \left(\frac{|z|}{r}\right)^{N+1} \cdot 2\pi r = \frac{M(f,r)}{r - |z|} \frac{|z|^{N+1}}{r^N}$$

which proves the result. ∎

9. (UIC, Spring 2016) Describe all entire functions such that $f(1/n) = f(-1/n) = 1/n^2$ for all $n \in \mathbf{Z}$.

Solution The function must be z^2. Note that $f(z) - z^2$ has an accumulation point of zeros hence must be identically zero. ∎

10. Let $U \subset \mathbf{C}$ be a connected open set. Consider $U \subset \mathbf{C}$ with the subspace topology (open subset of U are the intersection of open subsets of $\mathbf{C}$ with U and closed subset are closed subset of $\mathbf{C}$ intersected with U). Show that the only subset of U which are open, closed and nonempty is U itself.

    Solution Suppose that $A \subset U$ is open and closed. Then A^c is both open and closed and $A \cap A^c = \emptyset$. We also have $U = A \cup A^c$ which is the union of two proper open sets. By definition of connected, we have $U = A$. ∎