Dupuy — Complex Analysis — Spring 2016 — Homework 02

1. (CUNY, Fall 2005) Let D be the closed unit disc. Let g_n be a sequence of analytic functions converging uniformly to f on D.
   (a) Show that g_n' converges.
   (b) Conclude that f is analytic.

2. Here is a first example of an analytic continuation "from the wild".
   (a) Show that the Riemann Zeta function

$$\zeta(z) := \sum_{n \geq 1} \frac{1}{n^z}$$

   converges for $\operatorname{Re} z > 1$ and is analytic on this domain. (You need to use the "analytic convergence theorem, which states that a uniform limit of analytic functions is analytic. This is just the previous problem.)
   (b) (Whittaker and Watson, 2.8, problem 10)
      i. Show that when $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty} \left[\frac{1}{n^s} + \frac{1}{s-1} \left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}} \right) \right]$$

      ii. Show that the series on the right converges when $0 < \operatorname{Re} s < 1$. (This means the series above gives us access to the interesting part of the Riemann-Zeta function. Hint: $\int_n^{n+1} x^{-s} dx = \frac{(n+1^{-s+1}}{1-s} - \frac{n^{-s+1}}{1-s}$)

3. (New Mexico, not sure which year) Let f be analytic on $\mathbf{C}$. Assume that $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of valued r going to infinity. Show that f is a polynomial of degree less than or equal to n.

4. (a) Prove the Riemann Extension Theorem: Let $U \subset \mathbf{C}$ be a region containing a point z_0. Let $f \in \operatorname{hol}(U \setminus \{z_0\})$. If f is bounded on U show that there exists a unique $\widetilde{f} \in \operatorname{hol}(U)$ such that $\widetilde{f}|_U = f \in \operatorname{hol}(U)$.
   (b) Recall that a morphism of topological spaces $f : X \to Y$ is "proper" if and only if the inverse image of every compact set is compact. Show that an analytic map $f : \mathbf{C} \to \mathbf{C}$ is proper if and only if for all $z_j \to \infty$ we have $f(z_j) \to \infty$.
   (c) Show that the only proper maps $f : \mathbf{C} \to \mathbf{C}$ are polynomials. (see page 27 of McMullen, you need to consider the function $g(z) = 1/f(1/z)$ and show that $g(z) = z^n g_0(z)$ where $g_0(z)$ is analytic and non-zero. This will allows you to conclude $|g(z)| > c|z|^n$ for some n which will allows you to conclude behavious about the growth of $f(z)$ as $z \to \infty$.)

5. (New Mexico, not sure which year) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume that g is not identically zero. Show that f/g is rational.

6. Prove Goursat's theorem. Let γ be a simple contour. If $f : \overline{\gamma^+} \to \mathbf{C}$ is holomorphic (but whose derivative is not necessarily continuous) then

$$\int_\gamma f(\zeta) d\zeta = 0.$$

7. Suppose that $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ has a finite radius of convergence. Let $g(z) = a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots$. Show that $g(z)$ has the same radius of convergence as $f(z)$ at z_0. (Hint: don't think about this too much)

8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let R be the radius of convergence (which is possibly infinite). Let $S_N(f)(z) = \sum_{n=0}^{N} a_n z^n$. Show that for all $r < R$ and all $z \in \mathbf{C}$ with $|z| < r$ we have

$$|f(z) - S_N(f)(z)| \leq \frac{M(f,r)}{r - |z|} \frac{|z|^{N+1}}{r^N}$$

where $M(f,r) = \max_{|z|=r} |f(z)|$.

9. (UIC, Spring 2016) Describe all entire functions such that $f(1/n) = f(-1/n) = 1/n^2$ for all $n \in \mathbf{Z}$.

10. Let $U \subset \mathbf{C}$ be a connected open set. Consider $U \subset \mathbf{C}$ with the subspace topology (open subset of U are the intersection of open subsets of $\mathbf{C}$ with U and closed subset are closed subset of $\mathbf{C}$ intersected with U). Show that the only subset of U which are open, closed and nonempty is U itself.