Dupuy — Complex Analysis — Spring 2017 — Homework 03

Master's level

1. (Rice, May 2003) Find all entire functions $f(z)$ which satisfy $\operatorname{Re} f(z) \leq 2/|z|$ when $|z| > 1$. (Hint: Consider $e^{-f(z)}$ or $e^{f(z)}$. You will need the maximum modulus principle and Liouville's theorem.)

   Solution Consider the function $g(z) = e^{f(z)}$. We have $|g(z)| = e^{\operatorname{Re} f(z)} < e^{2/|z|} < e^2$ on all $\mathbf{C}$ by the given bound and the Maximum-Modulus principle. Every bounded entire function is constant so $g(z)$ is constant and hence $f(z)$ is constant. ∎

2. Let $u(z)$ be a real valued harmonic function on a domain $D \subset \mathbf{C}$

   (a) A **harmonic conjugate** is a function $v(x, y)$ such that $f(x + iy) := u(x, y) + iv(x, y)$ is holomorphic. Show that $u(x, y) = u(z)$ has a harmonic conjugate locally. (Hint: Use the fundamental theorem of line integrals $v(\vec{P}) - v(\vec{Q}) = \int_C \nabla v \cdot d\vec{r}$ if C is a path starting a $\vec{Q}$ and ending at $\vec{P}$)

   Solution We claim that

$$v(x, y) := \int_C -u_y dx + u_x dy$$

   where C is a path from a fixed point (x_0, y_0) to a variable point (x, y) does the trick.
   More precisely, let R_0 be an open retangle in D. Fix (x_0, y_0) in R_0. For each $(x, y) \in R_0$ we can form a rectangle R with diagonal vertices (x_0, y_0) and (x, y). Note that

$$\oint_{\partial R} -u_y dx + u_x dy = \iint_R (u_{xx} + u_{yy}) dx dy = 0$$

   by Green's Theorem. This implies that

$$\int_{C_1} -u_y dx + u_x dy = \int_{C_2} -u_y dx + u_x dy$$

   where C_1 is the path around the boundary of R given by $(x_0, y_0) \to (x_0, y) \to (x, y)$ and C_2 is the path around the boundary of R given by $(x_0, y_0) \to (x, y_0) \to (x, y)$.
   In coordinates we have

$$v(x, y) = \int_{C_1} -u_y dx + u_x dy = \int_{y_0}^{y} u_x(x_0, \eta) d\eta + \int_{x_0}^{x} -u_y(\xi, y) d\xi.$$

$$v(x, y) = \int_{C_2} -u_y dx + u_x dy = \int_{x_0}^{x} -u_y(\xi, y_0) d\xi + \int_{y_0}^{y} u_x(x, \eta) d\eta.$$

   The formula using C_1 implies $v_x(x, y) = -u_y(x, y)$. The formula using C_2 implies $v_y(x, y) = u_x(x, y)$. This shows that the pair of functions (u, v) satisfy the Cauchy-Riemann equations and hence that $f(z) = u(z) + iv(z)$ is locally an analytic function. ∎

   (b) Show that for all $D_r(z_0) \subset D$ we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

   (Hint: use a harmonic conjugate)

Solution By the existence of a harmonic conjugate we may write $f(z) = u(z) + iv(z)$ for some analytic function $f(z)$. By the mean value theorem for analytic functions we have

$$u(z_0)+iv(z_0) = f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(z_0+re^{i\theta})d\theta$$

Breaking into real an imaginary parts gives the result. $\blacksquare$

(c) If $z_0 \in D$ has the property that there exists some $r > 0$ with $D_r(z_0) \subset D$ and

$$u(z_0) \geq u(z)$$

for all $z \in D_r(z_0)$ then $u(z)$ is constant. (Hint: Consider a function such that $f(z) = u(z) + iv(z)$ then consider the maximum of $e^{f(z)}$.)

Solution Consider $f(z) = u(z) + iv(z)$ analytic and let $g(z) = e^{f(z)}$. Since $|g(z)| = e^{u(z)}$ if $u(z_0)$ is a maximum then $e^{f(z)}$ is a constant function by the maximum modulus principal for analytic function. This shows that $u(z)$ is constant as $|g(z)|$ would be. $\blacksquare$

Remark By the maximum modulus principle for analytic functions we have

$$\max_{|z-z_0|\leq r} |g(z)| \leq \max_{|z-z_0|=r} |g(z)|.$$

Since $|g(z)| = e^{u(z)}$ this implies

$$\max_{|z-z_0|\leq r} u(z) = \max_{|z-z_0|=r} u(z).$$

$\blacksquare$

3. Let $u(x+iy) = u(x,y)$ be a real valued harmonic function. A *harmonic conjugate* is a function $v(x,y)$ such that $f(x+iy) := u(x,y) + iv(x,y)$ is holomorphic.

   Find all of the harmonic conjugates of $u(x,y) = x^3 - 3xy^2 + 2x$.

   Solution We can see that $u = \mathrm{Re}(z^3 + 2z + ic)$ where c is a real number. This implies all harmonic conjugates have the form

   $$v = -y^3 + 3x^2y + c.$$

   $\blacksquare$

4. (Green and Krantz, Ch 11) A subset $S \subset \mathbf{R}^n$ is **path connected** if for all $a, b \in S$ there exists a continuous $\gamma : [0,1] \to S$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

   Let U be an open subset of $\mathbf{R}^n$. Show that U is path connected if and only if U is connected. (Hint: show that the collection of path connected elements is open and closed. Also, you can use that the only nonempty open and closed subset of a connected open set is the entire set itself.)

   Solution Clearly if U is path connected it is connected.

   Conversely, suppose that U is connected. Let $S_a \subset U$ be the set of points that is path connected to some point $a \in U$.

   - This set S_a is nonempty as it contains the neighborhood of a.
   - The set S_a is open because every open ball is path connected.
   - The set S_a^c is also open. Suppose not. Suppose that there existed some $b \in S_a^c$, some disc $D \ni b$ and some point $c \in D$ such that c was path connected to a. Well, b is path connected to c and c is path connected to a which implies b is path connected to a which is a contradiction.

   The set S_a is open, closed and non-empty in a connected open set and hence must be all of U (see Homework 02) $\blacksquare$

Ph.D. level

5. (New Mexico, not sure which year) Let $f(z)$ and $g(z)$ be entire functions. Show that if $f(g(z))$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials. (Hint: this relates to the problem on properness from the previous homework).

   Solution We will use the fact that polynomials are the only proper entire functions (Homework 02, Problem 4c). Suppose $g(z)$ is proper. There exists some $z_j \to \infty$ such that $g(z_j)$ is bounded. By passing to a subsequence, we can assume $g(z_j) \to w_0$ as $j \to \infty$. This means

   $$f(w_0) = f(\lim_{j\to\infty} g(z_j)) = \lim_{j\to\infty} f(g(z_j)) = \infty,$$

   which is a contradiction. The first equality follows from the definition of w_0, the second equality follows from continuity and the last equality follows from the assumption that $f \circ g$ is proper. Hence we have the $g(z)$ is a polynomial.

   Suppose now that $f(z)$ is not a polynomial. Then there exist some sequence $w_j \to \infty$ such that $f(w_j)$ is bounded. By passing to a subsequence we can assume that $f(w_j)$ converges to ζ_0. Since $g(z)$ is a polynomial and $\mathbf{C}$ is algebraically closed, $g(z) = w$ has a solution in z for all w, hence g is surjective. Let z_j be such that $g(z_j) = w_j$. By shifting if necessary, we can assume, without loss of generality, that $|z_j| > 1$.

   We claim that $z_j \to \infty$ as $j \to \infty$. To see this, write $g(z) = a_0 + a_1 z + \cdots a_d z^d$. If $g(z) = w$ with $|z| > 1$ then

   $$|w| \le |g(z)| \le |a_0| + |a_1||z| + \cdot + |a_d||z|^d \le A|z|^d$$

   where $A = \max |a_j|$. Then

   $$(|w|/A)^{1/d} \le |z|.$$

   Hence if $w_j \to \infty$ then $z_j \to \infty$.

   We now have a sequence $z_j \to \infty$ such that $f(g(z_j)) = f(w_j) \to \zeta_0$ which contradicts the fact that $f \circ g$ is proper. This implies that f must also be a polynomial. ∎

   Remark There is another way of doing this using the Castorati-Weierstrass Theorem which we haven't talked about yet. It states that if $f(z)$ has an essential singularity at z_0 then for all $c \in \mathbf{C}$ there exists some $z_j \to z_0$ such that $f(z_j) \to c$. ∎

6. Show that the following conditions are equivalent for a topological space X:

   (a) For all $a, b \in X$ there exists open sets $U \ni a$ and $V \ni b$ with $U \cap V = \emptyset$.

   (b) For all $a, b \in X$, if every neighborhood of a intersects every neighborhood of b then $a = b$.

   (c) The diagonal map $X \to X \times X$ given by $x \mapsto (x, x)$ is proper.

   (d) The diagonal subset is closed.

   If any of these conditions hold we call the topological space **separated** or **hausdorff**. (Hint: You should use the fact that a morphism f is proper if and only if f is closed and the inverse image of every point is compact.)

   Solution 6a $\iff$ 6b Taking the contrapositive of 6b we get

   $$a \neq b \iff \exists \text{ neighborhoods of } a \text{ and } b \text{ which don't intersect }.$$

   This is exactly condition 6b.

   6a $\implies$ 6d Let $\Delta \subset X \times X$ be the diagonal:

   $$X \times X \setminus \Delta = \{(a, b) : a \neq b\}.$$

We will show $X \times X \setminus \Delta$ is open. Suppose $a \neq b$. Then there exists U and V open such that

$$U \ni a \text{ and } V \ni b \text{ and } U \cap V = \emptyset.$$

This means

$$(U \times X) \cap (X \times V) = U \times V$$

is an open set containing (a, b) not contained in the diagonal.

6a $\implies$ **6d** Using the hint, if the diagonal map

$$\text{diag} : X \to X \times X$$

$$\text{diag}(x) = (x, x)$$

is proper, then the image of a closed set is closed. Hence, the diagonal is closed.

6d $\implies$ **6c** Suppose Δ is closed. To show diag is proper, is it enough to show that the it is closed and the preimage of every point is compact.[1] $\blacksquare$

- The preimage of every point under diag is compact as it is either a point or the empty set.
- Let $A \subset X$ be closed. Then $\text{diag}(A) = (A \times X) \cap \Delta$ which is the intersection of two closed sets and hence closed. This shows $\text{diag} : X \to X \times X$ is a closed map.

Background:

- Let X and Y be topological spaces. We define the topology on $X \times Y$ to be the smallest topology such that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous (this means the open sets are generated by sets of the form $U \times Y$ or $X \times V$ for $U \subset X$ open or $X \times V$ for $V \subset Y$ open.
- A topological space X is **compact** if every open cover has a finite subcover. An open cover is just a union of open sets that equal X.
- A **proper map** is a morphism of topological spaces such that the inverse image of compact sets is compact.

Side Remark: The third condition is interesting because Grothendieck realized we can use it to extend this definition to categories other than topological spaces. In particular to the category of "schemes".

[1] A proof of this can be found on Wikipedia