Dupuy — Complex Analysis — Spring 2017 — Homework 03

Master's level

1. Find all entire functions $f(z)$ which satisfy $\operatorname{Re} f(z) \le 2/|z|$ when $|z| > 1$. (Hint: Consider $e^{-f(z)}$ or $e^{f(z)}$. You will need the maximum modulus principle and Liouville's theorem.)

2. Let $u(z)$ be a real valued harmonic function on a domain $D \subset \mathbf{C}$

   (a) A **harmonic conjugate** is a function $v(x, y)$ such that $f(x + iy) := u(x, y) + iv(x, y)$ is holomorphic. Show that $u(x, y) = u(z)$ has a harmonic conjugate. (Hint: Use the fundamental theorem of line integrals $v(\vec{P}) - v(\vec{Q}) = \int_C \nabla v \cdot d\vec{r}$ if C is a path starting a $\vec{Q}$ and ending at $\vec{P}$)

   (b) Show that for all $D_r(z_0) \subset D$ we have
   $$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$
   (Hint: use a harmonic conjugate)

   (c) If $z_0 \in D$ has the property that there exists some $r > 0$ with $D_r(z_0) \subset D$ and
   $$u(z_0) \ge u(z)$$
   for all $z \in D_r(z_0)$ then $u(z)$ is constant. (Hint: Consider a function such that $f(z) = u(z) + iv(z)$ then consider the maximum of $e^{f(z)}$.)

3. Let $u(x + iy) = u(x, y)$ be a real valued harmonic function. A *harmonic conjugate* is a function $v(x, y)$ such that $f(x + iy) := u(x, y) + iv(x, y)$ is holomorphic.

   Find all of the harmonic conjugates of $u(x, y) = x^3 - 3xy^2 + 2x$.

4. (Green and Krantz, Ch 11) A subset $S \subset \mathbf{R}^n$ is **path connected** if for all $a, b \in S$ there exists a continuous $\gamma : [0, 1] \to S$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

   Let U be an open subset of $\mathbf{R}^n$. Show that U is path connected if and only if U is connected. (Hint: show that the collection of path connected elements is open and closed. Also, you can use that the only nonempty open and closed subset of a connected open set is the entire set itself.)

Ph.D. level

5. (New Mexico, not sure which year) Let $f(z)$ and $g(z)$ be entire functions. Show that if $f(g(z))$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials. (Hint: this relates to the problem on properness from the previous homework).

6. Show that the following conditions are equivalent for a topological space X:

   (a) For all $a, b \in X$ there exists open sets $U \ni a$ and $V \ni b$ with $U \cap V = \emptyset$.

   (b) For all $a, b \in X$, if every neighborhood of a intersects every neighborhood of b then $a = b$.

   (c) The diagonal map $X \to X \times X$ given by $x \mapsto (x, x)$ is proper.

   (d) The diagonal subset is closed.

If any of these conditions hold we call the topological space **separated** or **hausdorff**. (Hint: You should use the fact that a morphism f is proper if and only if f is closed and the inverse image of every point is compact.)

Background:

- Let X and Y be topological spaces. We define the topology on $X \times Y$ to be the smallest topology such that the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous (this means the open sets are generated by sets of the form $U \times Y$ or $X \times V$ for $U \subset X$ open or $X \times V$ for $V \subset Y$ open.
- A topological space X is **compact** if every open cover has a finite subcover. An open cover is just a union of open sets that equal X.
- A **proper map** is a morphism of topological spaces such that the inverse image of compact sets is compact.

Side Remark: The third condition is interesting because Grothendieck realized we can use it to extend this definition to categories other than topological spaces. In particular to the category of "schemes".