Dupuy — Complex Analysis — Spring 2017 — Homework 04

Master's Level

1. (Whittaker and Watson, 6.24,3) If $-1 < z < 3$ then

$$\int_0^\infty \frac{x^z}{(1+x^2)^2} dx = \frac{\pi(1-z)}{4\cos(\pi z/2)}$$

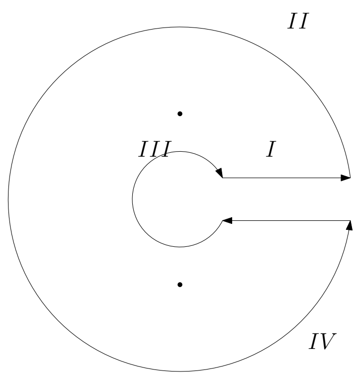

Figure 1: A bad drawing of the key-hole contour you need to use for the first problem.

Solution I am going to replace z with t to remind me that it is real. We will use z for the complex variable. Let $f(z) = \frac{z^t}{(1+z^2)^2}$. We use the "keyhole" contour and the branch of the log where $\mathrm{Arg}(z) \in [0, 2\pi)$. We have

$$\int_{I_{R,r}+II_R+III_{R,r}+IV_r} f(z)dz = 2\pi i(\mathrm{Res}(f; z=i) + \mathrm{Res}(f; z=-i)).$$

We will use the notation $\sum \mathrm{Res}$ for the right hand side of this equality.

- Estimate on outer circle:

$$\begin{aligned} \left|\int_{II_R} \frac{z^t}{(1+z^2)^2} dz\right| &\leq \int_{II_R} \frac{|z^t|}{|1+z^2|^2} |dz| \\ &\leq \frac{R^t}{(R^2-1)^2} 2\pi R \to 0 \text{ as } R \to \infty. \end{aligned}$$

  The vanishing follows from the fact that the last term is $O(R^{t+1-4})$ as $R \to \infty$ with $t < 3$.

- Estimate on the inner circle:

$$\left|\int_{IV_r} \frac{z^t}{(1+z^2)^2} dz\right| \leq \frac{r^t}{(1-r^2)^2} 2\pi r \to 0 \text{ as } r \to 0$$

- Computation of residues: Using $f(z) = z^t/(z-i)^2(z+i)^2$ we get that

$$\begin{aligned}
\mathrm{Res}(f; z=i) &= \frac{1}{2\pi i}\oint_{\partial D_r(i)} \frac{z^t}{(z-i)^2(z+i)^2}dz \\
&= \frac{d}{dz}\left[\frac{z^t}{(z+i)^2}\right]_{z=i} \\
&= \frac{-1}{4}\left(i^{t-1}t + i^{t+1}\right) \\
\mathrm{Res}(f; z=-i) &= \frac{1}{2\pi i}\oint_{\partial D_r(-i)} \frac{z^t}{(z-i)^2(z+i)^2}dz \\
&= \frac{d}{dz}\left[\frac{z^t}{(z-i)^2}\right]_{z=-i} \\
&= \frac{-1}{4}\left((-i)^{t-1}t + (-i)^{t+1}\right)
\end{aligned}$$

  The sum of the residues $\sum \mathrm{Res}$ then satisfies

$$-4\sum \mathrm{Res} = 2e^{\pi i t}\sin(\pi t/2))(1-t).$$

  This computation took me 9 lines which I'm omitting. In these computations we are constantly using things like $1 + e^{2ia} = e^{ia}(e^{-ia} + e^{ia}) = 2e^{ia}\cos(a)$. Equivalently, one can just expan everything out in terms of sine and cosine using Euler's formula and watch things cancel.

Using the monodromy we see that

$$\begin{aligned}
& (1-e^{2\pi i t})\int_0^\infty \frac{x^t}{(1+x^2)^2}dx = 2\pi i \sum \mathrm{Res} \\
\implies\quad & -ie^{i\pi t}(4\sin(\pi t/2)\cos(\pi t/2))\int_0^\infty \frac{x^t}{(1+x^2)^2}dx = 2\pi i \cdot \frac{-1}{4}\cdot 2e^{\pi i t}\sin(\pi t/2)(1-t) \\
\implies\quad & \int_0^\infty \frac{x^t}{(1+x^2)^2}dx = \frac{\pi(1-t)}{4\cos(\pi t/2)}
\end{aligned}$$

The computation of the left hand side used that $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$. ∎

2. (Whittaker and Watson, 6.21, Example 4) Let $a > b > 0$ be real numbers. Show that

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Solution We turn the computation into a residue integral. If $z = e^{i\theta}$ then $d\theta = dz/iz$:

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a+b\cos(\theta))^2} &= \oint_{\partial D_1(0)} \frac{1}{(a+\frac{b}{2}(z+\frac{1}{z}))^2}\frac{dz}{iz} \\
&= 2\pi \sum (\text{residues of } \tfrac{1}{(a+\frac{b}{2}(z+\frac{1}{z}))^2}\tfrac{1}{z} \text{ inside unit disc }).
\end{aligned}$$

It remains to compute the residues. We have

$$\frac{1}{(a+\frac{b}{2}(z+\frac{1}{z}))^2}\frac{1}{z} = \frac{z}{(az+\frac{b}{2}(z^2+1))^2} = \frac{4}{b^2}\left[\frac{z}{(z^2+2(a/b)z+1)^2}\right].$$

Note that since $a > b > 0$ we have $a/b > 1$. The roots of the denominator are $(a/b) \pm \sqrt{(a/b)^2-1}$ which are both real. One finds that $(a/b) + \sqrt{(a/b)^2-1} > 1$ and $(a/b) - \sqrt{(a/b)^2-1} < 1$. To see this second inequality, write $a/b = 1 + \varepsilon$, then one can see that $(1+\varepsilon) - \sqrt{(1+\varepsilon)^2 - 1} = (1+\varepsilon) - \sqrt{\varepsilon(2+\varepsilon)} < 1 + \varepsilon - \sqrt{\varepsilon^2} = 1$.

It remains to compute the residue at $z = (a/b) - \sqrt{(a/b)^2 - 1}$.

Let $c = a/b$. Then

$$\frac{4}{b^2}\left[\frac{z}{(z^2+2cz+1)^2}\right] = \frac{4}{b^2}\left[\frac{z}{(z-(c+\sqrt{c^2-1}))^2(z-(c-\sqrt{c^2-1}))^2}\right]$$

and the residue is

$$\frac{4}{b^2}\frac{d}{dz}\left[\frac{z}{(z-(c+\sqrt{c^2-1}))^2}\right]_{z=(c-\sqrt{c^2-1}}.$$

Computing this out gives

$$2\pi \cdot \frac{4}{b^2}\frac{c}{4(\sqrt{c^2-1})^3} = \frac{2\pi a}{a^2-b^2}.$$

(I wrote this last line out in four lines on my paper) ∎

3. (Whittaker and Watson, 6.23, 2) If $a > 0$ and $b > 0$ show that

$$\int_{-\infty}^{\infty} \frac{x^4 dx}{(a+bx^2)^4} = \frac{\pi}{16a^{3/2}b^{5/2}}.$$

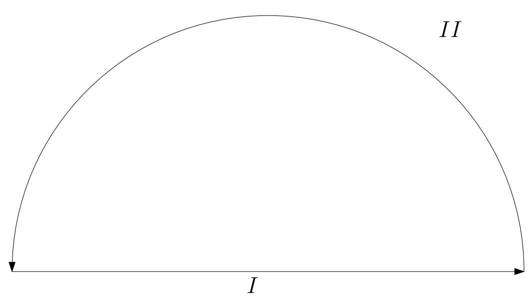

Figure 2: The semi-circle contour you need to use for this problem.

Solution For this problem we use an upper semicircle to relate our improper integral to our contour integral. Note that because $\frac{z^4}{(a+bz^2)^4} \sim 1/b^4z^4$ as $|z| \to \infty$ we have that

$$\lim_{R\to\infty} \int_{II_R} \frac{z^4}{(a+bz^2)^4} dz = 0.$$

This implies that

$$\int_{-\infty}^{\infty} \frac{x^4 dx}{(a+bx^2)^4} = \lim_{R\to\infty} \int_{I_R + II_R} \frac{z^4 dz}{(a+bz^2)^4} = \sum \operatorname{Res}.$$

Since $a + bx^2 = b(x - i(a/b)^{1/2})(x + i(a/b))^{1/2}$, we only have one residue and this gives

$$\int_{-\infty}^{\infty} \frac{x^4 dx}{(a+bx^2)^4} = 2\pi i \operatorname{Res}\left(\frac{z^4}{b^4(z^2+\frac{a}{b})^4}; z = i\sqrt{\frac{a}{b}}\right).$$

We just need to compute the series of $z^4/(z+c)^4$ for for $c = i\sqrt{\frac{a}{b}}$ out to $(z-c)^3$ in order to compute the residue. The series of $z^4/(z+c)^4$ at $z = c$ is

$$z^4/(z+c)^4 = \frac{1}{16} + \left(\frac{1}{8\,c}\right)(-c+z) + \left(\frac{1}{32\,c^2}\right)(-c+z)^2 + \left(-\frac{1}{32\,c^3}\right)(-c+z)^3 + \left(\frac{3}{256\,c^4}\right)(-c+z)^4 + \left(-\frac{1}{256\,c^6}\right)(-c+z)^6 + \cdots$$

(I used sage here because I'm lazy, the command `latex` will export output as latex. The command `show`, shows the display as it would in latex.)

Using $c = i\sqrt{a/b}$ and the third coefficient gives our residue:

$$2\pi i \cdot \frac{1}{b^4} \cdot \frac{-1}{32c^3} = \frac{\pi}{16} \frac{1}{b^4 (a/b)^{3/2}}$$

∎

4. (Whittaker and Watson, 6.22, 1) Show that if $a > 0$ then

$$\int_0^\infty \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}.$$

Solution We have

$$\int_0^\infty \frac{\cos(x)}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(x)}{x^2 + a^2} dx$$

since the function is even.

We then do the trick where we replace $\cos(x)$ by e^{ix} and then take the real part of the result. For the integral $\int_{-\infty}^\infty e^{ix}/(x^2 + a^2) dx$ we use the usual semi-circle in the upper-half plane. This gives

$$\frac{1}{2} \int_{-\infty}^\infty e^{ix}/(x^2 + a^2) dx = \frac{1}{2} \cdot 2\pi i \operatorname{Res}(\frac{e^{iz}}{(z - ia)(z + ia); z = ia}) = \pi i \cdot \frac{e^{iz}}{z + ia}|_{z=ia} = \frac{\pi e^{-a}}{2a}.$$

∎

Ph.D. Level

5. (Whittaker and Watson, 6.22) If the $\operatorname{Re} z > 0$ then

$$\int_0^\infty (e^{-t} - e^{-tz}) \frac{dt}{t} = \log z$$

Solution Observe that the integrand is analytic as $t \to 0$ so the integral is well defined. One can take the derivative of this function with respect to z and show that it is equal to $1/z$. This shows that it agrees with the logarithm up to a constant when $\operatorname{Re}(z) > 0$.

I couldn't see how this relates back to residues. I appears you don't need to shew anything. ∎

Here is actually what they had in mind.

Remarks on Whittaker and Watson's solution They use that a function is analytic and apply the Cauchy integral formula... *I will type this up when I get some time.*

6. (Whittaker and Watson, 6.24,2) If $0 \leq z \leq 1$ and $-\pi < a \leq \pi$ then

$$\int_0^\infty \frac{t^{z-1}}{t + e^{ia}} dt = \frac{\pi e^{i(z-1)a}}{\sin(\pi z)}$$

Solution This is another "keyhole" contour since there is a branch cut. One again gets

$$(1 - e^{2\pi i(z-1)}) \int_0^\infty \frac{t^{z-1}}{t + e^{ia}} dt = 2\pi i \operatorname{Res}(\frac{t^{z-1}}{t + e^{ia}}; t = -e^{ia}) = 2\pi i (e^{i(a+\pi)(z-1)})$$

which tells us

$$\begin{aligned} \int_0^\infty \frac{t^{z-1}}{t + e^{ia}} dt &= 2\pi i \frac{e^{i(a(z-1)} e^{i\pi z}(-1)}{1 - e^{2\pi i z}} \\ &= -2\pi i \frac{e^{i(a(z-1)}}{e^{-i\pi z} - e^{i\pi z}} \\ &= -2\pi i \frac{e^{i(a(z-1)}}{-2i \sin(\pi z)} \\ &= \frac{\pi e^{i(z-1)a}}{\sin(\pi z)} \end{aligned}$$

∎

7. (Whittaker and Watson 6.24, 1, pg118) If $0 < a < 1$ show that

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \pi \csc a\pi$$

Solution Yet another "keyhole" problem. Using the difference in phase we get

$$(1 - e^{(a-1)2\pi i}) \int_0^\infty \frac{x^{a-1}}{1+x} dx = 2\pi i e^{(a-1)\pi i} = -2\pi i e^{a\pi i}.$$

Using $e^{(a-1)2\pi i} = e^{2\pi a i}$ we get

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = 2\pi i \frac{e^{a\pi i}}{1 - e^{2a\pi i}} = 2\pi i \frac{1}{-e^{a\pi i} + e^{-a\pi i}} = \frac{2\pi i}{-2i\sin(a\pi)} = \frac{-\pi}{\sin(a\pi)}$$

Note that in computing the residue we used $e^{i\pi} = -1$. $\blacksquare$

8. (Whittaker and Watson, 6.24, 4)Show that if $-1 < p < 1$ and $-\pi < \lambda < \pi$ we have

$$\int_0^\infty \frac{x^{-p} dx}{1 + 2x\cos(\lambda) + x^2} = \frac{\pi}{\sin(p\pi)} \frac{\sin(p\lambda)}{\sin(\lambda)}$$

Solution Keyhole. The trick is to observe that

$$1 + 2x\cos(\lambda) + x^2 = (x + e^{i\lambda})(x + e^{-i\lambda}).$$

If we let $A = \int_0^\infty \frac{x^{-p} dx}{1+2x\cos(\lambda)+x^2}$ we get Using monodromy and decaying of the portions of the integral at infinity, we get

$$\int_{Contour} = (1 - e^{-p2\pi i})A = 2ie^{-p\pi i}\sin(p\pi)A$$

On the other hand, we may compute the residue

$$\begin{aligned} & 2\pi i \left[\mathrm{Res}(\frac{z^{-p}}{(z+e^{i\lambda})(z+e^{-i\lambda})}; z = -e^{i\lambda}) + \mathrm{Res}(\frac{z^{-p}}{(z+e^{i\lambda})(z+e^{-i\lambda})}; z = -e^{-i\lambda}) \right]. \\ = & 2\pi i \left(\frac{(-e^{i\lambda})^{-p}}{(-e^{i\lambda} + e^{-i\lambda})} + \frac{(-e^{-i\lambda})^{-p}}{(-e^{-i\lambda} + e^{i\lambda})} \right) \\ = & 2\pi i \left(\frac{(-e^{i\lambda})^{-p}}{-2i\sin(\lambda)} + \frac{(-e^{-i\lambda})^{-p}}{2i\sin(\lambda)} \right) \\ = & \pi \frac{e^{-i\pi p}(-e^{-i\lambda p} + e^{+i\lambda p})}{2i\sin(\lambda)} \\ = & \pi e^{-i\pi p} \frac{\sin(\lambda p) 2i}{\sin(\lambda)} \end{aligned}$$

which gives

$$A = \pi \frac{\sin(\lambda p)}{\sin(p\pi)\sin(\lambda)}$$

$\blacksquare$

9. (Whittaker and Watson, 6.21, Example 3) Let n be a positive integer. Show that

$$\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$$

Solution We convert the real periodic integral into a contour integral.

$$\begin{aligned}
\int_0^{2\pi} e^{\cos(\theta)} \cos(n\theta - \sin(\theta)) d\theta &= \operatorname{Re} \int_0^{2\pi} e^{\cos(\theta)} e^{i(n\theta - \sin(\theta))} d\theta \\
&= \operatorname{Re} \int_0^{2\pi} e^{\cos(\theta) - i\sin(\theta)} e^{in\theta} d\theta \\
&= \operatorname{Re} \int_0^{2\pi} e^{e^{-i\theta}} (e^{i\theta})^n d\theta \\
&= \operatorname{Re} - \int_0^{-2\pi} e^{e^{i\phi}} (e^{-i\phi})^n d\theta \\
&= \operatorname{Re} \int_{-2\pi}^{0} e^{e^{i\phi}} (e^{-i\phi})^n d\theta \\
&= \operatorname{Re} \int_0^{2\pi} e^{e^{i\phi}} (e^{-i\phi})^n d\theta \\
&= \operatorname{Re} i \oint_\gamma \frac{e^z}{z^{n+1}} \frac{dz}{i} \\
&= 2\pi \operatorname{Res}(\frac{e^z}{z^{n+1}}; z = 0) = \frac{2\pi}{n!}
\end{aligned}$$

∎