Dupuy — Complex Analysis — Spring 2017 — Homework 05

I am following some solutions from a course given by Pedro Embid at UNM. He made some great notes.

Conformal Maps

1. Find the points where $w = f(z)$ is conformal if

   (a) $w = \cos(z)$

   (b) $w = z^5 - 5z$

   (c) $w = 1/(z^2 + 1)$

   (d) $w = \sqrt{z^2 + 1}$.

   Solution The places where the map is conformal is where the derivative is non-zero and doesn't have a pole or branch point.

   (a) If $w = \cos(z)$, $dw/dz = -\sin(z)$ therefore, $dw/dz = 0$ if and only if $z \in \pi\mathbf{Z}$. Hence, $w = \cos(z)$ is conformal for all $z \neq n\pi$, $n \in \mathbf{Z}$.

   (b) If $w = z^5 - 5z$ then $dw/dz = 5(z^4 - 1)$ which means $dw/dz = 0$ only when $z = \pm 1, \pm i$ which means $w = z^4 - 5z$ is conformal when $z \in \mathbf{C} \setminus -1, 1, i, -i\}$.

   (c) $w = 1/(z^2 + 1)$ is differentiable when $z \neq \pm i$, and $\frac{dw}{dz} = \frac{-2z}{(z^2+1)^2}$ which means $dw/dz = 0$ if and only if $z = 0$. So $w = 1/(z^2 + 1)$ is conformal on $\mathbf{C} \setminus \{0, \pm i\}$.

   (d) For $w = \sqrt{z^2 + 1} = \sqrt{(z - i)(z + i)}$ we have two branch points $z = \pm i$, so we have to introduce branch cuts. The answer depends on the branch of $\sqrt{z^2 + 1}$ one has selected. Assume we take branch from i to ∞ and $-i$ to ∞ as indicated in the figure (so that $w = \sqrt{z^2 + 1}$ is differentiable at zero).
   Then on $\Omega = \mathbf{C} \setminus \{z = iy : |y| \geq 1\}$ and $w = \sqrt{z^2 + 1}$ is differentiable and $\frac{dw}{dz} = \frac{z}{\sqrt{z^2+1}}$ so $dw/dz = 0$ if and only if $z = 0$. We conclude that $w = \sqrt{z^2 + 1}$ is conformal on $\Omega \setminus \{0\}$.

2. Find a conformal map of the strip $0 < \operatorname{Re} z < 1$ onto the unit disc $|w| < 1$ in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

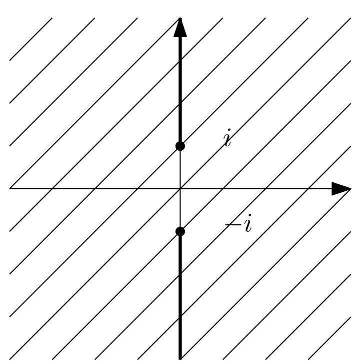

Figure 1: The branch cuts for $w = \sqrt{z^2 + 1}$.

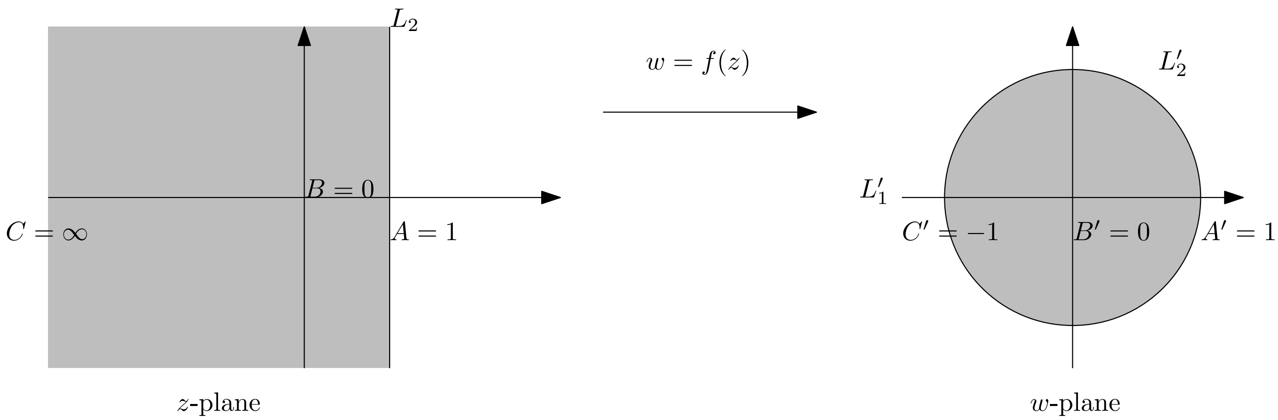

Figure 2: The conformal mapping $f(z) = \frac{z}{2-z}$.

Solution We worked this out in class: $f_1(z) = i\pi z$ rotates the horizontal strip and scales it. $f_2(z) = e^z$ moves the strip to the upper half plane. $f_2(z) = -(z-i)/(z+i)$ is the negative of the Cayley transformation.

$$f(z) = f_3(f_2(f_1(z))) = -\frac{e^{i\pi z} - i}{e^{i\pi z} + 1}$$

3. Find the Möbius transformation that maps the left have plane $\{z \in \mathbf{C} : \operatorname{Re} z < 1\}$ to the unit disc $\{w \in \mathbf{C} : |w| < 1\}$ and has $z = 0$ and $z = 1$ as fixed points.

   Solution We use that $\frac{w-1}{w} = k\frac{z-1}{z}$ hence $0 \mapsto 0$, $1 \mapsto 1$ and if $\infty \mapsto -1$ we need $k = 2$ so that $w = \frac{z}{2-z}$. The conformal map is pictured in figure 2.

4. Find a conformal map from the following regions onto the unit disc $D = \{z : |z| < 1\}$

   (a) $A = \{z : |z| < 2, \operatorname{Arg}(z) \in (0, \pi/4)\}$
   (b) $B = \{z : \operatorname{Re}(z) > 2\}$
   (c) $C = \{z : -1 < \operatorname{Re}(z) < 1\}$
   (d) $D' = \{z : |z| < 1 \text{ and } \operatorname{Re} z < 0\}$

   Solution These can be done with the handout that Christelle handed out with the conformal maps from Marsden and Hoffman book

   (a) We did this in class:

   $$f(z) = f_4(f_3(f_2(f_1(z)))) = \frac{\left(\frac{1+z^4/16}{1-z^4/16}\right)^2 - i}{\left(\frac{1+z^4/16}{1-z^4/16}\right)^2 + i}.$$

   (b) $f_1(z) = z - 2$ is a translation to the plane with $\operatorname{Re} z > 0$. $f_2(z) = T(iz)/i = (z-1)/(z+1)$ is a conjugate of the Cayley transformation. The composition is:

   $$f(z) = f_2(f_1(z)) = \frac{z-3}{z-1}.$$

   (c) $f_1(z) = (z+2)/2$ translates and scales. $f_2(z) = -\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$ is the conformal map from the second problem:

   $$f(z) = \frac{e^{i\pi(z+2)/2} - i}{e^{i\pi(z+2)/2} + i}$$

(d) $f_1(z) = -iz$ rotated the left half disc to the upper half disc. We then apply part (a)

$$f(z) = \frac{\left(\frac{1-iz}{1+iz}\right)^2 - i}{\left(\frac{1-iz}{1+iz}\right)^2 + i}.$$

5. Let D be the unit disc. Let $f : D \to D$ be a conformal map.

(a) If $f(0) = 0$ show that $f(z) = \omega z$ for some $\omega \in \partial D$.

(b) If $f(0) \neq 0$ show that there exists some $a \in D$ and $\omega \in \partial D$ such that

$$f(z) = \omega \frac{z-a}{1-\overline{a}z}.$$

Solution (a) This is exactly the conclusion of Schwarz Lemma.

(b) Let $g : D \to D$ be a conformal map and suppose that $g(0) = a$. Suppose $h(z) = \frac{z-a}{1-\overline{a}z}$ is a conformal map from D to D. If this is the case then because $h(a) = 0$ we have $h \circ g : D \to D$ a conformal map with $h(g(0)) = 0$ which implies $h \circ g(z) = \omega^{-1} z$ with $|\omega| = 1$ by the first part. This implies that $\omega h(z) = f(z)$ is an inverse to $g(z)$ as in the statement of the problem. It is hence enough to show that $|h(z)| \leq 1$ for $|z| \leq 1$.
Writing $z = x + iy$ and $a = \alpha + i\beta$ gives

$$|\frac{z-a}{1-\overline{a}z}| = \frac{(x-\alpha)^2 + (y-\beta)^2}{(1-\alpha x - \beta y)^2 + (y\alpha - \beta x)^2}$$

This is a mess. Don't do this.
This dude is a Blashke factor (Look at Proposition 9.1.1 of Green and Krantz). If $|z| = 1$ then $|\overline{z}| = 1$ and $|z\overline{z}| = 1$ this means

$$\left|\frac{z-a}{1-\overline{a}z}\right| = \frac{1}{1/|z|}\left|\frac{z-a}{1-\overline{a}z}\right| = \frac{|z-a|}{|\overline{z}-\overline{a}|} = 1$$

and since the function $h(z)$ is analytic on all of D and hence we can use the maximum modulus principle to conclude that $|h(z)| < 1$

6. (a) Show that $\mathrm{PSL}_2(\mathbf{Z})$ is generated by $S(z) = -1/z$ and $T(z) = z + 1$ and hence has the presentation

$$\langle S, T : S^2 = 1, (ST)^3 = 1\rangle.$$

(b) Show that a fundamental domain[1] for this action is the complement of the unit disc in a vertical strip of length 1 centered around zero in the upper half plane. In other words

$$\Omega = \{z : |z| \geq 1 \text{ and } -1/2 \leq \mathrm{Re}(z) \leq 1/2\}$$

is a fundamental domain for this action.

(c) Show that the following points are fixed points of $\overline{\Omega}$ with the following stabilizers:

i. $\mathrm{Stab}(i) = \{1, S\}$

ii. $\mathrm{Stab}(e^{2\pi i/3}) = \{1, ST, (ST)^2\}$

iii. $\mathrm{Stab}(e^{\pi i/3}) = \{1, TS, (TS)^2\}$

(Note: this exercise gives you an example of an action that is not free.)

[1] A fundamental domain for an action $\Gamma \times X \to X$ is a closed subset $\Omega \subset X$ such that

i. $X = \bigcup_{\gamma\in\Gamma} \gamma(\Omega)$

ii. For all $\gamma \neq 1$ the set $\gamma(\Omega) \cap \Omega$ has empty interior.

Note that this definition is different from what I had originally said in class. We had our fundamental domains have the property that $\gamma(\Omega) \cap \Omega = \emptyset$. Unfortunately, as this example shows, we can't always arrange for this.

Solution (a) Clearly $S^2(z) = z$. We have $S \circ T(z) = \frac{-1}{z+1}$, and hence

$$\begin{aligned}
(S \circ T)^2(z) &= \frac{-1}{\frac{-1}{z+1} + 1} = \frac{-(z+1)}{-1+z+1} = -\frac{z+1}{z} \\
(S \circ T)^3(z) &= -\frac{\frac{-1}{z+1} + 1}{\frac{-1}{z+1}} = \frac{-1+(z+1)}{1} = z.
\end{aligned}$$

It is enough to show that $\langle S, T \rangle \supset \mathrm{PSL}_2(\mathbf{Z})$.
If $\gcd(a, b) \neq 1$ then $ad - bc \neq 1$, which is a contradiction. Similarly, $\gcd(c, d) = 1$.
The following algorithm find a presentation of

$$f(z) = \frac{az+b}{cz+d}.$$

- (Division Algorithm Step) If $|a| \leq |b|$ then write $b = aq + r$ and consider

$$f \circ T^q(z) = \frac{a(z-q)+b}{c(z-q)+d} = \frac{az+r}{cz-cq+d}.$$

- If $|b| < |a|$ consider

$$f \circ S(z) = \frac{-a+bz}{-c+dz}.$$

  This reduces us to the previous division algorithm.
- By using these two steps we may reduce $f(z)$ to a transformation of the form

$$f(z) = \frac{az}{cz+d},$$

  since we must have $ad - c0 = 1$ we must have $a = d = \pm 1$ and without loss of generality we can assume $a = 1$ (since we are working projectively)

$$f(z) = \frac{z}{cz+1},$$

  By applying $f \circ S$ we are reduced to considering

$$\frac{-1}{-c+z}$$

  which by translation reduces to

$$\frac{-1}{z}$$

  which is inverted by S.

(b) We need to check the two axioms of a fundamental domain from the footnote.

i. We need to show that H is covered by $\{\gamma(\Omega) : \gamma \in \mathrm{PSL}_2(\mathbf{Z})\}$. By translation, it suffices to show that the $\{z : |z| \leq 1 \text{ and } \operatorname{Im} z > 0\}$ is covered by these sets.

ii. We can readily see that $T(\Omega) \cap \Omega$ is just along the line $|z| = 1/2$. It remains to check that $S(\Omega) \cap \Omega$ has empty interior. This is the case since only the circle $|z| = 1$ is fixed by S.

(c) We will make use of the equation $\frac{az+b}{cz+d} = z$ which becomes

$$cz^2 + (d-a)z - b = 0. \tag{1}$$

Also, the only fixed points $\overline{\Omega}$ must be on the boundary. Also, we know that z such that $|\operatorname{Re}(z)| = 1/2$ and $|z| > 1$ are not fixed by S or T so it must be the case that $|z| = 1$. From equation (1) we know that any fixed point z is algebraic of degree two. The only roots of unity in the upper half plane of degree two are $i, \rho = e^{\pi i/3}$ and $-\overline{\rho} = e^{2\pi i/3} = S(\rho) = T^{-1}(\rho)$.

i. In the case when $z = i$, the fixed point equations reduce to $b + c = 0$ and $d - a = 0$ which implies $a = d$ and $b = -c$. From this we find that $1 = ad - bc = a^2 + b^2$ only has solutions when $(a, b) = (\pm 1, 0), (0, \pm 1)$ which give us S and the identity respectively.

ii. The observations that ρ and $-\overline{\rho}$ are related by S and T give use two things. First

$$\mathrm{Stab}(\overline{\rho}) = S^{-1}\,\mathrm{Stab}(\rho) S.$$

and so we only need to compute the stabilizers of ρ. Second, it tells us that $TS \in \mathrm{Stab}(\rho)$. We can also see that this is an element of order three.
Suppose that $(a\rho + b)/(c\rho + d) = \rho$ Since $\mathrm{Im}(\frac{az+b}{cz+d}) = \frac{x}{|cz+d|^2}$ we must have that $\frac{1/2}{|cz+d|^2} = \sqrt{3}/2$ which implies that $|c\rho + d| < 1$
(Finish writing this down)
Another approach would be to look at the minimal polynomial of $\rho = e^{\pi i/3}$, which is $z^2 - z + 1 = 0$. We know that this polynomial must divide the fixed point polynomial. This implies things like $(d - a)/c = -1$ and $b/c = 1$. These give $c = b$ and $d - a = -b$ this means our matrix is of

Elliptic Functions

7. Show that

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

is elliptic with period lattice Λ.

Solution (Knapp) It is easy to show that $\wp'(z)$ is elliptic with period lattice Λ. Also observe that $\wp(z) = \wp(-z)$. Let $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. This means $\wp'(z + \omega_1) - \wp'(z) = 0$ which implies $\wp(z + \omega_1) - \wp(z) = C$ and evaluating at $z = \frac{-\omega_1}{2}$ shows that $C = 0$. Similarly for ω_2.

8. For a lattice $\Lambda \subset \mathbf{C}$ and $m \geq 3$ define $G_m = G_m(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-m}$.

(a) Show that $\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k$.[2]

(b) Conclude that

$$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2 = O(z^2),$$

as $z \to 0$, which shows that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2$ is analytic at the origin of $\mathbf{C}$. Here $g_2 = 60G_4$ and $g_3 = 140G_6$.

(c) Conclude that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2$ is constant. (Hint: use that elliptic functions without poles are constant.)

(d) Show the constant in the previous number is zero.

Solution (a) We have

$$\frac{1}{(z-\lambda)^2} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(n+1)z^n}{\lambda^{n+1}} = 1/\lambda^2 + 2z/\lambda^3 + 3z^2/\lambda^4 + \cdots$$

This shows that

$$\begin{aligned} \wp(z) - \frac{1}{z^2} &= \sum_{\lambda \in \Lambda^*} \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\lambda^{n+2}} \\ &= \sum_{n=1}^{\infty} (n+1) z^n \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^{n+2}}, \end{aligned}$$

where we used the footnote.

[2] You may need to use that you can interchange some series. If $f_n(z) = \sum a_j^{(n)} z^j$ and $A_j = \sum_{n=0}^{\infty} a_j^{(n)}$ converges then $\sum_{n=0}^{\infty} f_n(z) = \sum_{j=0}^{\infty} A_j z^j$.

(b) (Check using Sage or a long by-hand computation)

(c) $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_2$ is an elliptic function without poles. Hence it is constant.

(d) Evaluating at say $\omega_1/2$ shows that this constant must be zero.

9. The zeros of $\wp(z) - c$ are simple with precisely double zeros at the points congruent to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$. (Hint: what are the zeros of $\wp'(z)$ and what does this mean?)

Solution As usual, the double points are where the derivative vanishes.

Geometric Argument: if the $\wp'(z)$ vanishes then by our equation we have $4\wp(z)^3 - g_2\wp(z) - g_2 = 0$. There are precisely the 2-torsion points which under Weierstrass uniformization group isomorphism correspond to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$ in a fundamental domain of $\mathbf{C}/\Lambda$.