Dupuy — Complex Analysis — Spring 2017 — Homework 05

Conformal Maps

1. Find the points where $w = f(z)$ is conformal if
   (a) $w = \cos(z)$
   (b) $w = z^5 - 5z$
   (c) $w = 1/(z^2+1)$
   (d) $w = \sqrt{z^2+1}$.

2. Find a conformal map of the strip $0 < \operatorname{Re} z < 1$ onto the unit disc $|w| < 1$ in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

3. Find the Möbius transformation that maps the left have plane $\{z \in \mathbf{C} : \operatorname{Re} z < 1\}$ to the unit disc $\{w \in \mathbf{C} : |w| < 1$ and has $z = 0$ and $z = 1$ as fixed points.

4. Find a conformal map from the following regions onto the unit disc $D = \{z : |z| < 1\}$
   (a) $A = \{z : |z| < 2, \operatorname{Arg}(z) \in (0, \pi/4)\}$
   (b) $B = \{z : \operatorname{Re}(z) > 2$
   (c) $C = \{z : -1 < \operatorname{Re}(z) < 1\}$
   (d) $D' = \{z : |z| < 1 \text{ and } \operatorname{Re} z < 0\}$

5. Let D be the unit disc. Let $f : D \to D$ be a conformal map.
   (a) If $f(0) = 0$ show that $f(z) = \omega z$ for some $\omega \in \partial D$.
   (b) If $f(0) \neq 0$ show that there exists some $a \in D$ and $\omega \in \partial D$ such that
   $$f(z) = \omega \frac{z-a}{1-\overline{a}z}.$$

6. (a) Show that $\mathrm{PSL}_2(\mathbf{Z})$ is generated by $S(z) = -1/z$ and $T(z) = z+1$ and hence has the presentation
   $$\langle S, T : S^2 = 1, (ST)^3 = 1 \rangle.$$
   (b) Show that a fundamental domain[1] for this action is the complement of the unit disc in a vertical strip of length 1 centered around zero in the upper half plane. In other words
   $$\Omega = \{z : |z| \geq 1 \text{ and } -1/2 \leq \operatorname{Re}(z) \leq 1/2\}$$
   is a fundamental domain for this action.
   (c) Show that the following points are fixed points of $\overline{\Omega}$ with the following stabilizers:
      i. $\operatorname{Stab}(i) = \{1, S\}$
      ii. $\operatorname{Stab}(e^{2\pi i/2}) = \{1, ST, (ST)^2\}$
      iii. $\operatorname{Stab}(e^{\pi i/3}) = \{1, TS, (TS)^2\}$

   (Note: this exercise gives you an example of an action that is not free.)

[1] A fundamental domain for an action $\Gamma \times X \to X$ is a closed subset $\Omega \subset X$ such that
i. $X = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$
ii. For all $\gamma \neq 1$ the set $\gamma(\Omega) \cap \Omega$ has empty interior.
Note that this definition is different from what I had originally said in class. We had our fundamental domains have the property that $\gamma(\Omega) \cap \Omega = \emptyset$. Unfortunately, as this example shows, we can't always arrange for this.

Elliptic Functions

7. Show that

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

is elliptic with period lattice Λ.

8. For a lattice $\Lambda \subset \mathbf{C}$ and $m \geq 3$ define $G_m = G_m(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-m}$.

   (a) Show that $\wp(z) - \frac{1}{z^2} = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k$.[2]

   (b) Conclude that

   $$\wp'(z)^2 - 4\wp(z)^3 + g_2 \wp(z) + g_3 = O(z^2),$$

   as $z \to 0$, which shows that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is analytic at the origin of $\mathbf{C}$. Here $g_2 = 60G_4$ and $g_3 = 140G_6$.

   (c) Conclude that $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$ is constant. (Hint: use that elliptic functions without poles are constant.)

   (d) Show the constant in the previous number is zero.

9. The zeros of $\wp(z) - c$ are simple with precisely double zeros at the points congruent to $\omega_1/2, (\omega_1 + \omega_2)/2, \omega_2/2$. (Hint: what are the zeros of $\wp'(z)$ and what does this mean?)

[2] You may need to use that you can interchange some series. If $f_n(z) = \sum a_j^{(n)} z^j$ and $A_j = \sum_{n=0}^{\infty} a_j^{(n)}$ converges then $\sum_{n=0}^{\infty} f_n(z) = \sum_{j=0}^{\infty} A_j z^j$.