Dupuy — Complex Analysis — Spring 2017 — Homework 06

Some solutions come from notes from a course Pedro Embid gave at UNM.

Rouche's Theorem and Argument Principal

1. (New Mexico, Jan 1997) How many roots does $p(z) = z^4 + z + 1$ have in the first quadrant?

   Solution Let $w = p(z) = z^4 + z + 1$. To compute the number of zeros of $p(z)$ in the first consider the changes in the argument of $p(z)$ on the quarter circle C in the first quadrant where $C = C_1 \cup C_2 \cup C_3$:

   $$C_1 : z = x,\ 0 \leq x \leq R$$

   $$C_2 : z = Re^{i\theta},\ 0 \leq \theta \leq \pi/2$$

   $$C_3 : z = iy,\ 0 \leq y \leq R$$

   - On $C_1 : w = p(z) = x^4 + x + 1 \geq 1$ so the image C' stays on the u-axis. $A = 0 \mapsto A' = 1$, $B = R \mapsto B' = R^4 + R + 1$.
   - On $C_2 : w = p(z) = R^4 e^{i4\theta} + Re^{i\theta} + 1$, $0 \leq \theta \leq \pi/4$. If $R \geq 2$ then $|w| \geq |z^4 + z + 1| \geq |z|^4 - |z| - 1 = 14$ therefore the image C_1' is the outside of the disc of radius 14. Also, $C = Re^{i\pi/2} \mapsto C' = R^4 + 1 + 2R$ which is in the first quadrant.
   - On $C_3 : w = p(z) = y^4 + 1 + y$ therefore $u = \operatorname{Re} w \geq 1, v = \operatorname{Im} w \geq 0$, so w is in the first quadrant and to the left of $u = 1$. Since $D = A = 0$ we have $D' = 1$. We see that the image C' of C winds once around the origin so $p(z) = z^4 + z + 1$ has only one root there.

2. (New Mexico, Aug 1993) How many roots does $e^z - 4z^n + 1 = 0$ have inside the unit disc $|z| < 1$?

   Solution Let $f(z) = 4z^n$ and $g(z) = e^z + 1$. On $|z| = 1$: $|g(z)| = |e^z + 1| \leq e + 1 < 4 = |4z^n| = |f(z)|$ By Rouche's theorem, $f(z)$ and $g(z) - f(z)$ have the same number of roots. Since $f(z) = 4z^n = 0$ has n roots on $|z| = 1$ so does $g(z) - f(z) = e^z - 4z^n + 1$.

Riemann Surfaces

3. (a) Show that every automorphism of $\mathbf{C}$ extends to an automorphism of $\mathbf{P}^1$.

   (b) Show that $\mathrm{Aut}(\mathbf{C}) := \{az+b : a \in \mathbf{C}^\times \text{ and } b \in \mathbf{C}\}$ (This sometimes called the one dimensional affine linear group and is denoted $\mathrm{AL}_1(\mathbf{C})$.).

   Solution (a) Let $f(z)$ be an automorphism of $\mathbf{C}$. Since it is an automorphism then the map is proper (the inverse image of every compact set is compact). By our characterzation of proper maps $f(z)$ must be a polynomial. The only invertible polynomials have degree one so $f(z) = az + b$ for some $a, b \in \mathbf{C}$. This is a Möbius transformation and has a unique extension to $\mathbf{P}^1$.

   (This was not what I had in mind originally, maybe you guys will come up with something different — I think you can reduce to showing $g(z) = 1/f(1/z)$ is analytic in a neighborhood around zero and then concluding (via the automorphism property) that $g(0) = 0$ with multiplicaity one).

   (b) If $f(z)$ is an automorphism of $\mathbf{P}^1$ then $f(z) = \frac{az+b}{cz+d}$ and we must have $f(\infty) = \infty$ and this means that $c = 0$ which proves the result.

4. Show that $\mathbf{C}$ is not conformally equivalent to $D = \{z \in \mathbf{C} : |z| < 1\}$.

   Solution If $f : \mathbf{C} \to D$ then we have a bounded entire function which must be constant.

5. Show that $\mathrm{Aut}(H) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbf{R} \text{ and } ad - bc = 1\}$ (This is sometimes called the two dimensional projective special linear groups with coefficients in $\mathbf{R}$, and is denoted $\mathrm{PSL}_2(\mathbf{R})$).

   Solution The Cayley transform $T : H \to D$ is given by $T(z) = (z - i)/(z + i)$ and its inverse is $T(w) = -2iw/(w - 1)$. An automorphism of H is then given as $T^{-1} \circ S \circ T$ where $S(z) = \omega \frac{z-a}{1-\overline{a}z}$ where $|a| < 1$ and $|\omega| = 1$. In terms of matrices this is

$$\begin{bmatrix} -2i & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \omega & -\omega a \\ -\overline{a} & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{bmatrix} -2i(\omega - \omega a) & -2i(-i\omega - i\omega a) \\ \omega - \omega a + \overline{a} - 1 & i\omega - i\omega a - i\overline{a} - i \end{bmatrix}$$

   So the resulting Mobius transformation is of the form

$$\begin{aligned} T^{-1} \circ S \circ T(z) &= \frac{Az + B}{Cz + D} \\ A &= -2i\omega(1 - a), \\ B &= 2i(i\omega)(1 - a), \\ C &= \omega(1 - a) + (\overline{a} - 1), \\ D &= -i\omega(1 + a) - i(\overline{a} + 1). \end{aligned}$$

   We will multiply by $\omega^{-1/2}$ which gives... something isn't right here.

   (TODO: Use Sage to check these computations.)

   (This is also worked out in Green and Krantz by another method)

Harmonic Functions

6. Let $f(z) = u(z) + iv(z)$. Show that the level sets of $u(z)$ and $v(z)$ are orthogonal.

   Solution The normal vector of the level sets of $u(x, y) = c_1$ and $v(x, y) = c_2$ are given by $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ and we have

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0.$$

   Here we applied the Cauchy-Riemann equations.

7. Let $u_0(\theta)$ be a continuous 2π-periodic function. Let D be a disc of radius r. The Dirichlet boundary value problem asks to find a function $u(x, y)$ such that:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in D \\ u(e^{i\theta}) = u_0(\theta), \end{cases}$$

   Show that convolution with the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

   gives a solution to this problem.

   Solution I am going to ignore convergence issues for the moment. The solution of the Dirichlet problem is given by

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} \widehat{u}_0(n) r^{-|n|} e^{in\theta}.$$

where

$$\widehat{u}_0(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}u_0(t)dt.$$

We will examine the positive and negative terms of this series separately. In what follows we will be able to switch integration and summation because of uniform convergence on compact subsets when $r < 1$. In the case $r = 1$ we can assume that the Fourier representation is uniformly convergent representation of $u_0(\theta)$.

Positive terms:

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{u}_0(n) r^n e^{in\theta} &= \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} e^{-int}u_0(t)dt e^{in\theta} r^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} u_0(t)\sum_{n=0}^{\infty}\left(e^{in(\theta-t)}r\right)^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} u_0(t)\frac{1}{1-re^{i(\theta-t)}}dt
\end{aligned}$$

Negative terms:

$$\begin{aligned}
\sum_{n=1}^{\infty} \widehat{u}_0(-n) r^n e^{-in\theta} &= \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} e^{int}u_0(t)dt e^{-in\theta} r^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} u_0(t)\sum_{n=1}^{\infty}\left(e^{in(t-\theta)}r\right)^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} u_0(t)\frac{re^{i(t-\theta)}}{1-re^{i(t-\theta)}}dt
\end{aligned}$$

Putting these together we have

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u_0(t)\left[\frac{1}{1-re^{i(\theta-t)}} + \frac{re^{i(t-\theta)}}{1-re^{i(t-\theta)}}\right]dt = \frac{1}{2\pi}\int_0^{2\pi} u_0(t)P(r,\theta-t),$$

where

$$\begin{aligned}
P_r(\phi) = P_(r,\phi) &= \frac{1}{1-re^{i\phi}} + \frac{re^{i\phi}}{1-re^{i\phi}} \\
&= \frac{1-r^2}{1-2r\cos(\phi)+r^2}.
\end{aligned}$$