Dupuy — Complex Analysis — Spring 2017 — Homework 06

Rouche's Theorem and Argument Principal

1. (New Mexico, Jan 1997) How many roots does $p(z) = z^4 + z + 1$ have in the first quadrant?
2. (New Mexico, Aug 1993) How many roots does $e^z - 4z^n + 1 = 0$ have inside the unit disc $|z| < 1$?

Riemann Surfaces

3. (a) Show that every automorphism of $\mathbf{C}$ extends to an automorphism of $\mathbf{P}^1$.

   (b) Show that $\mathrm{Aut}(\mathbf{C}) := \{az+b : a \in \mathbf{C}^\times \text{ and } b \in \mathbf{C}\}$ (This sometimes called the one dimensional affine linear group and is denoted $\mathrm{AL}_1(\mathbf{C})$.).
4. Show that $\mathbf{C}$ is not conformally equivalent to $D = \{z \in \mathbf{C} : |z| < 1\}$.
5. Show that $\mathrm{Aut}(H) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbf{R} \text{ and } ad - bc = 1\}$ (This is sometimes called the two dimensional projective special linear groups with coefficients in $\mathbf{R}$, and is denoted $\mathrm{PSL}_2(\mathbf{R})$).

Harmonic Functions

6. Let $f(z) = u(z) + iv(z)$ be analytic. Show that the level sets of $u(z)$ and $v(z)$ are orthogonal.
7. Let $u_0(\theta)$ be a continuous 2π-periodic function. Let D be a disc of radius r. The Dirichlet boundary value problem asks to find a function $u(x, y)$ such that:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{for } (x, y) \in D \\ u(e^{i\theta}) = u_0(\theta), & \end{cases}$$

   Show that convolution with the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

   gives a solution to this problem.