Dupuy — Complex Analysis — Spring 2017 — Homework 07

1. Show the Gauss' formula for the gamma function:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)}.$$

(Take the definition of the gamma function to be from its product formula).

Solution

$$\begin{aligned}
\frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)} &= \frac{e^{z\log(n)}}{\frac{1}{n!}\prod_{j=0}^n (z+j)} \\
&= \frac{e^{z\log(n)}e^{-\sum_{j=1}^n z/j}}{\prod_{j=0}^n (1+\frac{z}{j})e^{-z/j}} \\
&= \frac{e^{z\left(\log(n)-\sum_{j=1}^n \frac{1}{j}\right)}}{\prod_{j=0}^n (1+\frac{z}{j})e^{-z/j}} \\
&\to \frac{e^{-\gamma z}}{\prod_{j=0}^\infty (1+\frac{z}{j})e^{-z/j}} = \Gamma(z)
\end{aligned}$$

2. Show that $\int_0^{2\pi} \log|1-e^{i\theta}|d\theta = 0$.

Solution We have $|1-e^{i\theta}| = |e^{-i\theta/2}-e^{i\theta/2}| = |2i\sin(\theta/2)| = 2|\sin(\theta/2)|$.
Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} \log|2\sin(\theta/2)|d\theta &= 2\int_0^{\pi} \log|2\sin(\theta/2)|d\theta \\
&= 4\int_0^{\pi/2} \log|2\sin(\phi)|d\phi \\
&= 2\pi\log(2) + 4\int_0^{\pi/2} \log|\sin(\phi)|d\phi
\end{aligned}$$

Where we used $\phi = \theta/2$; $2d\phi = d\theta$. The last integral is elementary and can be solved using the substitution $u = \sin(\phi)$:

$$\begin{aligned}
\int_0^{\pi/2} \log|\sin(\phi)|d\phi &= \int_0^1 \frac{\log(u)}{\sqrt{1-u^2}}du \\
&= -\frac{\pi}{2}\log(2).
\end{aligned}$$

This gives the result. (I didn't actually work this one out by hand, I used a Wolfram Alpha)

3. (New Mexico, Jan 2006) Consider $f(z) = \prod_{n=1}^\infty (1 - z/n^3)$. What is the order of $f(z)$?

Solution For infinite products the convergence exponent is the order.

$$\begin{aligned}
\rho = \alpha &= \inf\{\beta : \sum_{n\geq 0} |a_n|^{-\beta} \text{ converges }\} \\
&= \inf\{\beta : \sum_{n\geq 0} n^{-3\beta} \text{ converges }\} \\
&= 1/3
\end{aligned}$$

4. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of finite order ρ. Show that

$$\rho = \limsup_{n\to\infty} \frac{\log(n)}{\log|a_n|^{-1/n}}.$$

Solution (Buterin Freiling Yurko)

- Suppose $\alpha > \rho$. We have

$$\begin{aligned} |a_n| &= \left|\frac{1}{2\pi i}\oint_{\gamma_R} \frac{f(z)}{z^{n+1}} dz\right| \\ &\leq \frac{\exp(r^\alpha)}{r^n} = \exp(r^\alpha - n\log(r)). \end{aligned}$$

  We have the freedom to choose r, and $r^\alpha - n\log(r)$ is minimized when $r_n = (n/\alpha)^{1/\alpha}$ which gives

$$r_n^\alpha - n\log(r_n) = \frac{n}{\alpha} - \frac{n}{\alpha}\log\frac{n}{\alpha}.$$

  This implies

$$\log|a_n| < \frac{n}{\alpha} - \frac{n}{\alpha}\log\frac{n}{\alpha},$$

  which implies

$$\frac{1}{n}\log|a_n| < \frac{1}{\alpha}(1 - \log(n) + \log(\alpha)),$$

  which implies

$$\frac{\frac{1}{n}\log|a_n|}{\log(n)} < \frac{1}{\alpha}(-1 + o(1)).$$

  Since $\alpha > \rho$ was arbitrary we get

$$\limsup_{n\to\infty} \frac{\log(n)}{\log|a_n|^{-1/n}} < \rho.$$

- If $|a_{n_j}|^{1/n_j} > n_j^{1/\alpha}$ (I need work type this out. The solution is in the reference mentioned)

5. (a) Prove the Castorati-Weiestrass Theorem: Let $f(z)$ is analytic in a punctured disc of radius R at the origin. If $f(z)$ has an essential singularity at $z = 0$ show that for every r with $0 < r < R$ the set $f(D_r(0) \setminus \{0\})$ is dense in $\mathbf{C}$. (This is a corollary of Big Picard).

   (b) Let p be a polynomial. Show that there exists infinitely many z_j such that $p(z_j) = e^{z_j}$.

Solution (a) By the Big Picard Theorem $f(D_r(0) \setminus \{0\}) = \mathbf{C} \setminus \{a\}$ for some $a \in \mathbf{C}$. This set is dense. (This is a stupid proof. If you need this for a qualifying exam you should probably learn another one.)

(b) The function $f(z) = p(z) - e^z$ is entire and has an essential singularity at infinity. Let $g(z) = f(1/z)$.

6. The following exercise is intended to introduce you to the j function which plays a role in the proof of the Big Picard Theorem from class.

   Let H be the upper-half plane. A **modular function** of weight $2k$ and level $N = 1$ is a function $f : H \to \mathbf{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z). \tag{1}$$

   for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

   (a) Let M_{2k} denote the collection of modular forms of weight $2k$ and level 1. Show that $M = \bigoplus_{k\geq 0} M_{2k}$ is a graded ring (i.e. that $M_{2k_1} M_{2k_2} \subset M_{2k_1+2k_2}$).

(b) Show that $G_{2k}(\frac{az+b}{cz+d}) = (cz+d)^{2k} G_{2k}(z)$ has weight $2k$ (Hint: check this on the generators of $\mathrm{SL}_2(\mathbf{Z})$.)

Using the first part conclude that the we have the following modular forms of the indicated weights:

i. $g_2(\tau) = 60G_4(\tau)$, $k = 4$

ii. $g_3(\tau) = 140G_6(\tau)$, $k = 6$

iii. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, $k = 12$

iv. $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$, $k = 0$

Solution (a) Suppose that $f(z)$ has weight k and $g(z)$ has weight l and let $h(z) = f(z)g(z)$. We have

$$\begin{aligned} h(\frac{az+b}{cz+d}) &= f(\frac{az+b}{cz+d})g(\frac{az+b}{cz+d}) \\ &= (cz+d)^{-k} f(z)(cz+d)^{-l} g(z) \\ &= (cz+d)^{-(k+l)} h(z). \end{aligned}$$

It is also easy to see that M_{2k} is a $\mathbf{C}$-vector space.

(b) One needs to show that G_k is a modular form of weight k.

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}.$$

Observe that we may write the automorphy condition as

$$G(\gamma(z)) = j(\gamma, z)G(z)$$

where

$$j(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z) = (cz+d)^k.$$

is called the **automorphy factor**.

We wish to reduce the checking the automorphy condition on the generators. We claim that reducting to this case amounts to checking the following condition on the function j (which is sometimes called a "cocycle condition":

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z))j(\gamma_2, z) \tag{2}$$

This follows from the formal computation:

$$\begin{aligned} j(\gamma_1\gamma_2, z)G(z) &= G(\gamma_1(\gamma_2(z))) \\ &= j(\gamma_1, \gamma_2(z))G(\gamma_2(z)) \\ &= j(\gamma_1, \gamma_2(z))j(\gamma_2, z)G(\gamma_2(z)). \end{aligned}$$

We now check (2). If $\gamma_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ one can check that

$$\begin{aligned} j(\gamma_1, \gamma_2(z)) &= j(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \frac{Az+B}{Cz+D}) \\ &= (c\frac{Az+B}{Cz+D} + d)^k \\ &= \frac{1}{(Cz+D)^k}((Ac+Cd)z + Bc + Dd)^k \\ &= \frac{1}{j(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, z)} j(\begin{bmatrix} Aa+Cb & Ba+Db \\ Ac+Cd & Bc+Dd \end{bmatrix}, z) \\ &= \frac{1}{j(\gamma_2, z)} j(\gamma_1\gamma_2, z) \end{aligned}$$

which proves (2).[1]

It remains to check the automorphy condition on the generators. The generators are

$$\begin{aligned} T : z \mapsto z+1 &\quad\leftrightarrow\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, c=0, d=1 \\ S : z \mapsto -1/z &\quad\leftrightarrow\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, c=1, d=0. \end{aligned}$$

We now check the condition for S:

$$\begin{aligned} G_k(-1/z) &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m\left(\frac{-1}{z}\right)+n)^k} \\ &= \sum_{(m,n)\neq(0,0)} \frac{z^k}{(-m+nz)^k} \\ &= z^k G_k(z). \end{aligned}$$

We now check the condition for T:

$$\begin{aligned} G_k(z+1) &= \sum_{(m,n)\neq(0,0)} \frac{1}{(m(z+1)+n)^k} \\ &= \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+(m+n))^k} \\ &= G_k(z). \end{aligned}$$

To conclude the last line it remains to check:

$$A = \{(m, m+n) : (m,n) \in \mathbf{Z}^2 \setminus \{(0,0)\} = \mathbf{Z}^2 \setminus \{(0,0)\}.$$

Note that $A \subset \mathbf{Z}^2 \setminus \{(0,0)\}$— if $(m, m+n) = (0,0)$ then $m = 0$ which implies $n = 0$. We now show that $\mathbf{Z}^2 \setminus \{(0,0)\} \subset A$. To show $(a,b) = (m, m+n)$ we set $m = a$ and $n = a - b$.

The Roman items follow directly from the graded ring property.

I just wanted you to look at these because they are important.

7. Explain in words the ideas that go into the proof of Montel's Theorem in Green and Krantz (page 193). How is Arzela-Ascoli used?

   Solution We let $\mathcal{F}$ the bounded family of functions from the statement of Montel's Theorem. After restricting to a particular subset we show that it is equicontinuous using Cauchy's formula for derivatives. We then apply Arzela-Ascoli to this equicontinuous family.

8. Let $X = \mathbf{C}^\times = \mathbf{C} \setminus \{0\}$. What is the universal cover of X? What is group of deck transformations for this cover?

   Solution The universal cover is $\exp : \mathbf{C} \to \mathbf{C}^\times$ and the deck transformations are generated by $z \mapsto 2\pi i + z$; $G(\mathbf{C}/\mathbf{C}^\times) \cong \mathbf{Z}$.

9. Use Van Kampen's theorem to rigorously compute $\pi_1(\mathbf{P}^1 \setminus \{p_1, \ldots, p_r\}, z_0)$ for arbitrary r. (Hint: apply Van Kampen to open sets U, V where $U \cap V$ is simply connected).

   Solution I will omit notation for base points understanding that they are necessary for the computation.

[1] The condition for (2) is a group cocycle for group cohomology.

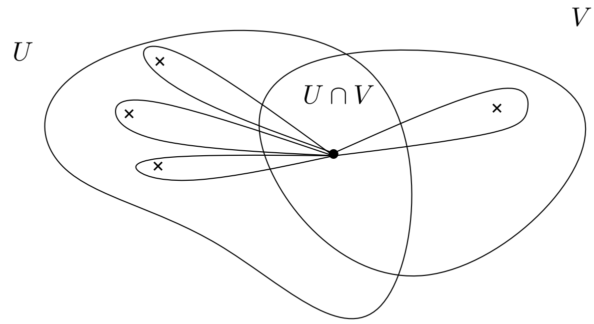

Figure 1: The open subsets used to compute $\pi_1(D \setminus \{p_1, \dots, p_r\})$

- Let's first compute that $\pi_1(D \setminus \{p_1, \dots, p_r\})$ since this is necessary for the computation. We let U contain $r-1$ punctures and V contain one punctures as in Figure 1. Since $U \cap V$ simply connected we have

$$\pi_1(D \setminus \{p_1, \dots, p_r\}) \cong \pi_1(D \setminus \{p_1, \dots, p_{r-1}\}) * \pi_1(D \setminus \{p_r\})$$

Using this formula we get

$$\pi_1(D \setminus \{p_1, p_2\}) \cong \pi_1(D \setminus \{p_2\}) * \pi_1(D \setminus \{p_1\}) \cong \mathbf{Z} * \mathbf{Z} = F_2$$
$$\pi_1(D \setminus \{p_1, p_2, p_3\}) \cong \pi_1(D \setminus \{p_1, p_2\}) * \pi_1(D \setminus \{p_3\}) \cong \mathbf{Z} * \mathbf{Z} * \mathbf{Z} = F_3$$
$$\vdots$$
$$\pi_1(D \setminus \{p_1, p_2, \dots, p_r\}) \cong \underbrace{\mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z}}_{r\text{-times}} = F_r$$

- One can use the homotopy equivalence: $\mathbf{P}^1 \setminus \{p_1, \dots, p_r\} \sim D \setminus \{p_1, \dots, p_{r-1}\}$ which proves

$$\pi_1(\mathbf{P}^1 \setminus \{p_1, \dots, p_r\}) \cong \underbrace{\mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z}}_{(r-1)\text{-times}} = F_{r-1}$$

- Alternatively, we could that $\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ as in Figure 2. Here we have $U \cong D \setminus \{p_1, \dots, p_{r-1}\}$ and $V \cong D \setminus \{p_r\}$ and $U \cap V \cong S^1$. We then have

$$\langle \beta_1, \dots, \beta_{r-1}, \alpha : \beta_1 \cdots \beta_r = \gamma_1 = \alpha \rangle \cong \langle \beta_1, \dots, \beta_{r-1} \rangle \cong F_{r-1}$$

10. Verify that $F(z) = \int_0^\infty t^{z-1} e^{-t} dt$ and $\Gamma(z)$ (via $1/\Gamma(z)$ being defined by the product formula) satify the hypotheses of Weilandt's Theorem. In particular that $F(z)$ and $\Gamma(z)$ are bounded when $1 < \operatorname{Re} z < 2$.

Solution The solution is worked out in McMullen. Also, it is worked out in Stein and Shakarchi.

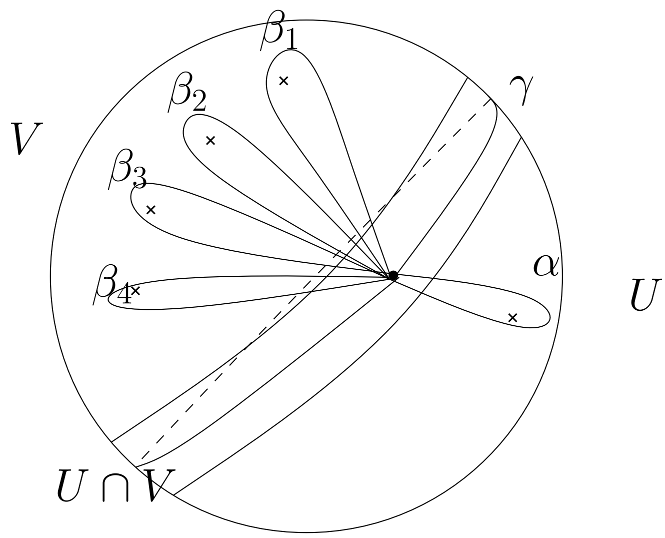

Figure 2: An open covering of $\mathbf{P}^1 - (5 \text{ points})$.