Dupuy — Complex Analysis — Spring 2017 — Homework 07

1. Show the Gauss formula for the Gamma function:
$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)\cdots(z+n)}.$$
(Take the definition of the Gamma function to be from its product formula).

2. Verify that $F(z) = \int_0^\infty t^{z-1}e^{-t}dt$ and $\Gamma(z)$ (via $1/\Gamma(z)$ being defined by the product formula) satify the hypotheses of Weilandt's Theorem. In particular that $F(z)$ and $\Gamma(z)$ are bounded when $1 < \operatorname{Re} z < 2$.

3. Show that $\int_0^{2\pi} \log|1-e^{i\theta}|d\theta = 0$.

4. (New Mexico, Jan 2006) Consider $f(z) = \prod_{n=1}^\infty (1 - z/n^3)$. What is the order of $f(z)$?

5. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an entire function of finite order ρ. Show that
$$\rho = \liminf_{n\to\infty} \frac{\log(n)}{\log|a_n|^{-1/n}}.$$

6. (a) Prove the Castorati-Weiestrass Theorem: Let $f(z)$ is analytic in a punctured disc of radius R at the origin. If $f(z)$ has an essential singularity at $z=0$ show that for every r with $0<r<R$ the set $f(D_r(0)\setminus\{0\})$ is dense in $\mathbf{C}$. (This is a corollary of Big Picard).
   (b) Let p be a polynomial. Show that there exists infinitely many z_j such that $p(z_j) = e^{z_j}$.

7. The following exercise is intended to introduce you to the j function which plays a role in the proof of the Big Picard Theorem from class.
   Let H be the upper-half plane. A **modular form** of weight k and level $N=1$ is a function $f : H \to \mathbf{C}$ such that
$$f(\frac{az+b}{cz+d}) = (cz+d)^{-2k} f(z). \tag{1}$$
   for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.
   (a) Let M_k denote the collection of modular forms of weight k and level 1. Show that $M = \bigoplus_{k\geq 0} M_k$ is a graded ring (i.e. that $M_{k_1}M_{k_2} \subset M_{k_1+k_2}$.
   (b) Show that $G_{2k}(\frac{az+b}{cz+d}) = (cz+d)^{2k}G_{2k}(z)$ has weight $2k$ (Hint: check this on the generators of $\mathrm{SL}_2(\mathbf{Z})$.)
   Using the first part conclude that the we have the following modular forms of the indicated weights:
      i. $g_2(\tau) = 60G_4(\tau)$, $k=4$
      ii. $g_3(\tau) = 140G_6(\tau)$, $k=6$
      iii. $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, $k=12$
      iv. $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$, $k=0$

8. Explain in words the ideas that go into the proof of Montel's Theorem in Green and Krantz (page 193). How is Arzela-Ascoli used?

9. Let $X = \mathbf{C}^\times = \mathbf{C}\setminus\{0\}$. What is the universal cover of X? What is group of deck transformations for this cover?

10. Use Van Kampen's theorem to rigorously compute $\pi_1(\mathbf{P}^1\setminus\{p_1,\dots,p_r\}, z_0)$ for arbitrary r. (Hint: apply Van Kampen to open sets U, V where $U\cap V$ is simply connected).