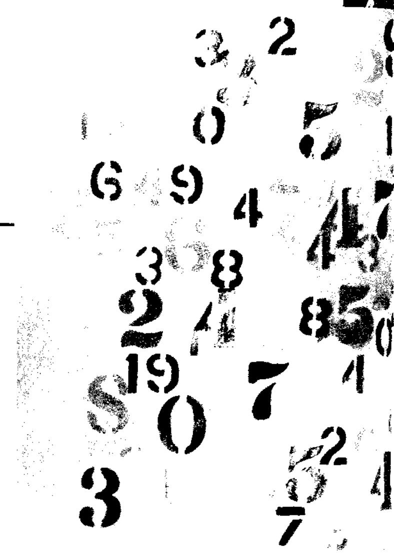
Chapter 3



The Subtraction Principle

There are 26 letters in the English alphabet, divided into vowels and consonants. For the purposes of this discussion, we'll say the letters

A, E, I, O and U

are vowels, and the letters

B, C, D, F, G, H, J, K, L, M, N, P, Q, R, S, T, V, W, X, Y and Z.

are consonants. Now, quickly: *how many consonants are there?*

How many of you counted out the letters in the sequence B, C, D, F, ...? Probably not many: it's just a lot easier to count the vowels, and subtract the number 5 of vowels from total number 26 of letters to arrive at the answer that there are $26 - 5 = 21$ consonants.

And that's all there is to the subtraction principle, which is the second of the basic counting tools we'll be using, after the multiplication principle. It's not at all deep—it amounts to nothing more than an observation, really—but we'll give it a box anyway:

> The number of objects in a collection that satisfy some condition is equal to the total number of objects in the collection minus the number of those that don't.

The point being, it's often easier to count the latter than the former. It hardly warrants a box of its own, but—in conjunction with the multiplication principle—it gives us a number of different ways of approaching a lot of counting problems. In fact, as we'll see in this chapter and the next, it greatly broadens the scope of the problems we can solve.

We start with a few simple examples ...

3.1 Back to the Video Store

Once more, you're going to have a triple feature in your room: one action film, one lighthearted romantic comedy and one movie based on a cartoon or video game. The House of Videos, again, has in stock a thousand copies each of seven

action movies (five of which feature car chases), five lighthearted romantic comedies (two of which feature car chases) and 23 movies based on cartoons or video games (eight of which feature car chases). But there's one restriction: some of your roommates have informed you that if you return with three movies featuring car chases they're officially kicking you out of the room. Now how many triple bills are possible?

Well, we could try to do this with the multiplication principle, as we did before the anti-car chase faction in your room raised its voice. But it's easy to see this isn't going to work. We can pick the action movie freely, of course; we have seven choices there. And we can pick the lighthearted romantic comedy freely as well; that's a free choice among five movies. But when it comes time to pick the last movie, how many choices we have depends on what our choices up to that point have been: if either of the first two movies is without car chases, we can choose the third movie freely among the 23 movies based on cartoons or video games; but if both of our first two choices do feature car chases, the choice of the third movie is limited to those 15 that don't. Changing the order of selection doesn't help, either: any way we work it, the number of choices available to us for the last movie depends on our first two selections.

So what do we do? It's simple enough. We already know how many total choices we'd have if there were no restrictions: as we worked it out, it's just

$$7 \times 5 \times 23 = 805.$$

At the same time, it's easy enough to count the number of triple bills that are excluded if we want to stay in the room: we can choose any of the five action movies featuring car chases, either of the two lighthearted romantic comedies featuring car chases, and any of the eight movies based on cartoons or video games featuring car chases, for a total of

$$5 \times 2 \times 8 = 80$$

disallowed triple features. The number of allowable choices is thus

$$805 - 80 = 725.$$

Here's a similar problem (some might say the same problem). We've already counted the number of four-letter words, by which we mean arbitrary sequences of four of the 26 characters in the English alphabet. Suppose we ask now, *how many such words have at least one vowel*? (Here we'll stick to the convention that "Y" isn't a vowel.)

As in the last problem, the multiplication principle seems to work fine until we get to the last letter, and then it breaks down. We have 26 choices for the first letter, 26 for the second and 26 for the third. But when it comes to choosing the last letter, we don't know how many choices we'll have: if any of the preceding three choices happened to be a vowel, we are now free to choose any letter for the last one in our word; but if none of the first three was a vowel we can only choose among the five vowels for the last.

Instead, we use the subtraction principle: we know how many words there are altogether, and *we'll subtract from that the number of words consisting entirely of consonants*. Both are easy: the number of all possible words is just 26^4, and the number of four-letter words consisting only of consonants is 21^4, so the answer to our problem is

$$26^4 - 21^4 = 456{,}976 - 194{,}481 = 262{,}495.$$

One more example: in the first section, we saw how to answer questions like, "How many numbers are there between 34 and 78?" and "How many numbers between 34 and 78 are divisible by 5?" Well, suppose now someone asks, "How many numbers between 34 and 78 are *not* divisible by 5?"

It's pretty clear this is a case for the subtraction principle. We know the number of numbers between 34 and 78 is

$$78 - 34 + 1 = 45.$$

Moreover, since the first and last numbers between 34 and 78 that are divisible by 5 are $35 = 7 \times 5$ and $75 = 15 \times 5$, the number of numbers in this range divisible by 5 is the number of numbers between 7 and 15; that is,

$$15 - 7 + 1 = 9.$$

So by the subtraction principle, the number of numbers between 34 and 78 that are not divisible by 5 is $45 - 9$, or 36.

3.2 Some More Problems

With the subtraction principle, we've doubled the number of techniques we can apply to counting problems. One downside to having more than one technique, though, is that it's no longer unambiguous how to go about solving a problem: we may need to use one technique, or the other, or a combination. This is the beginning of the art of counting, and to develop our technique we'll work out a few more examples.

For a start, let's go back to that first-grade classroom we were at before in Example 2.4.1, the one with 15 kids that need to be lined up. This time, though, let's make the problem a little more difficult: let's suppose that two of the kids in the class, Bobby and Jason, are truly obnoxious little brats. Either one individually is unruly to the point of psychopathy; the last thing in the world you'd want is the two of them standing next to each other in line. So, the problem we're going to deal with is,

EXAMPLE 3.2.1 How many ways are there of lining up the class so Bobby and Jason are *not* next to each other?

SOLUTION Well, we can certainly apply the subtraction principle here: we know there are 15! ways there are of ordering the class if we pose no restrictions; so if we can figure out how many ways there are of lining them up so that Bobby and Jason *are* next to each other, we can subtract that from the total and get the answer that way.

So, how do we figure out the number of lineups with Bobby and Jason adjacent? It seems we haven't exactly solved the problem yet: the next thing we see is that the multiplication principle isn't going to work here, at least not as we applied it in Example 2.4.1. We can certainly choose any of the 15 students to occupy the first place in line, but then the number of choices for the second place in line depends on whether the first choice was Bobby or Jason, or one of the other 13 kids. What's more, this ambiguity persists at every stage thereafter: whom we can put in each place in line depends on who we put in the preceding spot.

But there are other ways of applying the multiplication principle in this setting. In the solution we gave to Example 2.4.1, we made our choices one place at a time—that is, we chose one of the 15 kids to occupy the first place in line; then we chose one of the remaining 14 kids to occupy the second place, and so on. But we could have done it the other way around: we could have taken the kids one at a time, and assigned each a place in line. For example, we could start with Bobby, and assign him any of the 15 places in line; then go on to Jason and assign him any of the remaining 14 places in line, and so on through all 15 kids.

As long as we're dealing with the version of the problem given in Example 2.4.1, it doesn't matter which approach we take; both lead us to the answer $15 \cdot 14 \cdot 13 \ldots 3 \cdot 2 \cdot 1 = 15!$. But in the current situation—where we're trying to count the number of lineups with Bobby and Jason adjacent, say—it does make a difference.

At first it may not seem like it. Doing it this way, we can assign Bobby to any of the 15 places in line; but then the number of choices we have for Jason depends on where we assigned Bobby: if Bobby was placed in either the first or the fifteenth place in line, we will have no choice but to place Jason in the second or fourteenth, respectively; but if Bobby was placed in any of the interior slots, then we can choose to place Jason either immediately ahead of him or immediately behind him. So it seems that the multiplication principle doesn't work this way, either.

But there is a difference. Approaching the problem this way—taking the students one at a time, and assigning each in turn one of the remaining places in line—we see that once we've got Bobby and Jason assigned their places, the multiplication principle takes over: there are 13 choices for where to place the next kid, 12 choices of where to place the one after that, and so on. In other words, if we break the problem up into first assigning Bobby and Jason their places, and then assigning the remaining 13 kids theirs, we see that

$$\left\{\begin{array}{l}\text{the number of lineups}\\ \text{of the class with Bobby}\\ \text{and Jason adjacent}\end{array}\right\} = \left\{\begin{array}{l}\text{the number of ways of}\\ \text{assigning Bobby and Jason}\\ \text{adjacent places in line}\end{array}\right\} \times 13!$$

It remains to count the number of ways of assigning Bobby and Jason adjacent places in line. This is not hard: as we saw above, there are two ways of doing this with Bobby occupying an end position, and $13 \times 2 = 26$ ways of doing it with Bobby occupying an interior position (second through fourteenth), for a total of 28 ways. Or we could count this way: to specify adjacent places in line for Bobby and Jason, we could first specify the *pair* of positions they're to occupy—first and second, or second and third, and so on up to fourteenth and fifteenth—and then say which of the pair Bobby's to occupy. For the first, there are 14 choices, and for the latter 2 choices, so by the multiplication principle we see again there are 28 ways of assigning Bobby and Jason adjacent places in line.

In conclusion, we see that

$$\left\{\begin{array}{l}\text{the number of lineups}\\ \text{of the class with Bobby}\\ \text{and Jason adjacent}\end{array}\right\} = 28 \times 13!$$

and correspondingly,

$$\left\{\begin{array}{l}\text{the number of lineups} \\ \text{of the class with Bobby} \\ \text{and Jason apart}\end{array}\right\} = 15! - (28 \times 13!)$$

$$= 1{,}133{,}317{,}785{,}600. \quad \blacksquare$$

Exercise 3.2.2 Do Example 3.2.1 over, using the approach followed above but without the subtraction principle: that is, count the number of lineups of the class with Bobby and Jason apart by counting the number of ways you can assign Bobby and Jason to two *nonadjacent* places in line, and the number of ways you can assign the remaining 13 students to the remaining 13 places. Does your answer agree with the one above?

Before we go on, we want to emphasize one point that is illustrated by Example 3.2.1 and its solution. It's an important aspect of learning and doing mathematics, and the failure to appreciate it is the cause of a lot of the frustration that everyone experiences in reading math books. Simply put, it's this: *formulas don't work*. At least, they don't usually work in the sense that you can just plug in appropriate numbers, turn the crank and arrive at an answer. It's better to think of formulas as guides, suggesting effective ways of thinking about problems.

That's probably not what you wanted to hear. When it's late at night and your math homework is the only thing standing between you and bed, you don't want to embark on a glorious journey of exploration and discovery. You just want someone to tell you what to do to get the answer, and formulas may appear to do exactly that. But, really, that's not what they're there for, and appreciating that fact will spare you a lot of aggravation.

Now you try it.

Exercise 3.2.3 A new-style Massachusetts license plate has two letters (which can be any letter from A to Z) followed by four numbers (which can be any digits from 0 to 9).

1. How many new-style Massachusetts license plates are there?
2. How many new-style Massachusetts license plates are there if we require no repeated letters and no repeated numbers?
3. How many new-style Massachusetts license plates are there that have at least one 7?

Exercise 3.2.4 Let's assume that a phone number has seven digits, and cannot start with a 0.

1. How many possible phone numbers are there?
2. How many phone numbers are there with at least one even digit?

Exercise 3.2.5 Getting dressed: suppose you own eight shirts, five pairs of pants and three pairs of shoes.

1. Assuming you have no fashion sense whatsoever, how many outfits can you make?

2. Suppose now that one shirt is purple, one pair of pants is red, and you can make any combination *except* ones including the red pants and the purple shirt. How many outfits can you make?
3. Now suppose that any time you wear the purple shirt you *must* also wear the red pants. How many outfits can you make?

The following problem is hard, but doesn't use any ideas that we haven't introduced.

Exercise 3.2.6 Let's go back to the problem of lining up our class of 15 students. Suppose that Bobby and Jason are so wired that for the sake of everyone's sanity we feel there should be at least two other kids between them. Now how many possible lineups are there?

3.3 Multiple Subtractions

Even as simple an idea as the subtraction principle sometimes has complications. In this section, we'll discuss some of what can happen when we have to exclude more than one class of object from a pool. As with the subtraction principle itself, the basic concept is more common sense than arithmetic, and to emphasize that point we'll start with an edible example.

Consider the following list of 17 vegetables:

artichokes
asparagus
beets
broccoli
cabbages
carrots
cauliflower
celery
corn
eggplant
lettuce
onions
peas
peppers
potatoes
spinach
zucchini

Of these, four—beets, carrots, onions and potatoes—are root vegetables. Two—corn and potatoes—are starchy. Now we ask the question: how many are neither root vegetables nor starchy?

Well, the obvious thing to do would be to subtract the number of root vegetables and starchy vegetables from the total, getting the answer

$$17 - 4 - 2 = 11.$$

But a moment's thought (or, for that matter, actual counting) shows you that isn't right: because a potato is both a root vegetable and a starchy one, you've subtracted it twice. The correct answer is accordingly 12.

And that's the point of this section. It amounts to the observation that when you want to exclude two classes of objects from a pool and count the number left, you can start with the total number of objects in the pool and subtract the number of objects in each of the two excluded categories; *but then you have to add back in the number of objects that belong to both classes and have therefore been subtracted twice.*

Here's a more mathematical example:

EXAMPLE 3.3.1 How many numbers between 100 and 1,000 are divisible by neither 2 nor 3?

SOLUTION We know that the number of numbers between 100 and 1,000 is simply

$$1{,}000 - 100 + 1 = 901.$$

Likewise, we can count the numbers in this range divisible by 2: these are just the even numbers between 100 and 1,000, or in other words twice the numbers between 50 and 500; so there are

$$500 - 50 + 1 = 451$$

of them. Similarly, the numbers divisible by 3 are just 3 times the numbers between 34 and 333; so there are

$$333 - 34 + 1 = 300$$

of those. So, naively, we want to subtract each of 451 and 300 from the total 901.

But, as you've probably figured out—we've stepped all over this punchline—that would be wrong. Because there are numbers divisible by both 2 and 3, and these will have been subtracted twice; to rectify the count we have to add them back in once.

Now, what numbers are divisible by both 2 and 3? The answer is that a number divisible by 2 and by 3 is necessarily divisible by 6, and vice versa.[1] So the numbers between 100 and 1,000 that are divisible by both 2 and 3 are just the numbers in that range divisible by 6, which is to say 6 times the numbers between 17 and 167. There are thus

$$167 - 17 + 1 = 151$$

of them, and so the correct answer to our problem will be

$$901 - 451 - 300 + 151 = 301. \quad \blacksquare$$

Here's one more involved example of the same idea. Again, we're keeping the convention that by a "word" we mean an arbitrary sequence of letters of the English alphabet.

EXAMPLE 3.3.2 How many four-letter words are there in which no letter appears three or more times in a row?

SOLUTION This clearly calls for the subtraction principle. We know how many four-letter words there are in all—the number is

$$26 \times 26 \times 26 \times 26 = 456{,}976.$$

[1] Is this clear? Think about it, but if you don't see why this is the case, relax; take our word for it now, and in Chapter 8 we'll work it out.

We just have to subtract the number of words in which a letter appears three or more times in a row.

Now, there are two kinds of four-letter words in which a letter appears three times in a row: those in which the first three letters are the same, and those where the last three letters are the same. In each case, the number of such words is easy to count by the multiplication principle. For example, to specify a word in which the first three letters are the same, we have to specify that letter (26 choices) and the last letter (26 choices again), so there are

$$26 \times 26 = 676$$

of this type. By the same token, there are 676 four-letter words in which the last three letters are the same; so naively we want to exclude $2 \times 676 = 1{,}352$ words.

But once more that's not quite right: the 26 words in which all four letters are the same belong to both classes, and so have been subtracted twice! So to correct the count, we have to add them back in three times. The correct answer is therefore

$$456{,}976 - 1{,}352 + 26 = 455{,}650. \quad \blacksquare$$

Actually, there's another way to do this, morally the same but avoiding the issue of multiple subtractions. We can count the number of words in which one letter appears *exactly* three times in a row, and the number of words in which one letter appears four times, add them up and subtract the total from the number of all four-letter words. For the first, there are again two classes of such words; but within each class the number is different: we choose the repeated letter among the 26 letters of the alphabet as before, but since that letter is to appear exactly three times the remaining letter must be chosen among the remaining 25 letters of the alphabet. There are thus a total of

$$2 \times 26 \times 25 = 1{,}300$$

such words. There are again 26 words in which one letter appears all four times; so the correct answer is

$$456{,}976 - 1{,}300 - 26 = 455{,}650$$

as before.

This last exercise represents another level of complexity in the subtraction principle, but you should be able to do it if you keep your wits about you.

Exercise 3.3.3 How many five-letter words are there in which no letter appears 3 or more times in a row?