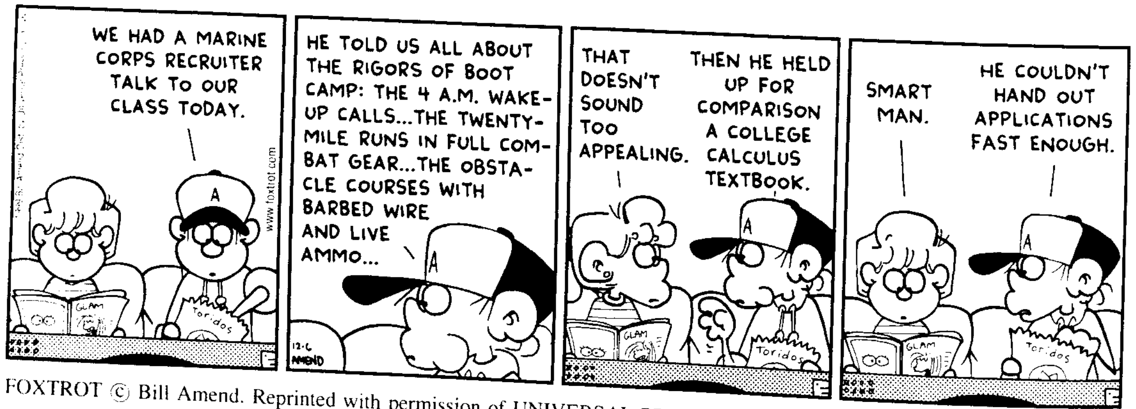

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Chapter 2

The Multiplication Principle

2.1 Choices

Let's suppose you climb out of bed one morning, still somewhat groggy from the night before. You grope your way to your closet, where you discover that your cache of clean clothes has been reduced to four shirts and three pairs of pants. It's far too early to exercise any aesthetic judgment whatsoever: any shirt will go with any pants; you only need something that will get you as far as the dining hall and that blessed, life-giving cup of coffee. The question is,

How many different outfits can you make out of your four shirts and three pairs of pants?

Admittedly the narrative took a sharp turn toward the bizarre with that last sentence. Why on earth would you or anyone care how many outfits you can make? Well, bear with us while we try to answer it anyway.

Actually, if you thought about the question at all, you probably have already figured out the answer: each of the four shirts is part of exactly three outfits, depending on which pants you choose to go with it, so the total number of possible outfits is $3 \times 4 = 12$. (Or, if you like to get dressed from the bottom up, each of the three pairs of pants is part of exactly four outfits; either way the answer is 3×4.) If we're feeling really fussy, we could make a table: say the four shirts are a golf shirt, an oxford, a tank top and a T-shirt extolling the virtues of your favorite athletic wear, and the pants consist of a pair of jeans, some cargo pants and a pair of shorts. Then we can arrange the outfits in a rectangle:

golf shirt w/ jeans	oxford shirt w/ jeans	tank top w/ jeans	T-shirt w/ jeans
golf shirt w/ cargo pants	oxford shirt w/ cargo pants	tank top w/ cargo pants	T-shirt w/ cargo pants
golf shirt w/ shorts	oxford shirt w/ shorts	tank top w/ shorts	T-shirt w/ shorts

Now, you know we're not going to stop here. Suppose next that, in addition to picking a shirt and a pair of pants, you also have to choose between two pairs of shoes. Now how many outfits are there?

Well, the idea is pretty much the same: for each of the possible shirt/pants combinations, there are two choices for the shoes, so the total number of outfits is $4 \times 3 \times 2 = 12 \times 2 = 24$. And if in addition we had a choice of five hats, the total number of possible outfits would be $4 \times 3 \times 2 \times 5 = 120$—you get the idea.

Now it's midday and you head over to the House of Pizza to order a pizza for lunch. You feel like having one meat topping and one vegetable topping on your pizza; the House of Pizza offers you seven meat toppings and four vegetable toppings. How many different pizzas do you have to choose among?

"That's the same problem with different numbers!" you might say, and you'd be right: to each of the seven meat toppings you could add any one of the four vegetable toppings, so the total number of different pizzas you could order would be 7×4, or 28.

Evening draws on, and your roommates send you out to the local House of Videos to rent some videos. You're going to have a triple feature in your room: one action film, one lighthearted romantic comedy and one movie based on a cartoon or video game. The House of Videos, following its corporate plan, has in stock a thousand copies each of seven action movies, five lighthearted romantic comedies and 23 movies based on cartoons or video games. How many triple features can you rent?

"That's the same problem again!" you might be thinking: the answer's just the number 7 of action movies times the number 5 of lighthearted romantic comedies times the number 23 of movies based on a cartoon or video game, or $7 \times 5 \times 23 = 805$. Trust us!—we are going somewhere with this. But you're right, it's time to state the general rule that we're working toward, which is called the *multiplication principle*:

The number of ways of making a sequence of independent choices is the product of the number of choices at each step.

Here "independent" means that how you make the first choice doesn't affect the number of choices you have for the second, and so on. In the first case above, for example—getting dressed in the morning—it corresponds to having no fashion sense whatsoever.

The multiplication principle is easy to understand and apply, but awkward to state in reasonably coherent English, which is why we went through three examples before announcing it. In fact, you may find the examples more instructive than the principle itself; if the boxed statement seems obscure to you, just remember: "4 shirts, 3 pants, 12 outfits."

2.2 More Examples

An old-style Massachusetts license plate has on it a sequence of three numbers followed by three letters. How many different old-style Massachusetts license plates can there be?

This is easy enough to answer: we have 10 choices for each of the numbers, and 26 choices for each of the letters; and since none of these choices is constrained in any way, the total number of possible license plates is

$$10 \times 10 \times 10 \times 26 \times 26 \times 26 = 17{,}576{,}000$$

A similar question is this. Suppose for the moment that by "word" we mean any sequence of the 26 letters of the English alphabet—we're not going to make a distinction between actual words and arbitrary sequences. How many three-letter words are there?

This is just the same as the license plate problem (or at least the first half): we have 26 independent choices for each of the letters, so the number of three-letter words is $26^3 = 17{,}576$. In general,

$$\begin{aligned}
\text{\# of 1-letter words} &= 26\\
\text{\# of 2-letter words} &= 26^2 = 676\\
\text{\# of 3-letter words} &= 26^3 = 17{,}576\\
\text{\# of 4-letter words} &= 26^4 = 456{,}976\\
\text{\# of 5-letter words} &= 26^5 = 11{,}881{,}376\\
\text{\# of 6-letter words} &= 26^6 = 308{,}915{,}776
\end{aligned}$$

and so on.

Next, let's suppose that there are 15 students in a class, and that they've decided to choose a set of class officers: a president, a vice president, a secretary and a treasurer. How many possible slates are there? That is, how many ways are there of choosing the four officers?

Actually, there are two versions of this question, depending on whether or not a single student is allowed to hold more than one of the positions. If we assume first that there's no restriction of how many positions one person can hold, the problem is identical to the ones we've just been looking at: we have 15 choices each for the four offices, and they are all independent, so that the total number of possible choices is

$$15 \times 15 \times 15 \times 15 = 50{,}625.$$

Now suppose on the other hand we impose the rule that no person can hold more than one office. How many ways are there of choosing officers?

Well, this can also be done by the multiplication principle. We start (say) by choosing the president; we have clearly 15 choices there. Next, we choose the vice president. Now our choice is restricted by the fact that our newly selected president is no longer eligible, so that we have to choose among the 14 remaining students. After that we choose a secretary, who could be anyone in the class except the two officers already chosen; so we have 13 choices here; and finally we choose a treasurer from among the 12 students in the class other than the president, vice president and secretary. Altogether, the number of choices is

$$15 \times 14 \times 13 \times 12 = 32{,}760.$$

Note one point here: in this example, the actual choice of, say, the vice president *does* depend on who we chose for president; the choice of a secretary does depend on who we selected for president and vice president, and so on. But the *number* of choices doesn't depend on our prior selections, so the multiplication principle still applies.

In a similar vein, we could modify the question we asked a moment ago about the number of three-letter words, and ask: how many 3 letter words have no repeated letters? The solution is completely analogous to the class-officer problem: we have 26 choices for the first letter, 25 for the second and 24 for the third, so that we have a total of

$$26 \times 25 \times 24 = 15{,}600$$

such words. In general, we can calculate

$$\text{\# of 1-letter words} = 26$$

$$\text{\# of 2-letter words w/o repeated letters} = 26 \cdot 25 = 650$$

$$\text{\# of 3-letter words w/o repeated letters} = 26 \cdot 25 \cdot 24 = 15{,}600$$

$$\text{\# of 4-letter words w/o repeated letters} = 26 \cdot 25 \cdot 24 \cdot 23 = 358{,}800$$

$$\text{\# of 5-letter words w/o repeated letters} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7{,}893{,}600$$

$$\text{\# of 6-letter words w/o repeated letters} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 165{,}765{,}600$$

and so on.

Now, here's an interesting (if somewhat tangential) question. Let's compare the numbers of words of each length to the number of words with no repeated letters. What percentage of all words have repetitions, and what percentage don't? Of course, as the length of the word increases, we'd expect a higher proportion of all words to have repeated letters—relatively few words of two or three letters have repetitions, while of course every word of 27 or more letters does. We could ask, then: when does the fraction of words without repeated letters dip below one-half? In other words, for what lengths do the words with repeated letters outnumber those without?

Before we tabulate the data and give the answer, you might want to take a few minutes and think about the question. What would your guess be?

STOP.

CLOSE THE BOOK.

GRAB A PAD OF PAPER AND A PEN.

WORK OUT SOME EXAMPLES ON YOUR OWN.

THINK.

length	number of words	without repeats	% without repeats
1	26	26	100.00
2	676	650	96.15
3	17,576	15,600	88.76
4	456,976	358,800	78.52
5	11,881,376	7,893,600	66.44
6	308,915,776	165,765,600	53.66
7	8,031,810,176	3,315,312,000	41.28
8	208,827,064,576	62,990,928,000	30.16
9	5,429,503,678,976	1,133,836,704,000	20.88

Now that's bound to be surprising: among six-letter words, those with repeated letters represent nearly half, and among seven-letter words they already substantially outnumber the words without repeats. In general, the percentage of words without repeated letters drops off pretty fast: by the time we get to twelve-letter words, fewer than 1 in 20 has no repeated letter. We'll see another example of this phenomenon when we talk about the birthday problem in Section 5.6.

2.3 Two Formulas

There are two special cases of the multiplication principle that occur so commonly in counting problems that they're worth mentioning on their own, and we'll do that here. Neither will be new to us; we've already encountered examples of each.

Both involve repeated selections from a single pool of objects. If there are no restrictions at all on the choices, the application of the multiplication principle is particularly simple: each choice in the sequence is a choice among all the objects in the collection. If we're counting three-letter words in an alphabet of 26 characters, for example—where by "word" we again mean an arbitrary sequence of letters—there are 26^3; if we're counting four-letter words in an alphabet of 22 characters, there are 22^4; and so on. In general, we have the following rule:

> The number of sequences of k objects chosen from a collection of n objects is n^k.

The second special case involves the same problem, but with a commonly applied restriction: we're again looking at sequences of objects chosen from a common pool of objects, but this time we're not allowed to choose the same object twice. Thus, the first choice is among all the objects in the pool; the second choice is among all but one, the third among all but two, and so on; if we're looking at a sequence of k choices, the last choice will be among all but the $k-1$ already chosen. Thus, as we saw, the number of three-letter words without repeated letters in an alphabet of 26 characters is $26 \cdot 25 \cdot 24$; the number of four-letter words without repeated letters in an alphabet of 22 characters is $22 \cdot 21 \cdot 20 \cdot 19$; and so on. In general, if the number of objects in our pool is n, the first choice will be among all n; the second among $n-1$, and so on. If we're making a total of k choices, the last choice will exclude the $k-1$ already chosen; that is, it'll be a choice among the $n-(k-1) = n-k+1$

objects remaining. The total number of such sequences is thus the product of the numbers from n down to $n - k + 1$. We write this as

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1),$$

where the dots in the middle indicate that you're supposed to keep going multiplying all the whole numbers in the series starting with n, $n - 1$ and $n - 2$, until you get down to $n - k + 1$. Time for a box:

> The number of sequences of k objects chosen without repetition from a collection of n objects is
>
> $$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1).$$

Exercise 2.3.1 In one of the Massachusetts state lotteries, the winning number is chosen by picking six ping-pong balls from a bin containing balls labeled "1" through "36" to arrive at a sequence of six numbers between 1 and 36. Ping-pong balls are not replaced after they're chosen; that is, no number can appear twice in the sequence. How many possible outcomes are there?

Note that in this last exercise, the order in which the ping-pong balls are chosen is relevant: if the winning sequence is "17-32-5-19-12-27" and you picked "32-17-5-19-12-27," you *don't* get to go to work the next day and tell your boss what you really think of her.

Exercise 2.3.2 The Hebrew alphabet has 22 letters. How many five-letter words are possible in Hebrew? (Again, by "word" we mean just an arbitrary sequence of five characters from the Hebrew alphabet.) What fraction of these have no repeated letters?

2.4 Factorials

The two formulas we described in the last section were both special cases of the multiplication principle. There is in turn a special case of the second formula that crops up fairly often and that's worth talking about now. We'll start, as usual, with an example.

EXAMPLE 2.4.1 Suppose that we have a first-grade class of 15 students, and we want to line them up to go out to recess. How many ways of lining them up are there—that is, in how many different orders can they be lined up?

SOLUTION Well, think of it this way: we have 15 choices of who'll be first in line. Once we've chosen the line leader, we have 14 choices for who's going to be second, 13 choices for the third, and so on. In fact, all we're doing here is choosing a sequence of 15 children from among the 15 children in the class, without repetition; whether we invoke the formula in the last section or do it directly, the answer is

$$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1{,}307{,}674{,}368{,}000,$$

or about 1.3×10^{12}—more than a trillion orderings. ■

In general, if we ask how many ways there are of placing n objects in a sequence, the answer is the product of all the whole numbers between 1 and n. This is a quantity that occurs so often in mathematics (and especially in counting problems) that it has its own symbol and name:

> The product $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ of the numbers from 1 to n is written $n!$ and called "n factorial."

Here's a table of the factorials up to 15:

n	n!
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880
10	3,628,800
11	39,916,800
12	479,001,600
13	6,227,020,800
14	87,178,291,200
15	1,307,674,368,000

There are many fascinating things to be said about these numbers. Their size alone is an interesting question: we've seen that 15 factorial is over a trillion; approximately how large a number is, say, 100 factorial? But we'll leave these questions aside for now. At this point, we'll be using factorials for the most part just as a way of simplifying notation. We'll start with the last formula of the preceding section.

It's pretty obvious that writing $15!$ is a whole lot easier than writing out the product $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. But there are other, less obvious uses of the notation. Suppose, for example, that we wanted to make up a baseball team out of the 15 kids in the class—that is, choose a sequence of nine of the kids in the class of 15, without repetition. We'd have 15 choices for the pitcher, 14 for the catcher, 13 for the first baseman, and so on. When you choose the ninth and last player, you'll be choosing among the $15 - 8 = 7$ kids left at that point, so that the total number of teams would be

$$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7.$$

But there's a faster way to write this number, using factorials. Basically, we could think of this product as the product of all the numbers from 15 down to 1, except we leave off the numbers from 6 down to 1—in other words, the product of the numbers from 15 to 1 divided by the product of the numbers from 6 to 1, or

$$\frac{15!}{6!}.$$

Now, this may seem like a strange way of writing out the product: it seems inefficient to multiply all the numbers from 15 to 1 and then divide by the product of the numbers you didn't want in the first place. And it is—no one in his right mind would calculate out the number that way. But just as notation, "$\frac{15!}{6!}$" takes up a whole lot less space than "$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$," and we'll go with it. For example, we'll rewrite the boxed formula from the last section:

The number of sequences of k objects chosen without repetition from a collection of n objects is $n!/(n-k)!$

One final note about factorial notation: it is the standard convention that $0! = 1$. You could think of this as the answer to a Zen koan: "How many ways are there of ordering no objects?" But we'll ignore the philosophical ramifications here and simply accept it as a notational convention: it just makes the formulas come out better, as we'll see.

2.5 Another Wrinkle

The multiplication principle itself is completely straightforward. But sometimes there may be more than one way to apply it; and sometimes one of those ways will work when another doesn't. We have to be prepared, in other words, to be flexible in applying the multiplication principle. We'll see lots of examples of this over the course of this part of the book; here's one of them.

To start with, let's take a simple problem: how many three-digit numbers can you form using the numbers 1 through 9, with no repeated digit? As we've seen already, this is completely straightforward: we have nine choices for the first digit, then eight choices for the second and finally seven choices for the third, for a total of

$$9 \times 8 \times 7 = 504$$

choices.

Now let's change the problem a bit: suppose we ask, "How many of those 504 numbers are odd?" In other words, how many have as their third digit a 1, 3, 5, 7 or 9?

We can try to do it the same way: as before, there are nine choices for the first digit and eight for the second. But when we get to the third digit, we're stuck. For example, if the first two digits we selected were 2 and 4, then the third digit could be any of the numbers 1, 3, 5, 7 or 9, so we have five choices. If the first two digits were 5 and 7, however, the third digit could only be a 1, 3 or 9; we have only three choices. The choices, in other words, don't seem to be independent.

But they are if we make them in a different order! Suppose that rather than choosing the first digit first and so on, we go from right to left instead—in other words, choose the third digit first, then the middle and finally the first. Now we can choose the last digit freely among the numbers 1, 3, 5, 7 and 9, for a total of five

choices. The choice of the middle digit is constrained only by the requirement that it not repeat the one we've already chosen; so there are eight choices for it, and likewise seven choices for the first digit. There are thus

$$5 \times 8 \times 7 = 280$$

such numbers.

Sometimes we find ourselves in situations where the multiplication principle may not seem applicable, but in fact its application is completely straightforward as long as we keep our wits about us. Here's an example:

EXAMPLE 2.5.1 Suppose that in the class we were discussing in Example 2.4.1 there are eight boys and seven girls, and we want to line them up so that no two boys are next to each other. How many ways are there of doing this?

SOLUTION Actually, before we go and give the solution, let's take a moment and see that the multiplication principle fails. In fact, if we try to use the same approach as we took to in solving Example 2.4.1, it screws up already at the second step. That is, we seemingly have as before 15 choices of who's to be first in line. But the numbers of possible choices for who goes second depends on our first choice: if we chose a girl to be first, there are no restrictions on who goes second, and there are 14 choices; but if we chose a boy to be first in line, the second in line must be chosen from among the seven girls.

We need, in other words, a different approach. But here we're in luck: if we think about it, we can see that since there are 8 boys out of 15 kids, and no two boys are to be next to each other in line, the line must alternate boy/girl/boy/girl until the final place, which must be a boy. In other words, the odd-numbered places in line must all be occupied by boys, and the even places by girls.

Thus, to choose an ordering of the whole class subject to the constraint that no two boys are next to each other, we have to choose a first boy, a second boy, and so on until we get to the eighth and last boy; and likewise we have to choose a first girl, a second girl, and so on to the seventh girl. Put another way, we simply have to order the boys and the girls separately. We know that there are 8! ways of ordering the boys and 7! ways of ordering the girls, so the multiplication principle tells us that the total number of ways of lining up the class is

$$8! \cdot 7! = 203{,}212{,}800.$$

■

EXAMPLE 2.5.2 One last puzzler: suppose that there were six boys and nine girls, and again we wanted to line up the class so that no two boys are next to each other. How many ways would there be of doing this?

In fact, this is a *much* harder problem, because we can't avail ourselves of the trick we used in the last example. But it is one you'll learn how to do. So think for a while about how you might try to approach it, and in Section 4.3 we promise we'll work it out.