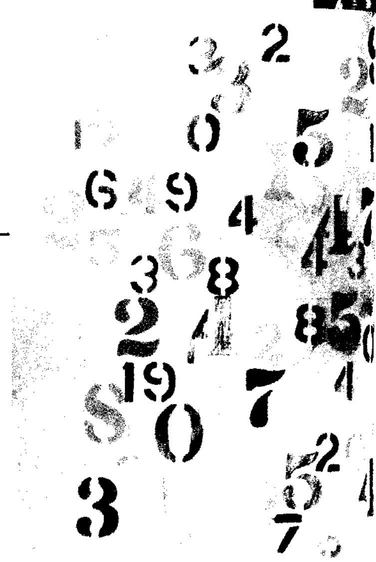
Chapter 5



Probability

In this chapter we're going to use our counting skills to discuss some basic problems in probability. We'll focus primarily on games—flipping coins, rolling dice, and playing poker and bridge—but it should be clear how the same ideas can be applied in other areas as well.

5.1 Flipping Coins

Suppose we flip a coin six times. What's the probability of getting three heads and three tails?

To answer this, we have to start with two hypotheses. The first is simply that we have a fair coin—that is, one that on average will come up heads half the time and tails half the time.

To express the second hypothesis, we have to introduce one bit of terminology. By the *outcome* of the process of flipping the coin six times we mean the sequence of six results, which we can think of as a six-letter word consisting of Hs and Ts. How many possible outcomes are there? That's easy: by the very first formula we worked out using the multiplication principle, the number of such sequences is 2^6, or 64.

Now, it is a fundamental hypothesis of probability that *all 64 outcomes are equally likely*. In effect, this means simply that the result of each coin flip is equally likely to be heads or tails, irrespective of what the result of the previous flips might have been—in English, "the coin has no memory." We should emphasize here that this is really a hypothesis: even though we're all brought up nowadays to see this as self-evident, and it's been verified extensively by experiment, it's not something we can logically prove. Indeed, there were long periods of human history when just the opposite was thought to be true: when people (and not just degenerate gamblers) believed, for example, that after a long run of heads a tail was more likely than another head.

So: let's adopt these hypotheses. What they mean is that any specific outcome—three heads followed by three tails (HHHTTT), or three tails followed by three heads (TTTHHH), or whatever—will occur $\frac{1}{64}$ of the time; in other words, the probability is 1 in 64 that any specified outcome will occur on a given experiment of six flips. Given this, to determine the likelihood of getting exactly three heads and three tails

on a given six flips, we have to answer a counting problem: of the 64 possible outcomes, how many include three Hs and three Ts?

This is also an easy problem: the number of six-letter words consisting of three Hs and three Ts is just the binomial coefficient

$$\binom{6}{3} = 20.$$

Now, if each of these outcomes occurs $\frac{1}{64}$ of the time, then in the aggregate we would expect one of these 20 outcomes to occur

$$\frac{20}{64} = \frac{5}{16} = .3125$$

of the time. In other words, when we flip six coins, we expect to get an equal number of heads and tails a little less than one-third of the time.

You can probably see the general rule here: if we flip a coin n times, there are 2^n possible outcomes—corresponding to the n-letter words consisting entirely of Hs and Ts—each of which will on average occur one in 2^n times. The number of these outcomes that involve exactly k heads and $n-k$ tails is $\binom{n}{k}$; and so our conclusion is that

The probability of getting exactly k heads in n flips is $\dfrac{\binom{n}{k}}{2^n}$.

We'll look further into other probabilities associated with flipping coins, but before we do we should take a moment to point out that this is the basic paradigm of probability: in general, when all possible outcomes of an experiment or process are equally likely, and we separate out the collection of outcomes into two kinds, favorable and unfavorable, the probability of a favorable outcome is simply

$$\text{probability of a favorable outcome} = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}.$$

Note again that this presupposes that all outcomes are equally likely. If that's not the case, or if we define "outcome" incorrectly, all bets are off, so to speak.

Here are some more examples. As with all probability problems, it's fun to think about them a little and try to estimate the odds before you actually go ahead and calculate them: sometimes they can surprise you (and you can come up with some lucrative bets).

EXAMPLE 5.1.1 Let's say you flip a coin eight times. What is the probability of getting three or more heads?

SOLUTION We have to figure out, of the $2^8 = 256$ eight-letter words consisting entirely of Hs and Ts, how many have at least three Hs. It's slightly easier to figure out how many don't, and use the subtraction principle: we have to count the number of such words that have zero, one or two Hs, and by what we've done the number is

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 1 + 8 + 28 = 37.$$

The number of such sequences that do have three or more heads is thus $256 - 37 = 219$, and so the probability of getting at least three heads is

$$\frac{219}{256} \sim .85.$$

In other words, you'll get three or more heads in eight flips about 85% of the time. ■

EXAMPLE 5.1.2 Say you and a friend are gambling. You flip nine coins; if they split 4/5—that is, if they come up either four heads and five tails or four tails and five heads—you pay him \$1; otherwise, he pays you \$1. Who has the better odds?

SOLUTION There are $2^9 = 512$ possible outcomes of the nine coin flips, of which

$$\binom{9}{4} + \binom{9}{5} = 126 + 126 = 252$$

involve either four or five heads. That leaves

$$512 - 252 = 260$$

outcomes that don't. Thus the odds are (very slightly) in your favor. ■

EXAMPLE 5.1.3 A variant of the last problem. You and your friend flip six coins; if three or more come up heads you pay him \$1; if two or fewer are heads, he pays you \$2. Who has the better odds?

SOLUTION This is more complicated than the last problem only in that the payoffs are different. We start the same way: figuring out, of the $2^6 = 64$ possible outcomes of the six coin flips, how many result in a win for you and how many in a win for your friend. First, the number of outcomes with fewer than three heads—that is, with 0, 1 or 2 heads—is the sum

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} = 1 + 6 + 15 = 22.$$

On these 22 outcomes, you win \$2. That leaves

$$64 - 22 = 42$$

outcomes where your friend wins \$1. You'll lose \$1, in other words, slightly less than twice as often as you win \$2; so once again the odds are slightly in your favor. ■

Exercise 5.1.4 You and a friend play the following game. You each flip three coins, and whoever gets more heads wins; if you get the same number, you win. If you win, your friend pays you \$1; if your friend wins, you pay her \$2. Who has the better odds?

Exercise 5.1.5 Say you flip a coin five times. What's the probability that some three in a row will come up the same?

5.2 Tumbling Dice

As far as mathematics goes, dice are not that different from coins; this section won't introduce any new ideas. But because dice have six faces rather than two, and you can do things like add the results of several dice rolls, they're more interesting. (Las Vegas casinos have tables for playing craps; they don't have coin-flipping tables.)

Let's start by rolling two dice and calculating some simple odds. Again, the hypotheses: first, the dice are fair; in other words, each of the six faces will on

average come up one-sixth of the time. Second, if we define the outcome of the roll to be a sequence of two numbers between 1 and 6, the $6 \times 6 = 36$ possible outcomes are all equally likely, that is, each occurs $\frac{1}{36}$ of the time.

We should stop for a moment here and try to clarify one potential misunderstanding. Most of the time, when we roll a pair of dice, the two dice are indistinguishable and we don't think of one as "the first die" and the other as "the second die." *But for the purposes of calculating odds, we should.* For example, there are two ways of rolling "a 3 and a 4": the first die could come up 3 and the second 4, or vice versa; "a 3 and a 4" thus comes up $\frac{2}{36}$, or $\frac{1}{18}$, of the time. By contrast, "two 3s" arises in only one way, and so occurs only $\frac{1}{36}$ of the time. This can be confusing, and even counter-intuitive: when we roll two identical dice, we may not even know whether we've rolled "a 3 and a 4" or "a 4 and a 3." It may help to think of the dice as having different colors—one red and one blue, say—or of rolling them one at a time, rather than together.

With this said, let's calculate some odds. To begin with, let's say we roll two dice and add the numbers showing. We could ask, for example: what's the probability of rolling a 7?

To answer that, we simply have to figure out, of the 36 possible outcomes of the roll, how many yield a sum of 7? This we can figure out by hand: we could get a 1 and a 6, a 2 and a 5, a 3 and a 4, a 4 and a 3, a 5 and a 2 or a 6 and a 1, for a total of six outcomes. The probability of rolling a 7, accordingly, is $\frac{6}{36}$, or $\frac{1}{6}$.

By contrast, there is only one way of rolling a 2—both dice have to come up 1—so that'll come up only $\frac{1}{36}$ of the time. Similarly, there are two ways of rolling a 3—a 1 and a 2, or a 2 and a 1—so that arises $\frac{2}{36}$, or $\frac{1}{18}$ of the time. You can likewise figure out of the odds of any roll; you should take a moment and verify the probabilities in the table below.

sum	# of ways to achieve the sum	probability
2	1	1 in 36
3	2	1 in 18
4	3	1 in 12
5	4	1 in 9
6	5	5 in 36
7	6	1 in 6
8	5	5 in 36
9	4	1 in 9
10	3	1 in 12
11	2	1 in 18
12	1	1 in 36

Now let's look at some examples involving three or more dice.

EXAMPLE 5.2.1 Suppose now you roll three dice. What are the odds that the sum of the faces showing will be 10? What are the odds of rolling a 12? Which is more likely?

SOLUTION There are 6^3, or 216, possible outcomes of the roll of three dice; we just have to figure out how many add up to 10, and how many add up to 12.

There are many ways of approaching this problem—we could even just write out all the possible outcomes, but it's probably better to be systematic. Here's one way: if the sum of the first two rolls is any number between 4 and 9, then there is one and only one roll of the third die that will make the sum of all three equal to 10. Thus, the number of ways we can get 10 is simply the sum of the number of outcomes of two dice rolls that add up to 4, the number of outcomes of two dice rolls that add up to 5 and so on up to the number of outcomes of two dice rolls that add up to 9. We worked all these out a moment ago; the answer is

$$3 + 4 + 5 + 6 + 5 + 4 = 27.$$

Thus the probability of rolling a 10 with three dice is 27 out of 216, or simply 1 in eight.

Similarly, the number of ways we can get 12 is simply the sum of the number of outcomes of two dice rolls that add up to 6, the number of outcomes of two dice rolls that add up to 7 and so on up to the number of outcomes of two dice rolls that add up to 11; that is,

$$5 + 6 + 5 + 4 + 3 + 2 = 25.$$

So the probability of rolling a 12 with three dice is slightly less than the probability of rolling a 10. ■

EXAMPLE 5.2.2 Let's again roll three dice; this time, calculate the probability of getting at least one 6.

SOLUTION This is actually simpler than the last problem, because it's easier to be systematic. Just use the subtraction principle: the number of outcomes that include at least one 6 is 216 minus the number of outcomes that don't involve a 6; that is,

$$216 - 5^3 = 216 - 125 = 91.$$

The probability of getting at least one 6 on three rolls is thus 91 out of 216. ■

EXAMPLE 5.2.3 For a final example, let's roll seven dice. What is the probability of getting exactly two 6s?

SOLUTION This time, it's the multiplication principle we want to use. We know there are $6^7 = 279{,}936$ sequences of seven numbers from 1 to 6; we have to count how many such sequences contain exactly two 6's. Well, we can specify such a sequence by choosing, in turn:

- Which two of the seven numbers in the sequence are to be the 6's; and
- What the other five numbers in the sequence are to be.

For the first, the number of choices is just $\binom{7}{2}$, or 21. The second involves simply specifying a sequence of five numbers other than 6, that is, from 1 to 5; the number of choices is thus $5^5 = 3{,}125$. The total number of the sequences we're counting is thus

$$21 \times 3{,}125 = 65{,}625$$

and the probability of rolling such a sequence is

$$\frac{65{,}625}{279{,}936} \sim .234.$$

In other words, our chances are slightly less than 1 in 4. ■

Exercise 5.2.4 Say you roll five dice. What are the odds that you'll get at least one 5 and at least one 6?

Exercise 5.2.5 In the game of Phigh, each player rolls three dice; his or her score is the highest number that appears.

1. What is the probability of scoring 1?
2. What is the probability of scoring 2?
3. Your opponent scored 4. What is the probability that you'll win (that is, score 5 or 6)?

5.3 Playing Poker

It's time to graduate from dice to cards, and we're going to focus here primarily on probabilities associated with poker.

To start with, let's establish the rules. A standard deck consists of 52 cards. There are four suits: spades (♠), hearts (♡), diamonds (◇) and clubs (♣). There are 13 cards of each suit, with denominations 2, 3, 4 up to 10, jack (J), queen (Q), king (K) and ace (A). A poker hand consists of five cards; the ranks of the various hands are as follows:

A pair: a hand including two cards of the same denomination
Two pair: a hand including two cards each of two denominations
Three of a kind: a hand including three cards of the same denomination
Straight: a hand in which the denominations of the five cards form an unbroken sequence. For this purpose an ace may be either high or low; that is, A 2 3 4 5 and 10 J Q K A are both straights.
Flush: a hand in which all five cards belong to the same suit
Full house: a hand consisting of three cards of one denomination and two cards of another denomination
Four of a kind: a hand including four cards of the same denomination
Straight flush: a hand consisting of five cards of the same suit forming an unbroken sequence

Note that when we talk about a hand whose rank is "exactly three of a kind," we'll mean a hand of that rank and no higher.

We're going to start with the basic question: if you're dealt five cards at random, what are the odds of getting a given type of hand? Here "at random" means that all the possible hands, of which there are

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960,$$

are equally likely to arise; so that the odds of getting a particular type of hand are just the total number of such hands divided by 2,598,960. Our goal, then, will be (as usual) to count the number of hands of each type.

We'll start at the top, with straight flushes. These are straightforward to count via the multiplication principle: to specify a particular straight flush, we simply have to specify the denominations and the suit, which are independent choices. There are four suits, obviously; and as for the denominations, a straight can have as its low card any card from A up to 10 (remember that we count A 2 3 4 5 as a straight), so there are 10 possible denominations. There are thus

$$4 \times 10 = 40$$

straight flushes, and the probability of being dealt one in five cards is accordingly

$$\frac{40}{2{,}598{,}960} \sim .0000153$$

or approximately 1 in 64,974. Not an everyday occurrence: if you play, for example, on the order of two hundred hands a week, it'll happen to you roughly once in six or seven years.

Next is four of a kind. Again, the multiplication principle applies more or less directly: to specify a hand with four of a kind, we have to specify first the denomination of the four, and then say which of the remaining 48 cards in the deck will be the fifth card of the hand. The number of choices is accordingly

$$13 \times 48 = 624,$$

and the probability of being dealt one in five cards is accordingly

$$\frac{624}{2{,}598{,}960} \sim .00024001$$

or, in cruder terms, approximately 1 in 4,000. A good bit more likely than a straight flush, in other words, but don't hold your breath; again, if you play on the order of two hundred hands a week, on average you'll get two or three of these a year.

Note that if we wanted to calculate the odds of getting "four of a kind or better" we'd have to add the number of hands with four of a kind and the number of hands with a straight flush. In general, we're going to calculate here the odds of getting a hand of exactly a given rank; to count the number of hands of a specified rank *or higher* you'll have to add up the numbers of hands of each rank above.

Full houses are also straightforward to count. Since a full house consists of three cards of one denomination and two cards of another, we have to specify first the denomination of which we have three cards, and the denomination of which we have two; then we have to specify which three of the four cards of the first denomination are in the hand, and which two of the second. Altogether, then, the number of choices is

$$\begin{aligned} 13 \times 12 \times \binom{4}{3} \times \binom{4}{2} &= 13 \times 12 \times 4 \times 6 \\ &= 3{,}744. \end{aligned}$$

The probability is

$$\frac{3{,}744}{2{,}598{,}960} \sim .0014406$$

or approximately 1 in 700.

Flushes are even easier to count: we specify a suit (4 choices), and then which five of the 13 cards in that suit will constitute the hand. The number of flushes is thus

$$\begin{aligned} 4 \times \binom{13}{5} &= 4 \times 1{,}287 \\ &= 5{,}148. \end{aligned}$$

Remember, though, that this includes straight flushes too! If we want to count the number of hands with *exactly* a flush and not any higher rank, we have to subtract those 40 hands, so that the number is

$$5{,}148 - 40 = 5{,}108.$$

The probability is thus

$$\frac{5{,}108}{2{,}598{,}960} \sim .0019654$$

or approximately 1 in 500.

If you're with us so far, straights are likewise simple to count: we have to specify the denominations of the cards—10 choices, as we counted a moment ago—and then the suits, which involve four choices for each of the five cards. The total number of straights is therefore

$$\begin{aligned} 10 \times 4^5 &= 10 \times 1{,}024 \\ &= 10{,}240; \end{aligned}$$

and if we exclude straight flushes, the number of hands whose rank is exactly a straight is

$$10{,}240 - 40 = 10{,}200.$$

The probability is

$$\frac{10{,}200}{2{,}598{,}960} \sim .0039246$$

or very approximately 1 in 250.

Next, we count hands with exactly three of a kind. Initially, this is similar to the cases we've done before: we have to specify the denomination of the three (13 choices); which three of the four cards of that denomination are to be in the hand $\left(\binom{4}{3} = 4 \text{ choices}\right)$, and finally which two of the remaining 48 cards of the deck will round out the hand.

But here there's one additional wrinkle: since we're counting only hands with the rank "three of a kind," and not "full house," the last two cards can't be of the same denomination. Now, if we were counting the number of sequences of two cards of different denominations from among those 48, the answer would be immediate: we have 48 choices for the first, and 44 for the second, for a total of $48 \times 44 = 2{,}112$ choices. Since the order doesn't matter, though, and because each collection of two such cards corresponds to two different sequences, the number of pairs of cards of different denominations from among those 48 is

$$\frac{48 \cdot 44}{2} = 1{,}056.$$

The number of hands with exactly three of a kind is thus

$$13 \times 4 \times 1{,}056 = 54{,}912$$

and the probability is

$$\frac{54{,}912}{2{,}598{,}960} \sim .021128$$

or roughly 1 in 50. In other words, if your typical night of poker consists of two hundred hands, you're likely to be dealt three of a kind four times.

Counting hands with exactly two pair is slightly easier. We specify the two denominations involved; we have $\binom{13}{2} = 78$ choices there. Then we have to say which two of the four cards of each of these denominations go in the hand: that's

$$\binom{4}{2}^2 = 6^2 = 36$$

choices. Finally, we have to say which of the remaining 44 cards in the deck (of the other 11 denominations) will complete the hand. The total number is, accordingly,

$$78 \times 36 \times 44 = 123{,}522$$

and the probability is

$$\frac{123{,}522}{2{,}598{,}960} \sim .047539$$

or approximately 1 in 20.

Finally, we come to the hands with exactly one pair. We can do this in a similar fashion to our count of hands with three of a kind; choose the denomination of the pair (13); choose two cards of that denomination $\left(\binom{4}{2} = 6\right)$, and finally choose three cards among the 48 cards not of that denomination. But, as in the case of hands with three of a kind, there's a wrinkle in that last step: the three cards not part of the pair must all be of different denominations. Again, if we were counting sequences, rather than collections, of cards, this would be straightforward: there'd be $48 \times 44 \times 40$ choices. Since we have to count collections, however, and since every collection of three cards corresponds to $3! = 6$ different sequences, the number of such collections is

$$\frac{48 \cdot 44 \cdot 40}{6} = 14{,}080.$$

The number of hands with exactly a pair is thus

$$13 \times 6 \times 14{,}080 = 1{,}098{,}240$$

and the probability is

$$\frac{1{,}098{,}240}{2{,}598{,}960} \sim .42256$$

or a little worse than half. As we remarked before, if we want to find the odds of being dealt a pair *or better*, we have to add up the numbers of all hands better than a pair: the total is

$$40 + 624 + 3{,}744 + 5{,}108 + 10{,}200 + 54{,}912 + 123{,}552$$
$$+ 1{,}098{,}240 = 1{,}296{,}420.$$

Now, there's another way to calculate this number, and it gives us a way to check a lot of our calculations. We can count the number of hands with a pair or better by the subtraction principle: that is, count the hands with no pairs, and subtract that from the total number of hands. To count the hands with no two cards of the same denomination, as in the calculation we did of hands with three of a kind or with a pair, we count first the sequences of five cards, no two of the same denomination; this number is simply

$$52 \times 48 \times 44 \times 40 \times 36.$$

But every such hand of cards corresponds to $5! = 120$ sequences, so the number of hands with no two cards of the same denomination is

$$\frac{52 \times 48 \times 44 \times 40 \times 36}{120} = 1{,}317{,}888.$$

But we're not quite done: these 1,317,888 hands include straights, flushes and straight flushes, and if we want to count hands that *rank below* a pair, we have also to exclude these. Thus the total number of poker hands ranking below a pair will be

$$1{,}317{,}888 - 40 - 5{,}108 - 10{,}200 = 1{,}302{,}540,$$

and the number of hands ranked a pair or better will be

$$2{,}598{,}960 - 1{,}302{,}540 = 1{,}296{,}420$$

as we predicted. Note that the probability of getting a pair or better is thus

$$\frac{1{,}296{,}420}{2{,}598{,}960} \sim .49843$$

or very nearly one in two.

Exercise 5.3.1 What are the odds of being dealt a *busted flush*—that is, four cards of one suit and a fifth card of a different suit?

Exercise 5.3.2 What are the odds that a five-card poker hand will contain at least one ace?

Exercise 5.3.3 This doesn't actually have anything to do with poker,[1] but what are the odds that a five-card poker hand will consist entirely of cards of the same color? Is this the same as the odds that 5 coins flipped will all come up the same? Why, or why not?

5.4 Really Playing Poker

This section is probably unnecessary, but our lawyers insisted that we include it.

The odds we've just calculated are obviously relevant to playing poker, but they're only the tip of the tip of the iceberg. In almost all versions of poker, your hand isn't simply dealt to you all at once; it comes in stages, after each of which there's a round of betting. Each time you have to bet, you have to calculate the

[1] Unless, of course, you're playing with extremely nearsighted people and can pass one of these off as a flush.

likelihood of winding up with each possible hand, based on what you have already and how many cards you have yet to receive.

What's more, most poker games involve at least some cards dealt face up, and every time a card is dealt face up, it changes the odds of what you're likely to receive on succeeding rounds, and what your hand is likely to wind up being. In addition, every time someone bets (or doesn't) or raises (or doesn't), it changes the (estimated) odds of what their hole cards are, and hence what cards you're apt to be dealt on succeeding rounds. In fact, every time it's your bet, you have to calculate the odds of your achieving each possible hand, and the amount you stand to win or lose depending on what you get (which depends in turn on other factors: what the other players get, how much is currently in the pot, how much the other players will contribute to the pot and how much you'll have to contribute to the pot).

To be really good at poker, you have to be able to calculate these odds accurately (and unsentimentally). At the same time, it can never be exact: for one thing, no one can make that many calculations that quickly. For another, figuring out how likely it is that the player across the table really does have a king under is necessarily an inexact science. In other words, serious poker exists somewhere in that gray area between mathematics and intuition. Those of us with weaknesses in either field should probably limit our bets.

5.5 Bridge

Bridge is a card game that calls, as much as poker, for estimations of odds. We're not going to discuss the game in any depth or detail at all, but there is one aspect of the game that makes for a beautiful problem in probability, which we'll describe.

In bridge, each player is dealt a hand of 13 cards from a standard deck. This means there are

$$\binom{52}{13} = 635{,}013{,}559{,}600$$

possible hands, all of which we're going to assume are equally likely on any given deal. Now, every hand has what is called a *distribution*, meaning how many cards it has from the four different suits: for example, a hand with four cards of one suit and three each of the others is said to have a 4333 distribution; a hand with four cards each of two suits, three of a third and two of the final suit is said to have a 4432 distribution, and so on.

The question we want to take up here is: what are the odds of a bridge hand having a given distribution? As a special case, we could ask: which is more likely to occur, a 4333 distribution, or a 4432? How do the odds of either compare to the odds of getting a relatively unbalanced distribution, like 5431?

Let's start by counting the number of hands with a 4333 distribution. Basically, we can specify such a hand in two stages: first, we specify which suit is to be the four-card suit; and then we have to specify which of the 13 cards from each suit we're to receive. To specify the four-card suit, there are clearly 4 choices; and as for specifying which of the 13 cards from each suit we're to receive, we have to choose four cards from one suit and three from each of the others. By the multiplication principle, then, the total number of choices is

$$4 \cdot \binom{13}{4} \cdot \binom{13}{3} \cdot \binom{13}{3} \cdot \binom{13}{3}$$

or, in factorials,

$$= 4 \cdot \frac{13!}{4!9!} \cdot \frac{13!}{3!10!} \cdot \frac{13!}{3!10!} \cdot \frac{13!}{3!10!}$$

which works out to

$$= 66{,}905{,}856{,}160.$$

The probability of being dealt a hand with a 4333 distribution is thus

$$\frac{66{,}905{,}856{,}160}{635{,}013{,}559{,}600} \sim .105,$$

or slightly better than one in ten.

Let's do the 4432 distribution next. The idea is the same: first we figure out how many ways we can match the four numbers with the four suits; then, once we've specified how many cards of each suit we're to receive, we calculate how many ways we can choose those cards. For the first part, we have to choose the suit with two cards (four choices) and then the suit with three (three choices); the remaining two suits will each get four cards. The total number of choices is thus

$$4 \cdot 3 \cdot \binom{13}{4} \cdot \binom{13}{4} \cdot \binom{13}{3} \cdot \binom{13}{2}$$

$$= 4 \cdot 3 \cdot \frac{13!}{4!9!} \cdot \frac{13!}{4!9!} \cdot \frac{13!}{3!10!} \cdot \frac{13!}{2!11!}$$

$$= 136{,}852{,}887{,}600.$$

The probability of being dealt a hand with a 4432 distribution is thus

$$\frac{136{,}852{,}887{,}600}{635{,}013{,}559{,}600} \sim .216,$$

or slightly better than one in five. So in fact we see that you're more than twice as likely to be dealt a hand with a 4432 distributions as one with a 4333 distribution!

By now you've probably got the idea; so you can do some yourself:

Exercise 5.5.1 Calculate the probability of being dealt a hand with

1. a 5332 distribution;
2. a 4441 distribution; and
3. a 7321 distribution.

Before you make the calculations, guess which will be most likely and which least.

The next exercise has to do with a basic problem in bridge: once you've seen your cards, what are the odds that govern what everyone else's hand looks like? Clearly what you've got has some effect on the odds: if you have 11 spades, for example, you can be certain that no one at the table has a 4333 distribution. It's a hard problem, but if you can do it you can call yourself a master counter.

Exercise 5.5.2 Say you're playing bridge, and you pick up your hand to discover you have a 7321 distribution. What are the odds that the player to your left has a 4333 distribution?

5.6 The Birthday Problem

Everyone has a birthday; and, leaving aside for the moment those unfortunate souls born on February 29 of a leap year, everyone's birthday is one of the 365 days of the standard year. The probability of two people selected at random having the same birthday is, accordingly, 1 in 365.

So, suppose now we get 10 people together at random. What are the odds that two have the same birthday? How about a group of 25, or 50, or 100? It's pretty clear that as the number of people in the group increases, so does the probability of two people having the same birthday—when you get up to 366 people, of course, it's a lock—so we might ask: *for what size group is there actually a better than 50% chance of two people having the same birthday*? We know how to calculate the odds by now, of course, but before we do so you might want to take a few moments out and think about it—take a guess.

Time's up; here we go. Suppose we line up a group of, say, 50 people, and list their birthdays. We get a sequence of 50 days of the year; and assuming the people were picked randomly—so that each one is as likely to have been born on one day as another—of the 365^{50} possible such sequences, all are equally likely.

So: how many of these 365^{50} possible sequences involve a repeated day? Well, we know how many don't: the number of sequences of 50 days without repetition is, by the standard formula, the product

$$\frac{(365)!}{(315)!} \quad \text{or} \quad 365 \cdot 364 \cdot 363 \cdot \cdots \cdot 317 \cdot 316.$$

The probability of there *not* being a repeated birthday among 50 people is thus

$$\frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot 317 \cdot 316}{365^{50}}.$$

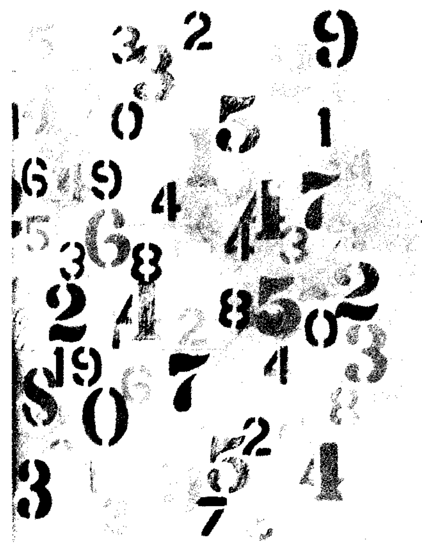
Now, these are some hefty numbers, and we have to be careful how we multiply them out: if we just ask our calculator to come up with 365^{50}, it'll never speak to us again. But we can rewrite this in a form that keeps the numbers reasonably sized:

$$\frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot 317 \cdot 316}{365^{50}} = \frac{365}{365} \cdot \frac{364}{365} \cdot \cdots \cdot \frac{317}{365} \cdot \frac{316}{365}$$

$$= 1 \cdot \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{48}{365}\right) \cdot \left(1 - \frac{49}{365}\right).$$

This is something we (or rather our computers) can evaluate, and the answer is that the probability of not having a repeat is 0.0296. In other words, if we take 50 people at random, the probability is better than 97% that two will share a birthday! Pretty surprising, when you think about it.

In fact, if you work it out, it's already the case with 23 people that the probability of a repeated birthday is 50.7%, or better than half; and by the time you get to 30 people the probability is 70.6% that two people will have the same birthday.



Chapter 6

Pascal's Triangle and the Binomial Theorem

If you're with us so far—if most of the calculations in the last chapter make sense to you—you've got a pretty good idea of what counting is about. In particular, you've seen all the ideas and techniques of counting that we're going to use in the rest of this book. From a strictly logical point of view, you could proceed directly to the second part.

But in the course of our counting, we've come across a class of numbers, the binomial coefficients, that are worth studying in their own right, both for the fascinating properties and patterns they possess and for the way they crop up in so many areas of mathematics. We're going to take some time out here, accordingly, and devote a chapter to the binomial coefficients themselves. These detours are common in mathematics—the tools that we develop to solve a particular problem often open up surprising areas of investigation in their own right.

6.1 Pascal's Triangle

Probably the best way to go about looking for patterns in binomial coefficients is simply to make a table of them and stare at it—maybe we'll be able to deduce something. (Mathematicians like to give the impression that they arrive at their conclusions by abstract thought, but the reality is more prosaic: most of us at least start with experimentation.) As for the form this table should take, there's a classic way of representing the binomial coefficients that is particularly well-suited to displaying their patterns, called *Pascal's triangle*.

Pascal's triangle consists of a sequence of rows, where each row gives the values of the binomial coefficients $\binom{n}{k}$ for a particular value of n. For example, the row with $n = 1$ has only two numbers in it:

$$\binom{1}{0} = 1 \quad \text{and} \quad \binom{1}{1} = 1.$$

The row with $n = 2$ has three:

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1,$$