Chapter 1

Simple Counting

It's hard to begin a math book. A few chapters in, it gets easier: by then, writer and reader have—or think they have—a common sense of the level of the book, its pace, language and goals; at that point, communication naturally flows more smoothly. But getting started is awkward.

As a consequence, it's standard practice in math textbooks to include a throw-away chapter or two at the beginning. These often have little or no content; rather, they're put there in the hope of establishing basic terminology and notation, and getting the reader used to the style of the book, before launching into the actual material. Unfortunately, the effect may be the opposite: a chapter full of seemingly obvious statements, expressed in vague language, can have the effect of making the reader generally uneasy without actually conveying any useful information.

Well, far be it from us to deviate from standard practice! The following is our introductory chapter. But here's the deal: you can skip it if you find the material too easy. Really. Just go right ahead to Chapter 2 and start there.

1.1 Counting Numbers

This is a book about numbers. We hope to show you, during its course, something of the wild beauty of numbers: the intricate patterns of their behavior, and the way that even simple operations on them can give rise to questions that people have wrestled with for centuries.

To start things off, we'd like to talk about counting, because that's how numbers first entered our world. It was four or five thousand years ago that people first developed the concept of numbers, probably in order to quantify their possessions and make transactions—my three pigs for your two cows and the like. And the remarkable thing that people discovered about numbers is that the same system of numbers—1, 2, 3, 4, and so on—could be used to count anything: beads, bushels of grain, people living in a village, forces in an opposing army. Numbers can count anything: numbers can even count numbers.

And that's where we'll start. The first problem we're going to pose is simply: how many numbers are there between 1 and 10?

At this point you may be wondering if it's too late to get your money back for this book. Bear with us! We'll get to stuff you don't know soon enough. In the meantime, write them out and count:

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10$$

—there are 10. How about between 1 and 11? Well, that's one more, so there are 11. Between 1 and 12? 12, of course.

Well, that seems pretty clear; and if we now asked you, for example, how many numbers there are between 1 and 57, you wouldn't actually have to write them out and count; you'd figure (correctly) that the answer would be 57.

OK, then, let's ramp it up a notch. Suppose we ask now: how many numbers are there between 28 and 83, inclusive? ("Inclusive" means that, as before, we include both 28 and 83 in the count.) Well, you could do this by making a list of the numbers between 28 and 83 and counting them, but you have to believe there's a better way than that.

Here's one: suppose you did write out all the numbers between 28 and 83:

$$28, \quad 29, \quad 30, \quad 31, \quad 32, \ldots, 82, \quad 83.$$

(Here the dots mean to imagine that we've written all the numbers in between in an unbroken sequence. We'll use this convention when it's not possible or desirable to write out a sequence of numbers in full.) Now subtract the number 27 from each of them. The list now starts at 1, and continues up to $83 - 27 = 56$:

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots, 55, \quad 56.$$

From what we just saw we know there are 56 numbers on this list; so there were 56 numbers on our original list as well.

It's pretty clear also that we could do this to count any string of numbers. For example, if we asked how many numbers there are between 327 and 573, you could similarly imagine the numbers all written out:

$$327, \quad 328, \quad 329, \quad 330, \quad 331, \ldots, 572, \quad 573.$$

Next, subtract the number 326 from each of them; we get the list

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \ldots, 246, \quad 247$$

and so we can conclude that there were $573 - 326 = 247$ numbers on our original list.

Now, there's no need to go through this process every time. It makes more sense to do it once with letters standing for arbitrary numbers, and in that way work out a formula that we can use every time we have such a problem. So: imagine that we're given two whole numbers n and k, with n the larger of the two, and we're asked: how many numbers are there between k and n, inclusive?

We do this just the same way: imagine that we've written out the numbers from k to n in a list

$$k, \quad k+1, \quad k+2, \quad k+3, \ldots, n-1, \quad n$$

and subtract the number $k - 1$ from each of them to arrive at the list

$$1, \quad 2, \quad 3, \quad 4, \ldots, n-1-(k-1), \quad n-(k-1).$$

Now we know how many numbers are on the list: it's $n - (k - 1)$ or, more simply, $n - k + 1$.[1] Our conclusion, then, is that

> The number of whole numbers between k and n inclusive is
>
> $$n - k + 1.$$

So, for example, if someone asked "How many numbers are there between 342 and 576?" we wouldn't have to think it through from scratch: the answer is $576 - 342 + 1$, or 235.

Since this is our first formula, it may be time to bring up the whole issue of the role of formulas in math. As we said, the whole point of having a formula like this is that we shouldn't have to recreate the entire argument we used in the concrete examples above every time we want to solve a similar problem. On the other hand, it's also important to keep some understanding of the process, and not to treat the formula as a "black box" that spews out answers. Knowing how the formula was arrived at helps us to know both when it's applicable, and how it can be modified to deal with other situations.

1.2 Counting Divisible Numbers

Now that we've done that, let's try a slightly different problem: suppose we ask "How many even numbers are there between 46 and 104?"

In fact, we can approach this the same way: imagine that we did make a list of all even numbers, starting with 46 and ending with 104:

$$46, \quad 48, \quad 50, \quad 52, \ldots, 102, \quad 104$$

Now, we've just learned how to count numbers in an unbroken sequence. And we can convert this list to just such a sequence if we just divide all the numbers on the list by 2: doing that, we get the sequence

$$23, \quad 24, \quad 25, \quad 26, \ldots, 51, \quad 52$$

of all whole numbers between $46/2$, or 23, and $104/2$, or 52. Now, we know by the formula we just worked out how many numbers there are on that list: there are

$$52 - 23 + 1 = 30$$

numbers between 23 and 52, so we conclude that there are 30 even numbers between 46 and 104.

One more example of this type: let's ask the question, "How many numbers between 50 and 218 are divisible by 3?" Once more we use the same approach: imagine that we made a list of all such numbers. But notice that 50 isn't the first such number, since 3 doesn't divide 50 evenly: in fact, the smallest number on our list that is divisible by 3 is $51 = 3 \times 17$. Likewise, the last number on our list is 218,

[1] Is it obvious that $n - (k - 1)$ is the same as $n - k + 1$? If not, take a moment out and convince yourself: subtracting $k - 1$ is the same as subtracting k and then adding 1 back. In this book we'll usually carry out operations like this without comment, but you should take the time to satisfy yourself that they make sense.

which isn't divisible by 3. The largest number on our list which is divisible by 3 is 216, which is ($216 = 3 \times 72$). So the list of numbers divisible by 3 would look like

$$51, \quad 54, \quad 57, \quad 60, \ldots, 213, \quad 216.$$

Now we can do as we did before, and divide each number on this list by 3. We arrive at the list

$$17, \quad 18, \quad 19, \quad 20, \ldots, 71, \quad 72$$

of all whole numbers between 17 and 72, and there are

$$72 - 17 + 1 = 56$$

such numbers.

Now it's time to stop reading for a moment and do some yourself:

Exercise 1.2.1

1. How many numbers between 33 and 97 are even?
2. How many numbers between 17 and 783 are divisible by 6?
3. How many numbers between 45 and 93 are odd?

1.3 "I've reduced it to a previously solved problem"

Note one thing about the sequence of problems we've just done. We started with a pretty mindless one—the number of numbers between 1 and n—which we could answer more or less by direct examination. The next problem we took up—the number of numbers between k and n—we solved by shifting all the numbers down to whole numbers between 1 and $n-k+1$. In effect we reduced it to the first problem, whose answer we knew. Finally, when we asked how many numbers between two numbers were divisible by a third, we answered the question by dividing all the numbers, to reduce the problem to counting numbers between k and n.

This approach—building up our capacity to solve problems by reducing new problems to ones we've already solved—is absolutely characteristic of mathematics. We start out slowly, and gradually accumulate a body of knowledge and techniques; the goal is not necessarily to solve each problem directly, but to reduce it to a previously solved problem.

There's even a standard joke about this:

> *A mathematician walks into a room. In one corner, he sees an empty bucket. In a second corner, he sees a sink with a water faucet. And, in a third corner, he sees a pile of papers on fire. He leaps into action: he picks up the bucket, fills it up at the faucet, and promptly douses the fire.*
>
> *The next day, the same mathematician returns to the room. Once more, he sees a fire in the third corner, but this time sitting next to it there's a full bucket of water. Once more he leaps into action: he picks up the bucket, drains it into the sink, places it empty in the first corner and leaves, announcing, "I've reduced it to a previously solved problem!"*

Well, maybe you had to be there. But there is a real point to be made here. It's simply this: the ideas and techniques developed in this book are cumulative, each

one resting on the foundation of the ones that have come before. We'll occasionally go off on tangents and pursue ideas that won't be used in what follows, and we'll try to tell you when that occurs. But for the most part, *you need to keep up*: that is, you need to work with the ideas and techniques in each section until you feel genuinely comfortable with them, before you go on to the next.

It's worth remarking also that the cumulative nature of mathematics in some ways sets it apart from other fields of science. The theories of physics, chemistry, biology and medicine we subscribe to today flatly contradict those held in the 17th and 18th centuries—it's fair to say that the medical texts dealing with the proper application of leeches are of interest primarily to historians, and we'd bet your high school chemistry course didn't cover phlogiston.[2] By contrast, the mathematics developed at that time is the cornerstone of what we're doing today.

1.4 Really Big Numbers

As long as we're talking about the origins of numbers, let's talk about another important early development: the capacity to write down really big numbers. Think about it: once you've developed the concept of numbers, the next step is to figure out a way to write them down. Of course, you can just make up an arbitrary new symbol for each new number, but this is inherently limited: you can't express large numbers without a cumbersome dictionary.

One of the first treatises ever written on the subject of numbers and counting was by Archimedes, who lived in Syracuse (part of what was then the Greek empire) in the 3rd century B.C. The paper, entitled *The Sand Reckoner*, was addressed to a local monarch, and in it Archimedes claimed that he had developed a system of numbers that would allow him to express as large a number as the number of grains of sand in the universe—a revolutionary idea at the time.

What Archimedes had developed was similar to what we would call *exponential notation*. We'll try to illustrate this by expressing a really large number—say, the approximate number of seconds in the lifetime of the universe. (Don't laugh: this number will actually come up in a practical context in the last part of this book.)

The calculation is simple enough. There are 60 seconds in a minute, and 60 minutes in an hour, so the number of seconds in an hour is

$$60 \times 60 = 3{,}600.$$

There are in turn 24 hours in a day, so the number of seconds in a day is

$$3{,}600 \times 24 = 86{,}400;$$

and since there are 365 days in a (nonleap) year, the number of seconds in a year is

$$86{,}400 \times 365 = 31{,}536{,}000.$$

Now, in exponential notation, we would say this number is roughly 3 times 10 to the 7th power—that is, a 3 with seven 0s after it. (A better approximation, of course,

[2] In case you're curious, phlogiston was the hypothetical principle of fire, of which every combustible substance was in part composed—at least until the whole theory was discredited by Antoine Lavoisier between 1770 and 1790.

would be to say the number is roughly 3.1×10^7, or 3.15×10^7; but we're going to go with the simpler estimate 3×10^7.)

Exponential notation is particularly convenient when it comes to multiplying large numbers. Suppose, for example, that we have to multiply $10^6 \times 10^7$. Well, 10^6 is just $10 \times 10 \times 10 \times 10 \times 10 \times 10$, and 10^7 is just $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, so when we multiply them we just get the product of 10 with itself 13 times: that is,

$$10^6 \times 10^7 = 10^{13}.$$

In other words, we simply add the exponents. So it's easy to take products of quantities that you've expressed in exponential notation.

For example, to take the next step in our problem, we have to say how old the universe is. Now, that very much depends on your model of the universe. Most astrophysicists believe that the universe is approximately 13.7 billion years old, with a possible error on the order of 1%. We'll write the age of the universe, accordingly, as

$$13{,}700{,}000{,}000 = 1.37 \times 10^{10}$$

years. So the number of seconds in the lifetime of the universe would be approximately

$$(1.37 \times 10^{10}) \times (3 \times 10^7) = 4.11 \times 10^{17};$$

or, rounding it off, the universe is 4×10^{17} seconds old.

You see how we can use this notation to express arbitrarily large numbers. For example, computers currently can carry out on the order of 10^{12} operations a second (a *teraflop*, as it's known in the trade). We could ask: if such a computer were running from the dawn of time to the present, how many operations could it have performed? The answer is, approximately

$$10^{12} \times (4 \times 10^{17}) = 4 \times 10^{29}.$$

Now, for almost all of this book, we'll be dealing with much smaller numbers than these; and we'll be doing exact calculations rather than approximations. But occasionally we will want to express and estimate larger numbers like these. (The last number above—the number of operations a computer running for the lifetime of the universe could perform—will actually arise later on in this book: we'll encounter mathematical processes that require more than this number of operations to carry out.) It's nice to know that we have a notation that can accommodate it.

1.5 It Could Be Worse

Look: this is a math book. We're trying to pretend it isn't, but it is. That means that it'll have jargon—we'll try to keep it to a minimum, but we can't altogether avoid using defined terms. That means that you'll encounter the odd mathematical formula here and there. That means it'll have long discussions aimed at solving artificially posed problems, subject to seemingly arbitrary hypotheses. Mathematics texts have a pretty bad reputation, and we're sorry to say it's largely deserved.

Just remember: it could be worse. You could, for example, be reading a book on Kant. Now, Immanuel Kant is a towering figure in Western Philosophy, a seminal

genius who shaped much of modern thought. "The foremost thinker of the Enlightenment and one of the greatest philosophers of all time," the Encyclopedia Britannica calls him. But just read a sentence of his writing:

> *If we wish to discern whether anything is beautiful or not, we do not refer the representation of it to the Object by means of understanding with a view to cognition, but by means of the imagination (acting perhaps in conjunction with understanding) we refer the representation to the Subject and its feeling of pleasure or displeasure.*

What's more, this is not a nugget unearthed from deep within one of Kant's books. It is, in fact, the first sentence of the first Part of the first Moment of the first Book of the first Section of Part I of Kant's *The Critique of Judgement*.

Now, we're not trying to be anti-intellectual here, or to take cheap shots at other disciplines. Just the opposite, in fact: what we're trying to say is that any body of thought, once it progresses past the level of bumper sticker catch phrases, requires a language and a set of conventions of its own. These provide the precision and universality that are essential if people are to communicate and develop the ideas further, and shape them into a coherent whole. But they also can have the unfortunate effect of making much of the material inaccessible to a casual reader. Mathematics suffers from this—as do most serious academic disciplines.

The point, in other words, is not that the passage from Kant we just quoted is babble; it's not. (Lord knows we could have dug up enough specimens of academic writing that are, if that was our intention.) In fact, it's the beginning of a serious and extremely influential attempt to establish a philosophical theory of aesthetics. As such, it may be difficult to understand without some mental effort. It's important to bear in mind that the apparent obscurity of the language is a reflection of this difficulty, not necessarily the cause of it.

So, the next time you're reading this book and you encounter a term that turns out to have been defined—contrary to its apparent meaning—some 30 pages earlier, or a formula that seems to come out of nowhere and that you're apparently expected to find self-explanatory, just remember: it could be worse.