Differential Modular Forms

Arnab Saha

Australian National University

arnab.saha@anu.edu.au

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$$\begin{array}{ccc} M^\infty & \xrightarrow{E^\infty} & S^\infty_{for} \\ \uparrow & & \uparrow \\ M & \xrightarrow{E} & R((q)) \end{array}$$

Various interesting examples of differential modular forms like $f^1, f^\partial, f^\sharp \in M^\infty$.

Introduction

p-derivations

Let A and B be rings. A tuple (u, δ) is called a *p-derivation* , where $u : A \to B$ is a ring homomorphism and $\delta : A \to B$ is a set-theoretic map satisfying

$$\delta(x + y) = \delta x + \delta y + C_p(u(x), u(y))$$

$$\delta(xy) = u(x)^p \delta y + u(y)^p \delta x + p\delta x \delta y$$

where $C_p(a, b) = \frac{a^p + b^p - (a+b)^p}{p}$, p a fixed prime.

Remark. The above is equivalent to giving a ring homomorphism $A \to W_1(B)$, $a \to (u(a), \delta a)$, where $W_1(B)$ is the ring of p-typical Witt vectors of length 2.

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Jet Rings

Let $R = \hat{\mathbb{Z}_p^{ur}}$. Let $A = R[\mathbf{x}]/(\mathbf{f})$ where $\mathbf{x}$ is a tuple of variables and $(\mathbf{f})$ is a tuple of polynomials in $R[\mathbf{x}]$. Define the n-th jet ring as

$$J^n A = \frac{R[\mathbf{x}, \mathbf{x}', ..., \mathbf{x}^{(n)}]\hat{}}{(\mathbf{f}, ..., \delta^n \mathbf{f})} \text{ where } \delta x^{(n)} = x^{(n+1)}$$

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Example

$A = R[x, y]/(xy)$ then $J^1 A = \left(\frac{R[x,y,x',y']}{(xy, x^p y' + y^p x' + p x' y')} \right)^{\hat{}}$.

Prolongation Sequence

We define a *prolongation sequence* of p-adic rings as $B^* = B^0 \to ... B^n \overset{(u_n,\delta_n)}{\to} B^{n+1} \to ...$ where (u_n, δ_n) is a p-derivation for each n. They form a category `prol` with morphisms defined holding the obvious compatibility condition.

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The n-th jet rings give us a prolongation sequence of rings $J^*A := \hat{A} \to \ldots J^nA \stackrel{(u,\delta)}{\to} J^{n+1}A \to \ldots$ where u and δ are induced by

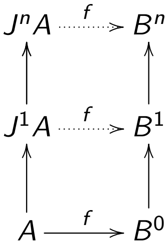
$u(\mathbf{x}^{(n)}) = \mathbf{x}^{(n)}$

$\delta\mathbf{x}^{(\mathbf{n})} = \mathbf{x}^{(n+1)}$ (p-derivation)

$\phi(\mathbf{x}) = \mathbf{x}^p + p\delta\mathbf{x}$ (lift of Frobenius)

Universal Property

Let B^* be a prolongation sequence. Then $Hom_{\texttt{rings}}(A, B^0) \simeq Hom_{\texttt{prol}}(J^*A, B^*)$.



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$$\begin{array}{ccc}
J^nA & \overset{f}{\dashrightarrow} & B^n \\
\uparrow & & \uparrow \\
J^1A & \overset{f}{\dashrightarrow} & B^1 \\
\uparrow & & \uparrow \\
A & \overset{f}{\longrightarrow} & B^0
\end{array}$$

Theorem (Borger, S)

$Hom\ (J^nA, B) \simeq Hom_\delta(A, W_n(B))$, where $W_n(B)$ is the ring of p-typical Witt vectors of length $n+1$.

Modular Forms

Let $p \nmid N$ and $N > 3$. Recall the modular curve $Y_1(N)$ over *Spec* $\mathbb{Z}[1/N, \zeta_N]$ that parametrizes elliptic curves E with level structures $i : \mu_N \hookrightarrow E[N]$.

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Fix an embedding $\mathbb{Z}[1/N, \zeta_N] \hookrightarrow R$ and base change $X_1(N)$ over R. Let $E \xrightarrow{\pi} X_1(N)$ be the universal elliptic curve and call $L := \pi_* \Omega_{E/X_1(N)}$.

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Consider $M^n := J^n M$ and let $E^n : M^n \to R((q))[q', ..., q^{(n)}]\hat{}$ be the prolongation of the Fourier expansion $E : M \to R((q))$.

Examples of δ-modular forms

1. f^{∂}: There exists a unique form $f^{\partial} \in M^1(\phi - 1)$ such that $E^1(f^{\partial}) = 1$ and its reduction mod p, $\overline{f^{\partial}}$, coincides with the image of the Hasse invariant $\overline{H} \in M^0(p-1)$.

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2. f^1: A unique form $f^1 \in M^1(-1-\phi)$ such that its Fourier expansion $E^1(f^1) = \Psi$ where

$$\Psi := \frac{1}{p} \log \frac{\phi(q)}{q^p} = \sum_{n \geq 1} (-1)^{n-1} n^{-1} p^{n-1} \left(\frac{q'}{q^p} \right)^n \in R((q))\hat{}[q']\hat{}$$

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3. $f^{\sharp}$: Given a classical cusp form f of weight 2 which is an eigen vector for all the Hecke operators and if $E(f) = \sum a_n q^n$, one can attach $f^{\sharp} \in M^2$ whose Fourier expansion is

$$E^2(f^{\sharp}) = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (\phi^2(q)^n - a_p \phi(q)^n + p q^n)$$

Remark. The above series infact has integral coefficients.

Differential Modular Forms

Set

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$$\begin{array}{ccc} M^{\infty} & \xrightarrow{E^{\infty}} & \mathbb{S}^{\infty} \subset S^{\infty}_{for} \\ \uparrow & & \uparrow \;\; \downarrow \pi \\ M & \xrightarrow[E]{} & R((q))\hat{} \end{array}$$

where $\pi : S^{\infty}_{for} \rightarrow R((q))\hat{}$ is the ring homomorphism obtained by setting $0 = \pi(q') = ... = \pi(q^{(n)}) = ...$

Theorem of Swinnerton-Dyer and Serre

Theorem

The kernel of the Fourier expansion map $\bigoplus_{\kappa \geq 0} M(\mathbb{F}_p, \kappa, N) \to \mathbb{F}_p[[q]]$ *is generated by* $E_{p-1} - 1$.

Kernel of E^∞

Theorem (Buium, S)

$X \subset X_1(N)^{ord}$ as before. Then
1) The inclusion $\mathbb{S}^\infty \subset S^\infty_{for}$ has torsion free cokernel.

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Hecke Operators away from p

Consider $f(q, ..., q') \in R((q))[q', ..., q^{(n)}]\hat{}$.

Let $(n, p) = 1$

For each integer $\kappa \in \mathbb{Z}$ the *Hecke* operator $f \mapsto T_\kappa(n)f$ on $R((q))[q', ..., q^{(r)}]\hat{}$ is defined as follows. For $f = f(q, q', ..., q^{(r)})$,

$$T_\kappa(n)f := n^{\kappa-1} \sum_{A,B,D} D^{-\kappa} f(\zeta_D^B q^{A/D}, \delta(\zeta_D^B q^{A/D}), ..., \delta^r(\zeta_D^B q^{A/D})).$$

where A, B, D belong to the following set

$$\{(A, B, D); A, B, D \in \mathbb{Z}_{\geq 0}, AD = n, (A, N) = 1, B < D\}$$

for each $n \geq 1$ and $N \geq 4$.

Hecke Operators at p

$A := R[[s_1, ..., s_p]][s_1', ..., s_p', ..., s_1^{(r)}, ..., s_p^{(r)}]\hat{}$

$B := R[[q_1, ..., q_p]][q_1', ..., q_p', ..., q_1^{(r)}, ..., q_p^{(r)}]\hat{}$

$A \to B$ induced by $s_j^{(i)} \mapsto \delta^i S_j$ where S_i's are the symmetric functions.

$S_1 = q_1 + ... q_p, \; ... \;, \; S_p = q_1 ... q_p.$

Definition

An element $G \in B$ will be called $\delta-$*symmetric* if it is the image of some element $G_{(p)} \in A$ (which is then unique).
An element $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ will be called $\delta - p$-*symmetric* if

$$\Sigma_p f := \sum_{j=1}^{p} f(q_j, ..., q_j^{(r)}) \in B$$

is $\delta-$symmetric.

Definition

For any $\delta - p$-symmetric $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ we define

$$Uf := p^{-1}(\Sigma_p f)_{(p)}(0, ..., 0, q, ..., 0, ..., 0, q^{(r)})$$

which is an element in $p^{-1}R[[q]][q', ..., q^{(r)}]\hat{}$.
For any $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ define

$$Vf := f(q^p, \delta(q^p), ..., \delta^r(q^p))$$

For any $\kappa \in \mathbb{Z}$ define

$$T_\kappa(p)f := Uf + p^{k-1}Vf$$

Remark. The restriction of $T_\kappa(p)$ on $R[[q]]$ gives us the usual "classical" Hecke operator on $R[[q]]$

$$T_\kappa(p)(\sum a_m q^m) = \sum a_{pm}q^m + p^{\kappa-1}\sum a_m q^{pm}$$

Lemma

δ-symmetry is independent of the parameter q.

That means that one can talk about δ-symmetry at a point ∞, for instance, we can define a δ-symmetric subspace in S^r, where $X = Spec\ S$.

Theorem (Buium, S)

There is a one-to-one correspondence between the following sets of objects:
i) Series in $qk[[q]]$ which are eigenvectors of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p) = 1$, and which are Fourier expansions of classical modular forms over k of weight $\equiv \kappa + 2 \mod p - 1$;
ii) Primitive δ-series in $k[[q]][q']$ which are eigenvectors of all Hecke operators $nT_{\kappa}(n)$, "$pT_{\kappa}(p)$", $(n,p) = 1$, and which are δ-Fourier expansions of δ-modular forms of some order $r \geq 0$ and weight w with $deg(w) = \kappa$.
This correspondence preserves the respective eigenvalues.

i) $\varphi = \sum_{m\geq 1} a_m q^m$ a classical modular form which is also an eigenform (means $a_1 \neq 0$).

$$\updownarrow$$

$$ii)\ \varphi^{\sharp,2} = \sum_{(n,p)=1} \frac{a_n}{n} q^n - \frac{a_p}{n} a_1 \left(\sum_{m\geq 1} a_m q^{mp} \right) \frac{q'}{q^p} + e \left(\sum_{m\geq 1} q^{mp^2} \right) \left(\frac{q'}{q^p} \right)^p$$

where $e = 0, 1$ depending on κ is 0 or ≥ 0.

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