Base rings for global (φ, Γ)-modules

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Overview

Kiran Kedlaya will be giving a follow-up talk.

Kedlaya's talk:

- Comparison between cohomologies using (φ, Γ)-modules
- Can the *p*-adic techniques work over $\mathbb{Q}$?

This talk:

- What sort of base rings do we expect in a theory of global (φ, Γ)-modules?

Comparison theorems and “big” rings

We start with a big ring familiar to everyone.

Theorem

Let X be a smooth, proper variety over $\mathbb{C}$. There is a comparison isomorphism

$$H^i_{sing}(X_{cl}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{dR}(X)$$

between singular cohomology and algebraic de Rham cohomology.

Comparison in the p-adic setting

Now assume X is smooth and proper over $\mathbb{Q}_p$.

- Maybe $H_{\mathrm{dR}}(X) \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$?
- Maybe $H_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$?
- Maybe $H_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^i_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$?

But in fact to get a natural comparison isomorphism, we need to tensor up to a bigger field, B_{dR}.

General lifting result

We're going to form our big field B_{dR} by lifting $\mathbb{C}_p$ to a DVR B_{dR}^+. The ring B_{dR} is the fraction field of B_{dR}^+.

Proposition

Let K denote a field of characteristic zero. A complete discrete valuation ring with residue field K is isomorphic to $K[[t]]$.

Proof.

See Serre's *Local Fields*. □

Preview of B_{dR}^{+}

Description of B_{dR}^{+}:

- B_{dR}^{+} is a complete discrete valuation ring with residue field $\mathbb{C}_p$.
- As a ring, $B_{dR}^{+} \cong \mathbb{C}_p[[t]]$.
- However, it is equipped with Galois action and topology, and those are not so natural if we describe B_{dR}^{+} as $\mathbb{C}_p[[t]]$.

Characteristic p construction

A perfect ring of characteristic p arising out of $\mathbb{C}_p$:

- Start with $\mathbb{C}_p$
- Take its valuation ring $\mathcal{O}_{\mathbb{C}_p}$
- Mod out by p: $\mathcal{O}_{\mathbb{C}_p}/(p)$
- Make it perfect:

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p)$$

- Denote this ring by $\tilde{E}^+$

The ring $\tilde{E}^+$

Lemma

The ring

$$\tilde{E}^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p)$$

is a perfect ring of characteristic p.

Proof.

- It is characteristic p: We have $p(1,1,1,\ldots) = (p,p,p,\ldots) = (0,0,0,\ldots)$.
- The map $z \mapsto z^p$ is surjective: If our element is $(z_0, z_1, z_2, \ldots)$, then $(z_1, z_2, z_3, \ldots)$ is its p-th root.
- The map $z \mapsto z^p$ is injective: If our element is $(z_0, z_1, z_2, \ldots)$, then its p-th power is $(z_0^p, z_0, z_1, \ldots)$. If this p-th power is zero, then our original element was zero.

□

The ring B_{dR}^{+}

We now briefly describe how to produce the ring B_{dR}^{+}. It is a complete discrete valuation ring with residue field $\mathbb{C}_p$.

- Begin with the perfect ring $\tilde{E}^{+} := \varprojlim \mathcal{O}_{\mathbb{C}_p}/(p)$
- Lift to $W(\tilde{E}^{+})$
- Invert p: $W(\tilde{E}^{+})\left[\frac{1}{p}\right]$.
- Let $\tilde{p} \in \tilde{E}^{+}$ be the following element:

$$\tilde{p} = (p, p^{1/p}, p^{1/p^2}, \ldots).$$

- Let $\xi \in W(\tilde{E}^{+})$ be the element $[\tilde{p}] - p$
- Define B_{dR}^{+} to be the completion of $W(\tilde{E}^{+})\left[\frac{1}{p}\right]$ at the principal ideal (ξ).

Alternate construction of $\tilde{E}^+$

Recall that $\tilde{E}^+ := \varprojlim \mathcal{O}_{\mathbb{C}_p}/(p)$. Here is an alternate description:

$$\tilde{E}^+ \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}.$$

Note. The transition maps are multiplicative but not additive, so it takes some thought to define addition on $\varprojlim \mathcal{O}_{\mathbb{C}_p}$.

- Let $K := \mathbb{Q}_p(\mu_{p^\infty})$
- Let L denote a finite extension of K
- Let $\widehat{K}$ and $\widehat{L}$ denote their p-adic completions
- To $\widehat{L}$ we associate

$$E_L^+ := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\widehat{L}}$$

  which we view as a subring of $\tilde{E}^+ \cong \varprojlim \mathcal{O}_{\mathbb{C}_p}$.
- The field $\operatorname{Frac} E_L^+$ is called the *perfect norm field* associated to L.

Galois correspondence

- Let $K := \mathbb{Q}_p(\mu_{p^\infty})$
- For L/K finite, let

$$E_L := \mathrm{Frac}\left(\varprojlim_{x \mapsto x^p} \mathcal{O}_{\widehat{L}}\right),$$

which is a perfect field of characteristic p.

Theorem (Fontaine-Wintenberger)

If L/K is a finite extension, then E_L/E_K is also a finite extension of the same degree. If L/K is a Galois extension, then E_L/E_K is also a Galois extension, and the two Galois groups are isomorphic.

Comparisons for number fields?

Question: How much of this can we adapt to the number field setting?

Goal: avoid characteristic p.

Proposition

Let F denote the Witt vector Frobenius. We have an isomorphism

$$W\left(\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}\right) \cong \varprojlim_{F} W(\mathcal{O}_{\mathbb{C}_p}).$$

The functor $\underleftarrow{W}^\dagger$

This construction avoids characteristic p by working with inverse limits of Witt vectors, with transition maps the Witt vector Frobenius.

- Continue to let $K = \mathbb{Q}_p(\mu_{p^\infty})$
- $W(L)$ isn't interesting when p is invertible in L, so we use overconvergent Witt vectors $W^\dagger(L)$
- Frobenius is more well-behaved for finite length Witt vectors. For example, $F : W_{p^n}(L) \to W_{p^{n-1}}(L)$ is surjective while $F : W(L) \to W(L)$ is not.
- Let $\underleftarrow{W}^\dagger(L)$ be shorthand for a certain overconvergent subring of

$$\varprojlim_F W_{p^n}(L)$$

Main result

Theorem

Let $K = \widehat{\mathbb{Q}(\mu_{p^\infty})}$ and let L denote a finite extension of K. Then $\underleftarrow{W}^\dagger(L)$ is a finite étale extension of $\underleftarrow{W}^\dagger(K)$.

In fact, the same is true without completion:

Theorem

Let $K = \mathbb{Q}(\mu_{p^\infty})$ and let L denote a finite extension of K. Then $\underleftarrow{W}^\dagger(L)$ is a finite étale extension of $\underleftarrow{W}^\dagger(K)$.

The fact that we have not p-adically completed here is key to our attempt to transition from the p-adic setting to the number field setting.

Thank you!

Thank you for your attention!

Any questions?