Witt vectors and total positivity

James Borger
Australian National University

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Semirings—basic definitions

1. $\mathbf{N} = \{0, 1, 2, \dots\}$
2. $\mathbf{N}$-module $=$ commutative monoid (written additively)
3. $\mathbf{N}$-algebra A $=$ semiring $=$ "ring without negatives"
4. A-modules, A-algebras, $\oplus$, $\times$, Hom, $\otimes$, base change, ...
5. $A \to B$. An A-model of a B-module N is an A-module M plus an isomorphism $\alpha\colon B \otimes_A M \to N$. Similarly for algebras, etc
6. There should be a beautiful world of algebraic geometry over $\mathbf{N}$, combining arithmetic algebraic geometry (over $\mathbf{Z}$) and semi-algebraic geometry (over $\mathbf{R}_{\geq 0}$).
7. Not many interesting theorems are known yet!
8. Today: Witt vectors and lambda-rings over $\mathbf{N}$ and $\mathbf{R}_{\geq 0}$.

   ~~Commuting Frobenius lifts~~ Combinatorics and positivity

Big Witt and Λ for rings, the formal theory

1. $W(A) = \{1 + a_1 t + a_2 t^2 + \cdots | a_i \in A\}$,
   addition on LHS := multiplication on RHS,
   multiplication on LHS determined by
   $(1 + at) * (1 + bt) = (1 + abt)$, functoriality, t-adic continuity
2. The functor W is represented by $\Lambda = \mathbf{Z}[e_1, e_2, \ldots]$, $e_i \mapsto a_i$
3. Clarifying point of view: If we write

   $$e_1 = x_1 + x_2 + \cdots, e_2 = x_1 x_2 + x_1 x_3 + \cdots, \ldots$$

   then Λ = all symmetric functions in $x_1, x_2, \ldots$
4. Power sum/Adams/Frobenius symmetric functions
   $\psi_n = x_1^n + x_2^n + \cdots$
5. Buium's p-derivations $\delta_p = \frac{1}{p}(\psi_p - \psi_1^p)$, p prime

The formal theory, continued

6. The ring structure on $W(A)$ is given by change of variables

$$\Delta^{+}\colon \Lambda \to \Lambda \otimes \Lambda,$$
$$f(\ldots, x_i, \ldots) \mapsto f(\ldots, x_i \otimes 1, 1 \otimes x_i, \ldots)$$

$$\Delta^{\times}\colon \Lambda \to \Lambda \otimes \Lambda,$$
$$f(\ldots, x_i, \ldots) \mapsto f(\ldots, x_i \otimes x_j, \ldots).$$

7. The comonad structure on W is "plethysm", which is determined by substituting monomials in variables

$$\Lambda \times \Lambda \xrightarrow{\circ} \Lambda$$
$$f \circ (m_1 + m_2 + \cdots) = f(m_1, m_2, \ldots).$$

$\Lambda_{\mathbf{N}}$

1. So Λ is a ~~plethory~~ composition algebra over $\mathbf{Z}$ (=composition object in the category of $\mathbf{Z}$-algebras), a collection of abstract operators which "knows how" to act on rings. $\rightsquigarrow$ Λ-rings
2. Main point: Λ descends to $\mathbf{N}$ as a composition algebra.
3. (Equivalently: it's possible to extend W to the category of $\mathbf{N}$-algebras, not just as a functor but as a representable comonad.)
4. $\rightsquigarrow$ $\Lambda_{\mathbf{N}}$-semirings
5. $\Lambda_{\mathbf{N}} = \{f \in \Lambda \mid \text{the coefficient of every monomial of } f \text{ is } \geq 0\}$
6. Proof: Clearly $\mathbf{Z} \otimes_{\mathbf{N}} \Lambda_{\mathbf{N}} = \Lambda$. Then observe that $\times$ preserves $\Lambda_{\mathbf{N}}$. So do Δ^{+}, $\Delta^{\times}$, and $\circ$ because they're given by simple changes of variables.

Witt vectors and positivity

1. For any $\mathbf{N}$-algebra, write $W(A) = \mathrm{Hom}(\Lambda_{\mathbf{N}}, A)$ of course.
2. If A is a ring, this agrees with the usual W:
$$\mathrm{Hom}_{\mathbf{N}}(\Lambda_{\mathbf{N}}, A) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z} \otimes_{\mathbf{N}} \Lambda_{\mathbf{N}}, A) = \mathrm{Hom}_{\mathbf{Z}}(\Lambda, A)$$
3. $W(A)$ has an $\mathbf{N}$-algebra structure induced by $\Delta^{+}, \Delta^{\times}$
4. W is a comonad: $W(A) \to W(W(A))$
5. If A has additive cancellation, then $W(A)$ is a sub-semiring of the ring of Witt vectors $W(\mathbf{Z} \otimes_{\mathbf{N}} A)$. It is the set of series $1 + a_1 t + a_2 t^2 + \cdots$ such that for all $P(e_1, e_2, \dots) \in \Lambda_{\mathbf{N}}$ we have $P(a_1, a_2, \dots) \in A$.

Λ_{Sch}

1. There is another model for Λ over $\mathbf{N}$!
2. $\Lambda_{\mathbf{N}}$ is the $\mathbf{N}$-linear span of the $\mathbf{Z}$-basis of Λ consisting of the "monomial" symmetric functions

   $$m_\lambda = m_{(\lambda_1,\ldots,\lambda_l)} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l} + \text{all permutations (no multiplicity)}$$

3. Now we'll do the same thing but with the basis consisting of the "Schur polynomials" s_λ:

   $$\Lambda_{\mathrm{Sch}} = \bigoplus_\lambda \mathbf{N} s_\lambda$$

4. What is s_λ? Jacobi–Trudi formula, e.g.,

   $$s_{(3,1,1,1)} = s_{(4,1,1)'} = \det \begin{pmatrix} e_4 & e_5 & e_6 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{pmatrix}$$

   ($e_0 = 1$ and $e_{-1} = e_{-2} = \cdots = 0$)
5. Representation-theoretic definition: There is a standard isomorphism $\Lambda \cong \bigoplus_{n\geq 0} K(S_n)$. The Schur polynomials correspond to the irreducible representations.

Λ_{Sch}, continued

6. Theorem: Λ_{Sch} is a model for Λ over **N** as a composition algebra, and we have $\Lambda_{\mathrm{Sch}} \subset \Lambda_{\mathbf{N}}$.
7. Proof: All coefficients are ≥ 0 below

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \quad \text{Littlewood–Richardson coefficients}$$

$$\Delta^+(s_\lambda) = \sum_\nu c^\lambda_{\mu\nu} s_\mu \otimes s_\nu \quad \text{Littlewood–Richardson coefficients}$$

$$\Delta^\times(s_\lambda) = \sum_\nu \gamma^\lambda_{\mu\nu} s_\mu \otimes s_\nu \quad \text{Kronecker coefficients}$$

$$s_\lambda \circ s_\mu = \sum_\nu a^\nu_{\lambda\mu} s_\nu \quad \text{a coefficients}$$

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \quad \text{Kostka numbers}$$

8. These are standard positivity facts in combinatorics.

Total positivity

1. A series $1 + e_1 t + e_2 t^2 + \cdots \in W(\mathbf{R}) = 1 + t\mathbf{R}[[t]]$ is totally positive if the infinite matrix

$$\begin{pmatrix} 1 & e_1 & e_2 & e_3 & \cdots \\ 0 & 1 & e_1 & e_2 & \cdots \\ 0 & 0 & 1 & e_1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

   is totally positive, i.e., all its minors are ≥ 0.
2. Fact: $W_{\mathrm{Sch}}(\mathbf{R}_{\geq 0}) =$ the totally positive series in $1 + t\mathbf{R}[[t]]$
3. So the Schur positivity structure on W is part of a well-studied area.

Total positivity, continued

4. Theorem (Edrei–Thoma): $W_{\mathrm{Sch}}(\mathbf{R}_{\geq 0})$ consists of series of the form
$$e^{\gamma t}\frac{\prod_i (1+\alpha_i t)}{\prod_i (1-\beta_i t)},$$
where $\gamma, \alpha_i, \beta_i \geq 0$, $\sum_i \alpha_i < \infty$, $\sum_i \beta_i < \infty$.
5. Proof: Nevanlinna theory. Non-trivial.
6. This theorem + a short argument $\Rightarrow$
$W(\mathbf{R}_{\geq 0}) = \{$series as above such that all $\beta_i = 0\}$.
7. Consequence: the Frobenius operators ψ_p on $W(\mathbf{R}_{\geq 0})$ interpolate to a continuous family ψ_s for $s > 1$ defined by
$$\psi_s \colon e^{\gamma t}\prod_i (1+\alpha_i t) \mapsto \prod_i (1+\alpha_i^s t).$$
8. ψ_s is a Frobenius flow! Could this be the holy grail?

Questions about W

1. "Calculate" $W(A)$ and $W_{\mathrm{Sch}}(A)$ for concrete $\mathbf{N}$-algebras A:

   $\mathbf{R}_{\mathrm{trop}}, \quad \mathbf{N}/(n+1=n), \quad$ etc.

2. E.g. Is $W_{\mathrm{Sch}}(\mathbf{N}/(1+1=1))$ countable or uncountable?

   (It is infinite. Ex: $\mathbf{N} \to W_{\mathrm{Sch}}(A)$ is injective $\Leftrightarrow A \neq 0$.)

3. If $A \to B$ is surjective, must $W_{\mathrm{Sch}}(A) \to W_{\mathrm{Sch}}(B)$ be surjective? Must $W(A) \to W(B)$?

4. Is the natural map $W(A) \to W_{\mathrm{Sch}}(A)$ always injective?

   (It obviously is if A has additive cancellation.)

5. Let $f \in \Lambda$ be a symmetric function at which every $a \in W_{\mathrm{Sch}}(\mathbf{R}_{\geq 0}) \subset \mathrm{Hom}(\Lambda, \mathbf{R})$ satisfies $a(f) \geq 0$. Does it follow that $f \in \Lambda_{\mathrm{Sch}}$? Same for $\Lambda_{\mathbf{N}}$ and $W(\mathbf{R}_{\geq 0})$.

6. Is there a natural class of $\mathbf{N}$-algebras A for which $W(A)$ has a continuous Frobenius ψ_s?

7. Classify the functorial additive operations on W_{Sch} and W. I.e., is there a Cartier theorem over $\mathbf{N}$?

Questions about Λ

1. Are $\Lambda_{\mathbf{N}}$ and Λ_{Sch} the only (flat, free, ...) $\mathbf{N}$-models for Λ?
2. Over $\mathbf{Q}_{\geq 0}$, there is a third model for $\mathbf{Q} \otimes \Lambda$, namely $\mathbf{Q}_{\geq 0}[\ldots, \psi_n, \ldots]$. Is there a fourth?
3. Do the algebras $\Lambda_n = \mathbf{Z}[e_1, \ldots, e_n]$ representing the truncated Witt vectors have models over $\mathbf{N}$ (with all but the $\circ$ structure)?
4. Does the composition algebra of p-typical symmetric functions $\Lambda^{(p)} = \mathbf{Z}[\ldots, \delta_p^{\circ n}, \ldots]$ have model over $\mathbf{N}$?
5. Let K be a number field embedded in $\mathbf{R}$. Does $\Lambda_{\mathcal{O}_K}$ (to be defined in Lance Gurney's talk) have a model over $\mathcal{O}_K \cap \mathbf{R}_{\geq 0}$? ($\Lambda_{\mathbf{Q}} = \Lambda$, so yes if $K = \mathbf{Q}$.)
6. Which of the familiar Λ-rings descend to $\mathbf{N}$? I.e. which are the base change of (flat) $\Lambda_{\mathbf{N}}$-semirings or Λ_{Sch}-semirings?