Differential lifts and differential symmetries

Alexandru Buium

Department of Mathematics and Statistics
University of New Mexico
buium@math.unm.edu

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Moral of talk

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Algebraic geometry in characteristic p

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Algebraic geometry in characteristic zero

↑ “wrong” lifts

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δ-geometry (characteristic zero)

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δ-geometry (characteristic zero)

$\downarrow$ using lifts as central fibers

δ-arithmetic deformation theory ?

Plan of talk

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Review of δ-geometry

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Review of δ-geometry

Review of δ-symmetry

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Review of δ-geometry

Review of δ-symmetry

6 examples in search of a general theory

δ

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Example: $p = 7$; $\delta 5 =$ "$\frac{d5}{d7}$" $= \frac{5 - 5^7}{7}$

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More generally δ-ring: a ring A with $\delta : A \to A$ satisfying axioms of the above δ; for A torsion free this means $\phi(x) = x^p + p\delta x$ is a ring homomorphism.

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Example $f : \mathbb{A}^1(R) = R \to R$, $f(x) = \sum_{n \geq 1} p^n x^n (\delta x)^{n^3} (\delta^2 x)^{n^n}$, δ-function of order 2

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3) $\overline{B}/\overline{A}$ a Γ-extension (in the usual sense)

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The homomorphism property should be viewed as an "extra-symmetry" of the function ψ

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$I^n(w) \subset M^n(w)$ space of isogeny covariant forms ("Hecke semi-invariant")

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$$f(q_1, ..., q_1^{(n)}) + ... + f(q_p, ..., q_p^{(n)}) = F(s_1, ..., s_p, ..., \delta^n(s_1), ..., \delta^n(s_p))$$

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where $s_1 = \sum_i q_i$, $s_2 = \sum_{i<j} q_i q_j$, $\ldots$, $s_p = q_1 \ldots q_p$.

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2. There exists $f^\partial \in M^1(\phi - 1)$ with δ-Fourier expansion 1

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1. There exists $f^1 \in M^1(-1-\phi)$ with $\delta - p-$symmetric expansion $\frac{1}{p}\log\left(1 + p\frac{q'}{q^p}\right)$
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1. f^1 above can be viewed as a lift to characteristic zero in δ-geometry of the Kodaira-Spencer class $\frac{2AdB-3BdA}{\Delta}$ of the universal family of elliptic curves $y^2 = x^3 + Ax + B$; the reduction mod p of f^1 is $H\frac{2AdB-3BdA}{\Delta} +$ (order zero terms) where H is the Hasse invariant.

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2. f^{∂} mod p is H so f^{∂} can be viewed as a lift to characteristic zero in δ-geometry of H; the lift E_{p-1} should be viewed as a wrong lift because it is not isogeny covariant.

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3. 6 and 7 above can be viewed as lifts to characteristic zero in $\delta-$geometry of the Serre-Swinnerton Dyer theorem on the kernel of the Fourier expansion map $\overline{M} \rightarrow k((q))$ in characteristic p.

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The theorem above can be viewed a lift to characteristic zero in $\delta-$geometry of the fact that the Igusa curve $I_1(N)_{/\mathbb{F}_p}$ in characteristic p is a $\mathbb{F}_p^{\times}-$cover of the modular curve $X_1(N)_{/\mathbb{F}_p}$. $I_1(N)_{/\mathbb{F}_p}$ has another lift to characteristic zero in usual algebraic geometry; that lift is a $\mathbb{F}_p^{\times}-$cover of the modular curve $X_1(N)_{/\mathbb{Z}_p}$ and should be viewed as the wrong one because the symmetry group is too small

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If $f = \sum a_n q^n$ is a new form of weight 2 over $\mathbb{Z}$ then the form $f^\sharp \in M^2(0)$ obtained by composing the Eichler-Shimura map $\Phi : X_1(N)(R) \to A_f(R)$ with the homomorphism $\psi : A_f(R) \to R$ has a δ-Fourier expansion that is $\delta - p-$symmetric and that is congruent mod p to

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Remark

$\sum n^{p-2} a_n q^n$ has a remarkable lift to characteristic zero as a p-adic modular form a la Serre, $\sum \frac{a_n}{n} q^n$; this should be viewed as a wrong lift while $f^\sharp$ above should be viewed as the "correct" lift because it has more symmetry "with respect to Heegner points"

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f^0 should be viewed as the "correct" lift to characteristic zero of Serre's weight one modular form mod p on the Igusa curve a; we skip this discussion; cf. B+Saha, JNT 2012

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there is evidence of connections between the above manifestations of δ-symmetry which may indicate the presence of a common concept

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There is evidence that Σ_δ/k could be central fibers for deformations in a (yet to be developed) δ-arithmetic deformation theory