A Torsor of Lifts of The Frobenius

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POINT OF THIS TALK

Theorem (Dupuy 2012). For curves over (+technical hypoth) the sheaf of formal lifts of the Frobenius are a torsor under a line bundle.

(Amounts to proving that the first p-jet space of a curve is a is an affine bundle with an "affine linear structure.")

(Statement really says that local lifts of the Frobenius are parametrized by something linear)

The Result

Theorem. Let X over $W_{p^\infty}(\overline{\mathbb{F}}_p)$ be a smooth projective curve of genus $g \geq 1$. If $p > 6g - 5$ then $J^1_p(X)$ is a torsor under a line bundle.

Remarks on the Proof

- Prove reduction of the structure group of the first p-jet spaces to the affine linear group.
- Uses are "pairing" between group cohomology and cech cohomology together with vanishing theorems to sucessively reduce the structure group.

Remarks on p-jets

- Introduced by Buium to study Lang conjecture in the Arithmetic setting --- using intersections in p-jets
- The nth p-jet space of a scheme is a scheme (or p-formal scheme) whose Zariski closed subsets correspond to Kolchin closed sets of the original scheme.
- Kolchin closed are just sets cut-out by "Arithmetic Differential Equations".

What is a p-derivation?

Fermat's Little Theorem

$\forall n \in \mathbb{Z}, \forall p$ prime

$$n \equiv n^p \mod p$$

$$n - n^p = p \cdot \text{CRAP}$$

This is a p-derivation

$$\text{CRAP} = \frac{n - n^p}{p}$$

$$\delta_p(n) = \frac{n - n^p}{p}$$

The Frobenius

$$F : A/p \mapsto A/p$$
$$a \mapsto a^p$$

Properties of p-derivations

Zero mod p

$$\delta_p(n) = \frac{n - n^p}{p}$$

Zero mod p

Proposition

Product Rule

$$\delta_p(ab) = \delta_p(a) b^p + a^p \delta_p(b) + p\delta_p(a)\delta_p(b)$$

Sum Rule

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j} b^j$$

non-linear

(Buium, Joyal ~1994)

Abstract Definition: $\delta_p : A \to B$ is a **p-derivation** provided that

Always an A algebra

Product Rule:

$$\delta_p(ab) = \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b)$$

Sum Rule:

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j}b^j$$

Proposition:

For $\delta_p : \mathbb{Z} \to \mathbb{Z}$
defined by $\delta_p(n) = \frac{n - n^p}{p}$

show $\delta_p(p^m) = p^{m-1} \cdot \ (\text{ unit mod } p\ )$

Example:

$$\delta_p(p) = \frac{p - p^p}{p}$$
$$= 1 - p^{p-1}$$

Idea: order of vanishing is "bumped down"

$$\delta_t = \frac{d}{dt}$$
$$\delta_t(t^n) = n \cdot t^{n-1}$$

derivations ⟷ ring homomorphisms

$\delta : A \to A$ ⟷ $f : A \to A[\varepsilon]/\langle \varepsilon^2 \rangle$

"dual numbers"

"infinitesimals"

p-derivations ⟷ ring homomorphisms

$\delta_p : A \to A$ ⟷ $f : A \to W_1(A)$

"Witt vectors"

"wittfinitesimals"

"Wittferentiation"

Analogies

Dual Numbers

$D_1(A) = A[t]/\langle t^2 \rangle$

Truncated Witt Vectors

$W_1(A)$

Power Series

$D(A) = A[[t]]$

Witt Vectors

$W(A)$

(p-typical, not like in Jim's Talk)

Lifts of the Frobenius

Definition: A lift of the Frobenius is a ring homomorphism $\phi : A \to B$ such that

$$\phi(a) \equiv a^p \mod p$$

Proposition: If $\delta_p : A \to B$ is a p-derivation then

$$\phi(a) := a^p + p\delta_p(a)$$

is a lift of the Frobenius.

Conversely, if B is p-torsion free ring with a lift of the Frobenius $\phi : A \to B$ then

$$\delta_p(a) := \frac{\phi(a) - a^p}{p}$$

defines a p-derivation.

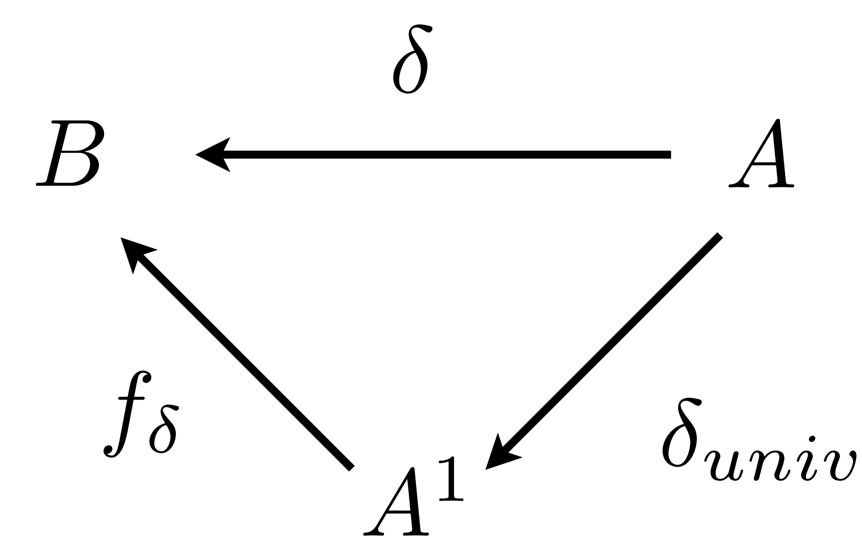
p-Jets I

$$A^1 = \Lambda_{p,1} \odot A = \mathcal{O}(J_p^1(\mathrm{Spec}(A)))$$

$$= \frac{A[\dot{a} : a \in A]}{(\text{relations for } p\text{-derivations})}$$

$$\delta_{univ} : A \to A^1$$

Universal Property:



p-Jets 2

Setup: $W = W(\overline{\mathbb{F}}_p) = \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$

X/W smooth

first p-jet space

$$\exists \quad J_p^1(X) \xrightarrow{\pi} X \supset U$$

s

local sections ⟷ local lifts of the Frobenius

$\phi : U \to U$

$$\{ \text{ lifts of Frobenius on } U \ \} = \Gamma(U, J_p^1(X))$$

How do we get a Torsor structure?

Philosophy: p-jets know all about lifts of the Frobenius.

$$\{ \text{ lifts of Frobenius on } U \ \} = \Gamma(U, J^1_p(X))$$

Recipe for torsor structure:

Step 1: Show that the first p-jet space of a smooth variety is an affine bundle which admits an additional structure.

Step 2: Reduce the structure group to the "affine linear group" (which is equivalent to being a torsor under a line bundle)

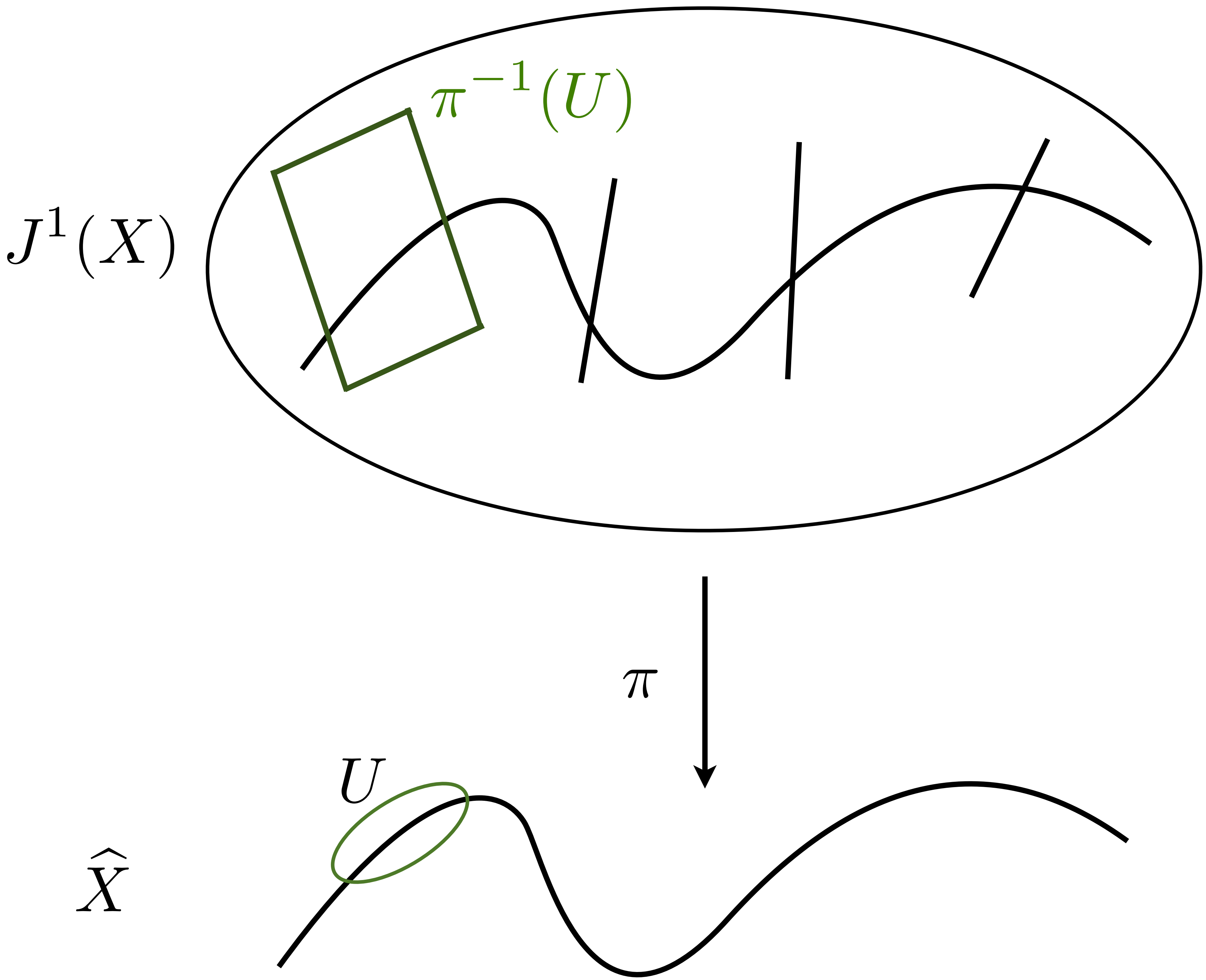
Defn. An affine bundle is a fiber bundle with fibers $\mathbb{A}^n$

Lemma. For $X/W(\overline{\mathbb{F}}_p)$ smooth the *p-adic completion* of the first p-jet space $\widehat{J^1_p(X)}$ is an affine bundle.

obligatory bundle diagram

$$\begin{array}{ccc} \widehat{J^1_p(X)} \supset \pi^{-1}(\widehat{U}) & \xrightarrow{\;\cong\;} & \widehat{U}\,\hat{\times}\,\widehat{\mathbb{A}}^n \\ \pi \big\downarrow & & \\ \widehat{X} \supset \widehat{U} & & \end{array}$$

obligatory bundle picture



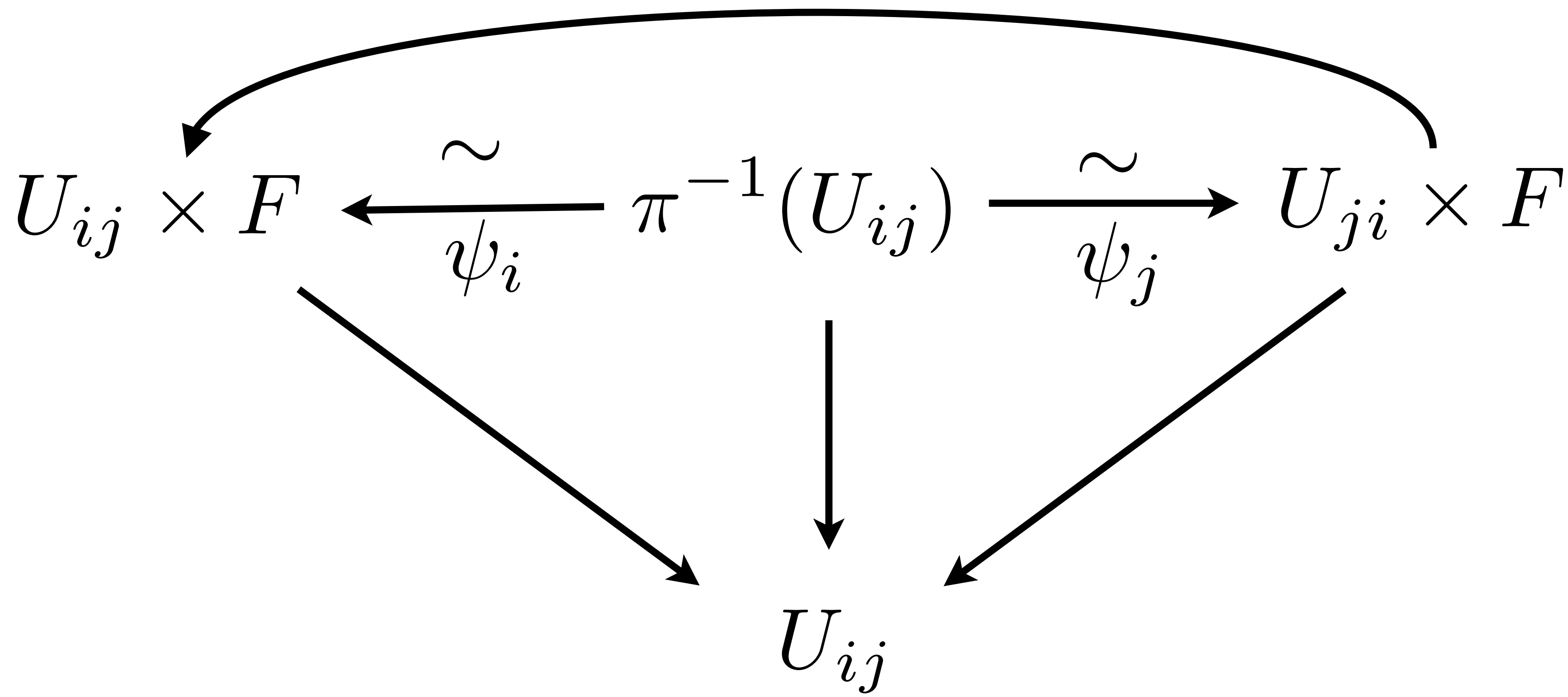
Fix an F-bundle E and a trivializing cover

$$
\begin{array}{ccc}
E \supset \pi^{-1}(U_i) & \xrightarrow[\psi_i]{\sim} & U_i \times F \\
\downarrow \pi & & \\
X \supset U_i & &
\end{array}
$$

$$\psi_{ij} := \psi_i \circ \psi_j^{-1} \in \underline{\mathrm{Aut}}(F)(U_{ij})$$

$$
\begin{array}{ccccc}
U_{ij} \times F & \xleftarrow[\psi_i]{\sim} & \pi^{-1}(U_{ij}) & \xrightarrow[\psi_j]{\sim} & U_{ji} \times F \\
& \searrow & \downarrow & \swarrow & \\
& & U_{ij} = U_i \cap U_j & &
\end{array}
$$

$$\psi_{ij} := \psi_i \circ \psi_j^{-1} \in \underline{\mathrm{Aut}}(F)(U_{ij})$$



cohomology class

$$\rightsquigarrow [\psi_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(F))$$

Our Particular Application

$$\begin{array}{ccc} J^1(X) \supset \pi^{-1}(\widehat{U}_i) & \xrightarrow[\psi_i]{\sim} & \widehat{U}_i \hat{\times} \widehat{\mathbb{A}}^{m} \\ \Big\downarrow \pi & & \\ \widehat{X} \supset \widehat{U}_i & & \end{array}$$

$$m = \dim(X)$$

$$\rightsquigarrow \quad \beta := [\psi_{ij}] \in H^1(\widehat{X}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^m))$$

Controls "Deligne-Illusie class"

Information Modulo p:

Theorem

$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

B: $\mathrm{DI}_0(\delta) \to a_{ij}$

A: $[F^* T_{X_0}] \to b_{ij}T$

idea used in A

$$\mathrm{AL}_1 = \underline{\mathrm{Aut}}(\mathbb{A}^1_{\mathbb{F}_p}) \xrightarrow{\sim} \mathcal{O}_{X_0} \rtimes \mathcal{O}^\times_{X_0}$$

$$a + bT \circ c + dT = a + bc + bdT \qquad (a, b) \cdot (c, d) = (a + bc, bd)$$

$$\mathcal{O}_{X_0} \rtimes \mathcal{O}^\times_{X_0} \xrightarrow{\pi} \mathcal{O}^\times_{X_0}$$

$$H^1(X, \mathcal{O}_X \rtimes \mathcal{O}^\times_X) \xrightarrow{\pi} H^1(X, \mathcal{O}^\times_X)$$

$$\pi(\beta_0) = [b_{ij}] = [F^* T_{X_0}]$$

■

Conventions

$$\varphi_i : \mathcal{O}(U_i) \to L(U_i)$$

$$\varphi_i(1) = v_i$$

$$b_{ij} v_i = v_j$$

$$[L] = [b_{ij}]$$

Remarks

- The fibers for the bundle structure on the first p-jet spaces are affine spaces
- The transition maps are univariate polynomial automorphisms
- The structure of these groups is extremely rich (p-formally or mod p^n)
- Similar “Arithmetic Kodaira-Spencer” classes originally introduced by Buium for Abelian Varieties

Ideas for Mod p^n

- Twisted (semi-direct product) cech-cocycles give cocycles in line bundles.
- Group cocycles applied to Cech cocycles produce twisted cocycles
- Vanishing theorems for line bundles give information about our twisted cocycles.
- “Triviality” of twisted cocycles allows us to reduce our structure group

Twisted Cocycles

Working out what a Cech cocycle in $\mathcal{O} \rtimes \mathcal{O}^\times$ looks like gives

$$(a_{ij}, b_{ij})(a_{jk}, b_{jk})(a_{ki}, b_{ki}) = 1$$

which gives

$$\begin{aligned} b_{ij}b_{jk}b_{ki} &= 1 \\ a_{ij} + b_{ij}a_{jk} + b_{ij}b_{jk}a_{ki} &= 0 \end{aligned}$$

Alternatively, one can get such pairs from a Cech cocycle for the line bundle $[b_{ij}] \in \mathrm{Pic}(Y) = H^1(Y, \mathcal{O}^\times)$

Strategy: Produce twisted cocycles from cocycles with values in more complicated groups to study them.

Structures!

Back to the abstract setting with an arbitrary fiber bundle

- Fix an F-bundle and subgroup $H \leq \underline{\mathrm{Aut}}(F)$
- An H-atlas is a trivializing cover whose transition maps lie in the subgroup

$$\{(U_i, \psi_i)\} \quad = \quad H\text{-atlas}$$

$$\psi_{ij} \in H(U_{ij})$$

- A H-structure is a maximal H-atlas.

Degree Structures

Naturally Occuring Structure Group

$$\{a_0 + a_1 T + p a_2 T^2 + \cdots + p^{n-1} a_n T^n \mod p^n\} \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{W/p^n})$$
$$= A_n$$

Affine Linear Group

$$\{a + bT \mod p^n\} \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{W/p^n})$$

Group cocycles that are involved

example: cocycle we use to get twisted cocycle from an A_d structure.

$$\psi(T) \mapsto \psi''(T)/\psi'(T) \mod p$$

NEW multivariate version!

$$C[f] = ((df^{-1})^j{}_l (d^2 f)^l{}_{jk})$$

example: in dimension 2

$$C_1[\psi] = \frac{f_{xx}g_y - f_{xy}g_x - g_{xx}f_y + g_{xy}f_x}{f_x g_y - g_x f_y}$$

$$C_2[\psi] = \frac{f_{xy}g_y - f_{yy}g_x - g_{xy}g_y + g_{yy}f_x}{f_x g_y - g_x f_y}$$

THE END