Witt-Burnside rings attached to $\mathbf{Z}_p^2$ as Lipschitz functions on $\mathbf{P}^1(\mathbf{Q}_p)$

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Introduction

This talk discusses generalizations of Witt vectors and how to recognize what kinds of rings they produce.

- Review the Dress and Siebeneicher's generalization involving profinite groups.
- Review the structure of Witt vectors attached to $\mathbf{Z}_p^d$ for $d \geq 2$.
- Give a concrete interpretation of a quotient in the $d = 2$ case in terms of $\mathbf{W}(k)$-valued functions on $\mathbf{P}^1(\mathbf{Q}_p)$.

Review of p-typical Witt Vectors

In $\mathbf{W}(A) = \{(a_0, a_1, \ldots) : a_i \in A\}$, the ring operations are

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (s_0(a_0; b_0), s_1(a_0, a_1; b_0, b_1), \ldots),$$

$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (m_0(a_0; b_0), m_1(a_0, a_1; b_0, b_1), \ldots),$$

where the polynomials s_i and m_i are defined via Witt polynomials

$$W_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n$$

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by solving for $\mathbf{s} = (s_0, s_1, \dots)$ and $\mathbf{m} = (m_0, m_1, \dots)$ in equations

$$W_n(\mathbf{x}) + W_n(\mathbf{y}) = W_n(\mathbf{s}), \quad W_n(\mathbf{x}) W_n(\mathbf{y}) = W_n(\mathbf{m})$$

for all $n \geq 0$. The s_i's and m_i's easily have $\mathbf{Z}[1/p]$-coefficients, but in fact have $\mathbf{Z}$-coefficients (Witt).

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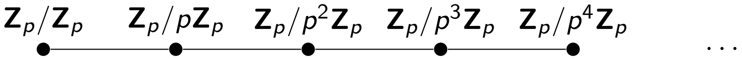
Example. The discrete transitive $\mathbf{Z}_p$-sets are $\mathbf{Z}_p/p^n\mathbf{Z}_p$ for $n \geq 0$. The frame of $\mathbf{Z}_p$ is like $\mathbf{N} = \{0, 1, 2, \dots\}$ or $\{1, p, p^2, \dots\}$.



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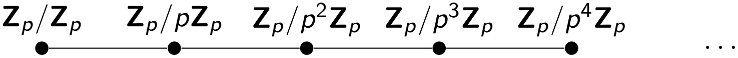
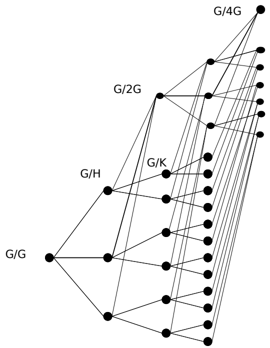


More generally, the frames of $\mathbf{Z}_p$ or $\mathbf{Z}/p^n\mathbf{Z}$ are totally ordered, but the frame of $\mathbf{Z}_p^2$ is not totally ordered...

Visualizing the Frame of $\mathbf{Z}_2^2$

The frame of $G = \mathbf{Z}_2^2$ is much more intricate than for $\mathbf{Z}_2$!



For pro-p G besides $\mathbf{Z}_p$ and $\mathbf{Z}/p^n\mathbf{Z}$, there is more than one (open) subgroup of index p.

Defining Generalized Witt Polynomials

For any discrete transitive G-set T, define the T-th Witt polynomial in $\mathbf{Q}[\ldots, x_T, \ldots] = \mathbf{Q}[\mathbf{x}]$ to be

$$W_T(\mathbf{x}) = \sum_{U \leq T} \#\mathrm{Map}_G(T, U) x_U^{\#T/\#U} = x_0^{\#T} + \cdots + \#\mathrm{Aut}_G(T) x_T.$$

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Therefore given independent variables $\mathbf{x} = (x_T)$ and $\mathbf{y} = (y_T)$, in $\mathbf{Q}[\mathbf{x}, \mathbf{y}]$ there are unique polynomial sequences $\mathbf{s} = (s_T(\mathbf{x}, \mathbf{y}))$ and $\mathbf{m} = (m_T(\mathbf{x}, \mathbf{y}))$ satisfying

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These polynomials s_T and m_T were defined by Dress and Siebeneicher (1988), who showed they have $\mathbf{Z}$-coefficients.

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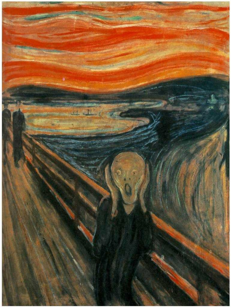
Cartier's "big" Witt vectors are recovered by $\mathbf{W}_{\widehat{\mathbf{Z}}}(A)$.

The Problem with Witt Vectors

Witt vector ring operations are a nightmare to work with explicitly.



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If $G = \mathbf{Z}_p^d$ for $d \geq 2$ and k is any field of characteristic p, the maximal ideal $\mathfrak{m}$ of $\mathbf{W}_G(k)$ is not finitely generated. In the case $d = 2$, $\mathbf{W}_G(k)$ is reduced.

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There is no concrete interpretation of $\mathbf{W}_G(k)$ in terms of known rings; not even for $G = \mathbf{Z}_p^2$.

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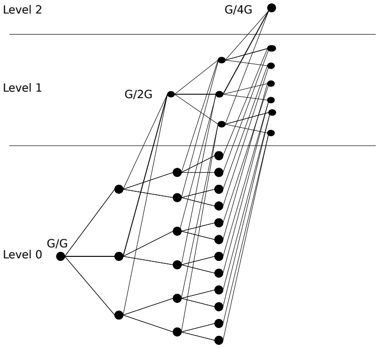
There is a natural family of prime ideals in $\mathbf{W}_{\mathbf{Z}_p^2}(k)$. In the diagram, we illustrate with $p = 2$.



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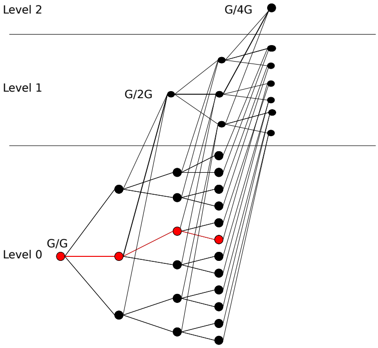

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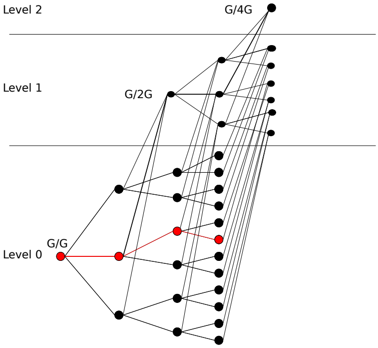

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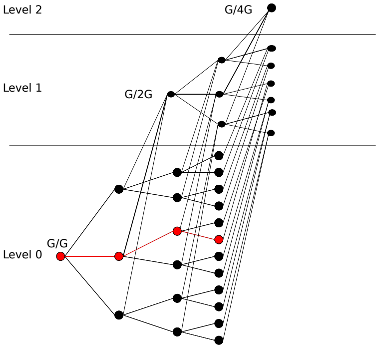

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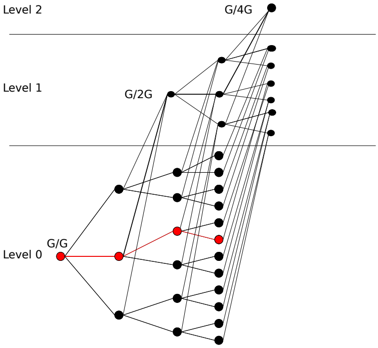

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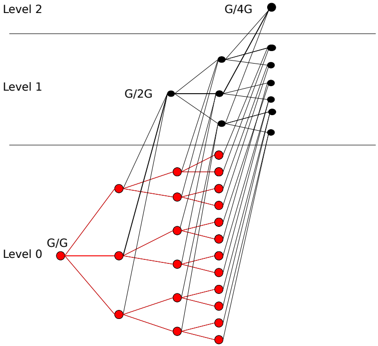

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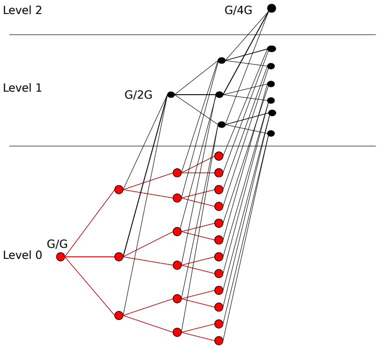

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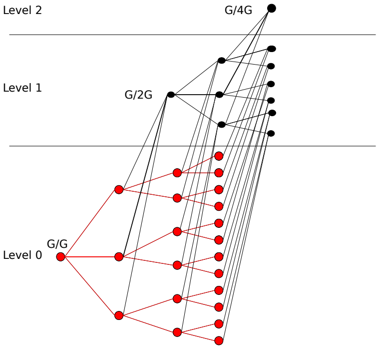

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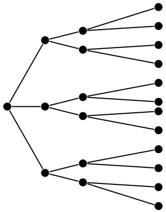
Interpreting $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ as functions

Set J the ideal of Witt vectors which are zero on the bottom tree portion of the frame of $\mathbf{Z}_p^2$. So $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ is the ring of "Witt vectors" on the tree:



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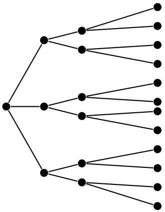
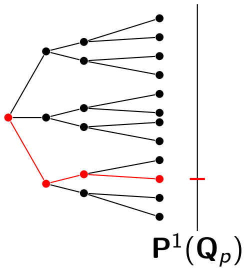


We can embed $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ into a more 'familiar' ring. The boundary of this tree, i.e., the space of all rooted paths.

Interpreting $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ as functions

The set of all rooted paths in this case is $\mathbf{P}^1(\mathbf{Q}_p)$.



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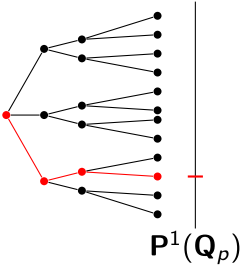

From a Witt vector $\mathbf{a} \in \mathbf{W}_{\mathbf{Z}_p^2}(k)/J$, we can associate a function on $\mathbf{P}^1(\mathbf{Q}_p)$ with values in $W(k)$.

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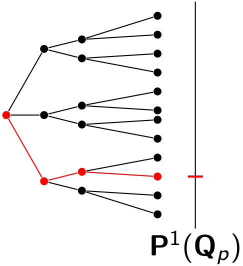

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$$\Phi\colon \mathbf{W}_{\mathbf{Z}_p^2}(k)/J \to \mathrm{Fun}(\mathbf{P}^1(\mathbf{Q}_p), W(k)),$$

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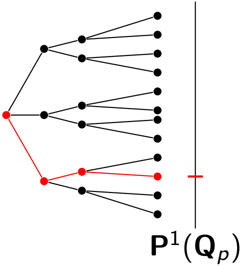

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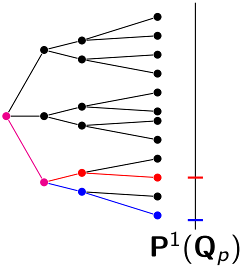
A norm on $\mathbf{P}^1(\mathbf{Q}_p)$

For two points in $\mathbf{P}^1(\mathbf{Q}_p)$, i.e., rooted paths defined the distance



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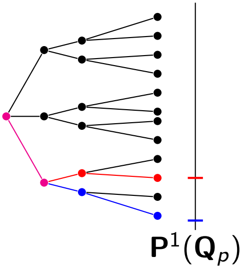
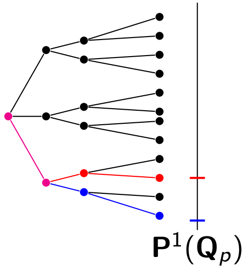


We measure distance between paths by counting number of edges in the overlap. So $|x-y| = p^{-n}$ where n is the number of edges of overlap.

A norm on $\mathbf{P}^1(\mathbf{Q}_p)$

For two points in $\mathbf{P}^1(\mathbf{Q}_p)$, i.e., rooted paths defined the distance



We measure distance between paths by counting number of edges in the overlap. So $|x - y| = p^{-n}$ where n is the number of edges of overlap.

This makes $\mathbf{P}^1(\mathbf{Q}_p)$ a metric space with diameter 1.

Interpreting $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ as functions

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We expect to exploit this to show that $\dim \mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ is infinite.

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Counting initial vanishing gives a norm, and hence a metric on $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$.

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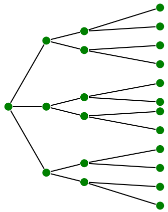

Elements of norm 1

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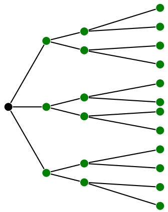

Elements of norm p^{-1}

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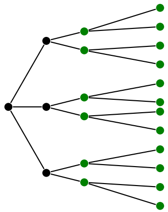

Elements of norm p^{-2}

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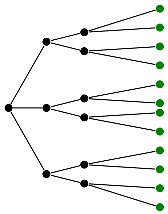

Elements of norm p^{-3}

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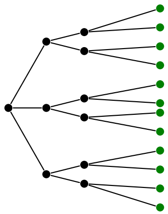

Elements of norm p^{-3}

This recovers the usual Witt-vector metric on paths.

Φ is an isometry

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Theorem (M, Steinhurst)

In the initial vanishing norm on $\mathbf{W}_{\mathbf{Z}_p^2}(k)/J$ and the sup norm on $p^{-1} - Lip(\mathbf{P}^1(\mathbf{Q}_p), \mathbf{W}(k))$, Φ is an isometry.

Thank you.
Questions?