The Oort Conjecture and the local lifting problem

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Outline

1. The problem
2. The proof
3. Further questions

Setup

- Notation:
  - p a prime number.
  - k an algebraically closed field of characteristic p.
  - G a finite group.
- Let G act faithfully on $k[[z]]$ by (z-adically) continuous k-algebra automorphisms.
  - Note: the action of each $g \in G$ is specified by $g(z)$, which is a power series of the form

$$a_1 z + \sum_{i=2}^{\infty} a_i z^i,$$

and $a_1 \neq 0$.

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The local lifting problem

- The local lifting problem (LLP) asks: Does there exist a finite extension $R/W(k)$, and a local action of G on $R[[Z]]$ by R-algebra automorphisms, which lifts the given action on $k[[z]]$?
- If any such R exists, the local G-action is said to *lift to characteristic zero*.
- The *local Oort conjecture*: If G is *cyclic*, the local G-action lifts to characteristic zero.
- Easy to reduce to the case $G \cong \mathbb{Z}/p^n$.

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Motivation: Lifting of curves with automorphisms

- Let X/k be a smooth, projective curve, with a group action $H \hookrightarrow \text{Aut}(X)$.
- We say X (with H-action) lifts to characteristic zero if there exists $R/W(k)$ finite, X_R a smooth projective R-scheme with special fiber X, and an H-action on X_R reducing to the given H-action on X on the special fiber.
- Local-Global Principle: X (with H-action) lifts to characteristic zero iff for each $x \in X$, the action of the stabilizer H_x on $\hat{O}_{X,x} \cong k[[z]]$ lifts to characteristic zero.
- Oort Conjecture (1987): If H is cyclic, X (with H-action) lifts to characteristic zero. Equivalent to local Oort conjecture.

Theorem (Obus-Wewers, Pop, 2012)

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Example

- If G is *cyclic*, a G-action on $k[[z]]$ is given by a single power series of order $|G|$. A lift is then given by lifting this power series to a power series in $R[[Z]]$ of the same order.
- $G \cong \mathbb{Z}/4 = \langle \sigma \rangle$, $p = 2$.
  - Simplest expression is

$$\sigma(z) = z + z^2 + \sum_{j=0}^{\infty} \sum_{\ell=0}^{2^j-1} z^{6\cdot 2^j + 2\ell}.$$

  - Lifting this explicitly to something of order 4 seems prohibitively difficult.
  - When $p > 2$, hard even to write down an action of order p^2 explicitly in characteristic p, let alone lift it.

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Galois-theoretic setup

- The more fruitful approach is Galois-theoretic.
- Know $k[[z]]^G \cong k[[t]]$, and $R[[Z]]^G \cong R[[T]]$ for a lift.
- Rephrase local lifting problem: When does a G-Galois extension $k[[z]]/k[[t]]$ lift to a G-Galois extension $R[[Z]]/R[[T]]$?
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The $G = \mathbb{Z}/p$ case (Kummer-Artin-Schreier theory)

- For $G \cong \mathbb{Z}/p$, local Oort conjecture is due to Oort-Sekiguchi-Suwa (1988).
- In characteristic p, all extensions can given by taking the normalization of $k[[t]]$ in
$$k((t))[y]/(y^p - y - t^{-m}),$$
for some $m \in \mathbb{N} \backslash p\mathbb{N}$.
- Take $R = W(k)[\zeta_p]$. One finds an explicit Kummer extension of $R[[T]]$ giving the correct reduction.
- Form is suggested by the *Kummer-Artin-Schreier theory*, which gives an exact sequence interpolating the standard Kummer and Artin-Schreier exact sequences.

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- Instead of the Kummer-Artin-Schreier theory, one uses the *Kummer-Artin-Schreier-Witt theory* (AKA *Sekiguchi-Suwa* theory).
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Proof of the local Oort conjecture: the work of Obus-Wewers

Theorem (Obus-Wewers, 2012)

The local Oort conjecture is true, given a certain condition on the higher ramification filtration for $k[[z]]/k[[t]]$.

- Condition is vacuous when $G \cong \mathbb{Z}/p^3$.
- Condition satisfied when $k[[z]]/k[[t]]$ has "no essential ramification." That is, for each upper jump u_i, have $u_{i+1} < pu_i + p$.
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Proof of the local Oort conjecture: the work of Obus-Wewers (cont'd)

- Main Idea: Given a $\mathbb{Z}/p^n$-extension $k[[z]]/k[[t]]$, instead of trying to write down a lift, instead try to write down *some* Kummer extension of $R[[T]]$ that reduces to a Galois extension of $k[[t]]$.
  - In general, one gets something inseparable, but can measure the inseparability (Kato's differential Swan conductor).
  - Given a guess where the inseparability is small enough, can make a deformation to reduce it.
  - Can show that this deformation process terminates using harmonic function theory on rigid-analytic spaces.
- Once we have one lift, can deform it to get them all.
- The entire proof is an induction, which breaks down if our condition on the higher ramification groups is not satisfied.

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- Main Idea: Given a $\mathbb{Z}/p^n$-extension $k[[z]]/k[[t]]$, deform it (in equal characteristic!) to an extension whose generic fiber has no essential ramification (but might be ramified at several points).
- In some sense, the generic fiber lifts to characteristic zero, using result of Obus-Wewers.
- Use this to show that the original extension lifts over a DVR S with residue field $k((s))^{\mathrm{ac}}$.
- Use model theory (Robinson's theorem) to show that original extension lifts over some finite extension $R/W(k)$.

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- Main Idea: Given a $\mathbb{Z}/p^n$-extension $k[[z]]/k[[t]]$, deform it (in equal characteristic!) to an extension whose generic fiber has no essential ramification (but might be ramified at several points).
- In some sense, the generic fiber lifts to characteristic zero, using result of Obus-Wewers.
- Use this to show that the original extension lifts over a DVR S with residue field $k((s))^{\text{ac}}$.
- Use model theory (Robinson's theorem) to show that original extension lifts over some finite extension $R/W(k)$.

Further questions

- Local lifting problem for other groups? In particular, where G has cyclic p-Sylow.
- Effectiveness (determine R or upper bounds on R).
- Relation to moduli space of curves with automorphisms.

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- Thank you!