Character Sheaves

## Geometrizing Quasi-characters of Tori

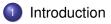
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Joint Mathematics Meetings: 2013

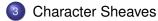
**Character Sheaves** 







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# Objective

- K a finite extension of  $\mathbb{Q}_p$ ,
- **T** an algebraic torus over K (e.g.  $\mathbb{G}_m$ ),
- $\ell$  a prime different from *p*.

### Goal

Construct "geometric avatars" for (quasi)-characters in

 $\text{Hom}(\textbf{T}(\textbf{\textit{K}}),\bar{\mathbb{Q}}_{\ell}^{\times})$  :

sheaves on some space functorially associated to T.

- Try to push characters forward along maps such as  $\mathbf{T} \hookrightarrow \mathbf{G}$ ;
- Deligne-Lusztig representations → character sheaves;
- Give a new perspective on class field theory.

# Approach

- Associate to **T** a projective system  $\mathfrak{T}$  of commutative group schemes  $\mathfrak{T}_d$  over the residue field *k* of *K*.
- 2 Define *character sheaves* on  $\mathbf{T}$  following Deligne.
- Solution Map from character sheaves on  $\mathfrak{T}$  to characters on T(K).

# The Néron model of $\mathbb{G}_m$

- **R** ring of integers of K with uniformizer  $\pi$
- $R_d R/\pi^{d+1}R$
- $T_R$  The Néron model of **T**: a separated, smooth commutative group scheme over *R*, locally of finite type with the Néron mapping property.

$$\mathbf{T}_{\boldsymbol{R}}(\boldsymbol{R}) = \mathbf{T}(\boldsymbol{K})$$

In the  $\mathbb{G}_m$  case the Néron model is a union of copies of  $\mathbb{G}_m/R$ , glued along the generic fiber.

 $\mathbf{T}_d - \mathbf{T}_R \times_R R_d.$ 

## Components

- The geometric component group of  $\mathbf{T}_R$  is  $X_*(\mathbf{T})_{\mathcal{I}_K}$ , where  $\mathcal{I}_K$  is the inertia group of K.
- π<sub>0</sub>(**T**<sub>R</sub>) is a constant group scheme after base change to the maximal unramified extension of *K*, but Frobenius may act nontrivially.
- The sequence of commutative *R*-group schemes

$$\mathbf{1} \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to \mathbf{1}$$

splits if **T** is unramified.

## The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathbf{t}_d = \operatorname{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \lim_{\leftarrow} \mathfrak{T}_d.$$

 $\mathbf{\tau}$  is a commutative group scheme over k with

$$\mathbf{T}(\mathbf{k}) = \mathbf{T}(\mathbf{K}).$$

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## Character Sheaves on ${f au}_d^\circ$

### Definition

A character sheaf on  $\mathbb{T}_d^\circ$  is an  $\ell$ -adic local system  $\mathcal{E}^\circ$  on  $\mathbb{T}_d^\circ$  so that  $m^*\mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}$ , where  $m \colon \mathbb{T}_d^\circ \times \mathbb{T}_d^\circ \to \mathbb{T}_d^\circ$  is multiplication.

Alternatively, a character sheaf on  ${\mathfrak T}_d^\circ$  is a short exact sequence

$$1 \rightarrow A \rightarrow H \rightarrow \mathbb{T}_d^{\circ} \rightarrow 1$$

together with a character  $A \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ , so that

- $H \to \mathbb{T}_d^\circ$  is a finite étale cover,
- 2  $Fr_q$  acts trivially on A.

Proposition

Base change to  $\bar{\mathbf{\tau}}_d^{\circ}$  defines an equivalence of categories

 $\mathcal{E}^{\circ} \mapsto (\bar{\mathcal{E}}^{\circ}, \mathsf{Fr}_{\mathcal{E}^{\circ}})$ 

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## Characters of $\mathbf{T}_d^{\circ}(\mathbf{k})$

Let  $x \in \mathbf{T}_d(k)$  and  $\overline{\mathcal{E}}_x^\circ$  be the stalk of  $\overline{\mathcal{E}}^\circ$  at x. Define a character  $\chi_{\mathcal{E}}^\circ$  of  $\mathbf{T}_d^\circ(k)$  by

$$\chi^{\circ}_{\mathcal{E}}(\mathbf{x}) = \mathsf{Tr}(\mathsf{Fr}_{\mathcal{E}^{\circ}}, \bar{\mathcal{E}}^{\circ}_{\mathbf{x}}).$$

Theorem (Deligne, SGA 4.5)

The map

$$\begin{aligned} \{ characters \ sheaves \ on \ \mathbf{\mathfrak{C}}_{d}^{\circ} \} &\to \operatorname{Hom}(\mathbf{\mathfrak{C}}_{d}^{\circ}(k), \bar{\mathbb{Q}}_{\ell}^{\times}) \\ \mathcal{E}^{\circ} &\mapsto \chi_{\mathcal{E}}^{\circ} \end{aligned}$$

is an isomorphism.

Since  $\mathbf{T}_{d}^{\circ}(k) = \mathbf{T}_{d}^{\circ}(R)$ , this theorem gives a geometrization of depth *d* characters of  $\mathbf{T}_{R}^{\circ}$ .

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### Character Sheaves on $\mathbf{\tau}_d$

### Definition

A character sheaf  $\mathcal{E}$  on  $\mathfrak{T}_d$  is a character sheaf  $\mathcal{E}^\circ$  on  $\mathfrak{T}_d^\circ$  plus an action of  $X_*(\mathbf{T})_{\mathcal{I}_K} \rtimes W_k$  on  $\mathcal{E}^\circ$  compatible with  $\operatorname{Fr}_{\mathcal{E}^\circ}$  Introduction

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# Characters of $\mathbf{T}_d(\mathbf{k})$

Suppose now that T is unramified so that

$$1 \to \mathbf{T}_R^{\circ} \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$

splits. A splitting defines an extension of  $\chi_{\mathcal{E}}^{\circ}$  to

$$\mathbf{t}_d(k) = T(K)/T_R(R)_{d+}.$$

From the action of  $X_*(\mathbf{T})_{\mathcal{I}_K}$  on  $\mathcal{E}^\circ$  one can produce a character of

$$(X_*(\mathbf{T})_{\mathcal{I}_K})^{W_k} = \mathbf{T}(K)/\mathbf{T}_R^{\circ}(R).$$

Thus we may associate to  $\mathcal{E}$  a depth *d* character  $\chi_{\mathcal{E}}$  of **T**( $\mathcal{K}$ ): the product of these two.

# Characters of T(K)

We define a character sheaf on  $\mathbf{T}$  as the pullback of a character sheaf on  $\mathbf{T}_d$  under the projection  $\mathbf{T} \to \mathbf{T}_d$  for some *d*. A character of  $\mathbf{T}(K)$  is *admissible* if it has depth *d* for some *d*: factors through the quotient  $\mathbf{T}(K)/\mathbf{T}(K)_{d+}$ .

### Theorem

The map

 ${character sheaves on \mathbb{C}} \rightarrow Hom(\mathbf{T}(K), \mathbb{Q}_{\ell}^{\times})$ 

defined above is surjective

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## Extra Character sheaves

### Let Y be the cokernel of the composition

$$(X_*(\mathsf{T})_{\mathcal{I}_{\mathcal{K}}})^{W_{\mathcal{K}}} \hookrightarrow X_*(\mathsf{T})_{\mathcal{I}_{\mathcal{K}}} \twoheadrightarrow (X_*(\mathsf{T})_{\mathcal{I}_{\mathcal{K}}})_{W_{\mathcal{K}}}$$

### Theorem

The map

```
\{\text{character sheaves on } \mathbf{t}\} \to \text{Hom}(\mathbf{T}(\mathcal{K}), \bar{\mathbb{Q}}_{\ell}^{\times})
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is surjective with fibers parameterized by  $Hom(Y, \overline{\mathbb{Q}}_{\ell}^{\times})$ .

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## Questions

Is

$$\mathbf{1} \to \mathbf{T}_R^{\circ} \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to \mathbf{1}$$

split for ramified tori?

 Is there some category containing all Néron models of tori on which the Greenberg functor is exact?