Geometrizing Quasi-characters of Tori

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Joint Mathematics Meetings: 2013

Outline

Objective

K – a finite extension of $\mathbb{Q}_p$,
$\mathbf{T}$ – an algebraic torus over K (e.g. $\mathbb{G}_m$),
ℓ – a prime different from p.

Goal

Construct "geometric avatars" for (quasi)-characters in

$$\mathrm{Hom}(\mathbf{T}(K), \bar{\mathbb{Q}}_\ell^\times):$$

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\Longrightarrow$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. Associate to $\mathbf{T}$ a projective system $\mathfrak{T}$ of commutative group schemes $\mathfrak{T}_d$ over the residue field k of K.
2. Define *character sheaves* on $\mathfrak{T}$ following Deligne.
3. Map from character sheaves on $\mathfrak{T}$ to characters on $T(K)$.

The Néron model of $\mathbb{G}_m$

R – ring of integers of K with uniformizer π

R_d – $R/\pi^{d+1}R$

$\mathbf{T}_R$ – The Néron model of $\mathbf{T}$: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_R(R) = \mathbf{T}(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.

$\mathbf{T}_d$ – $\mathbf{T}_R \times_R R_d$.

Components

- The geometric component group of $\mathbf{T}_R$ is $X_*(\mathbf{T})_{\mathcal{I}_K}$, where $\mathcal{I}_K$ is the inertia group of K.
- $\pi_0(\mathbf{T}_R)$ is a constant group scheme after base change to the maximal unramified extension of K, but Frobenius may act nontrivially.
- The sequence of commutative R-group schemes

$$1 \to \mathbf{T}_R^{\circ} \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$

splits if $\mathbf{T}$ is unramified.

The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \varprojlim \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over k with

$$\mathfrak{T}(k) = \mathbf{T}(K).$$

Character Sheaves on $\mathfrak{T}_d^\circ$

Definition

A *character sheaf* on $\mathfrak{T}_d^\circ$ is an ℓ-adic local system $\mathcal{E}^\circ$ on $\mathfrak{T}_d^\circ$ so that $m^*\mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}$, where $m\colon \mathfrak{T}_d^\circ \times \mathfrak{T}_d^\circ \to \mathfrak{T}_d^\circ$ is multiplication.

Alternatively, a character sheaf on $\mathfrak{T}_d^\circ$ is a short exact sequence

$$1 \to A \to H \to \mathfrak{T}_d^\circ \to 1$$

together with a character $A \to \bar{\mathbb{Q}}_\ell^\times$, so that

1. $H \to \mathfrak{T}_d^\circ$ is a finite étale cover,
2. Fr_q acts trivially on A.

Proposition

Base change to $\bar{\mathfrak{T}}_d^\circ$ defines an equivalence of categories

$$\mathcal{E}^\circ \mapsto (\bar{\mathcal{E}}^\circ, \mathrm{Fr}_{\mathcal{E}^\circ})$$

Characters of $\mathfrak{T}_d^\circ(k)$

Let $x \in \mathfrak{T}_d(k)$ and $\bar{\mathcal{E}}_x^\circ$ be the stalk of $\bar{\mathcal{E}}^\circ$ at x. Define a character $\chi_{\mathcal{E}}^\circ$ of $\mathfrak{T}_d^\circ(k)$ by

$$\chi_{\mathcal{E}}^\circ(x) = \mathrm{Tr}(\mathrm{Fr}_{\mathcal{E}^\circ}, \bar{\mathcal{E}}_x^\circ).$$

Theorem (Deligne, SGA 4.5)

The map

$$\{\textit{characters sheaves on } \mathfrak{T}_d^\circ\} \to \mathrm{Hom}(\mathfrak{T}_d^\circ(k), \bar{\mathbb{Q}}_\ell^\times)$$
$$\mathcal{E}^\circ \mapsto \chi_{\mathcal{E}}^\circ$$

is an isomorphism.

Since $\mathfrak{T}_d^\circ(k) = \mathbf{T}_d^\circ(R)$, this theorem gives a geometrization of depth d characters of $\mathbf{T}_R^\circ$.

Character Sheaves on $\mathfrak{T}_d$

Definition

A character sheaf $\mathcal{E}$ on $\mathfrak{T}_d$ is a character sheaf $\mathcal{E}^\circ$ on $\mathfrak{T}_d^\circ$ plus an action of $X_*(\mathbf{T})_{\mathcal{I}_K} \rtimes W_k$ on $\mathcal{E}^\circ$ compatible with $\mathrm{Fr}_{\mathcal{E}^\circ}$

Characters of $\mathfrak{T}_d(k)$

Suppose now that $\mathbf{T}$ is unramified so that

$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$

splits. A splitting defines an extension of $\chi_{\mathcal{E}}^\circ$ to

$$\mathfrak{T}_d(k) = T(K)/T_R(R)_{d+}.$$

From the action of $X_*(\mathbf{T})_{\mathcal{I}_K}$ on $\mathcal{E}^\circ$ one can produce a character of

$$(X_*(\mathbf{T})_{\mathcal{I}_K})^{W_k} = \mathbf{T}(K)/\mathbf{T}_R^\circ(R).$$

Thus we may associate to $\mathcal{E}$ a depth d character $\chi_{\mathcal{E}}$ of $\mathbf{T}(K)$: the product of these two.

Characters of $\mathbf{T}(K)$

We define a character sheaf on $\mathfrak{T}$ as the pullback of a character sheaf on $\mathfrak{T}_d$ under the projection $\mathfrak{T} \to \mathfrak{T}_d$ for some d.
A character of $\mathbf{T}(K)$ is *admissible* if it has depth d for some d: factors through the quotient $\mathbf{T}(K)/\mathbf{T}(K)_{d+}$.

Theorem

The map

$$\{\textit{character sheaves on } \mathfrak{T}\} \to \mathrm{Hom}(\mathbf{T}(K), \bar{\mathbb{Q}}_\ell^\times)$$

defined above is surjective

Extra Character sheaves

Let Y be the cokernel of the composition

$$(X_*(\mathbf{T})_{\mathcal{I}_K})^{W_k} \hookrightarrow X_*(\mathbf{T})_{\mathcal{I}_K} \twoheadrightarrow (X_*(\mathbf{T})_{\mathcal{I}_K})_{W_k}$$

Theorem

The map

$$\{\textit{character sheaves on } \mathfrak{T}\} \to \mathrm{Hom}(\mathbf{T}(K), \bar{\mathbb{Q}}_\ell^\times)$$

is surjective with fibers parameterized by $\mathrm{Hom}(Y, \bar{\mathbb{Q}}_\ell^\times)$.

Questions

- Is
$$1 \to \mathbf{T}_R^\circ \to \mathbf{T}_R \to \pi_0(\mathbf{T}_R) \to 1$$
split for ramified tori?
- Is there some category containing all Néron models of tori on which the Greenberg functor is exact?