Derived Invariants of Calabi-Yau Threefolds in Positive Characteristic

Matthew Ward

University of Washington

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Motivation of Height

Height 1 (Ordinary)	$\leftrightarrow$	Canonical Lift
Finite Height	$\leftrightarrow$	Quasi-canonical Lift
Infinite Height (Supersingular)	$\leftrightarrow$	Poor lifting properties

Question: Is liftability preserved under derived equivalence?

Partial Answer (Lieblich-Olsson): For K3 surfaces yes!

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Definition of Height

Suppose X/k is a smooth, projective n-dimensional variety over a perfect field of characteristic $p > 0$.

Theorem (Artin-Mazur, Schlessinger)

The functor $\Phi : \mathrm{Art}_k \to \mathrm{Grp}$ *given by*

$$\Phi(S) = \ker[H^n_{\acute{e}t}(X \otimes S, \mathbf{G}_m) \to H^n_{\acute{e}t}(X, \mathbf{G}_m)]$$

is pro-representable by a formal group.

If Φ is p-divisible, then $\mathrm{ht}(X)$ is the height of the corresponding formal group. Otherwise Φ is $\widehat{\mathbf{G}_a}$ in which case we say X has infinite height.

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Examples

If X is an elliptic curve, then $\Phi \simeq \widehat{\mathrm{Pic}}_X$ is the formal Picard group.
Possible heights: 1 or 2

If X is a K3 surface, then $\Phi \simeq \widehat{\mathrm{Br}}_X$ is the formal Brauer group.
Possible heights: 1-10, infinite

If X is a Calabi-Yau threefold, then we just say Φ is the **Artin-Mazur formal group** and there is no known bound on the height if it is finite. Call height 1 **ordinary** and infinite height **supersingular**.

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Now assume X/k is a Calabi-Yau threefold meaning $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Define $D(X) := D^b_{Coh}(X)$ to be the bounded derived category of coherent sheaves on X.

Theorem (W)

If Y/k is a projective variety such that there is an exact, k-linear equivalence $D(X) \simeq D(Y)$ of triangulated categories, then $\mathrm{ht}(X) = \mathrm{ht}(Y)$.

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Sketch of Proof

Relate height to crystalline cohomology:
$\mathrm{ht}(X) = \dim_K H^3_{\mathrm{crys}}(X/W) \otimes_k K_{[0,1)}$

The derived equivalence is (naturally isomorphic to) a Fourier-Mukai transform associated to some kernel $\mathcal{P}^\bullet \in D(X \times Y)$.

This gives a crystalline class via Gillet-Messing whose correspondence induces an isomorphism on the odd crystalline cohomology (the action of Frobenius is twisted).

In our case the odd cohomology is just H^3, and so we get an isomorphism of F-isocrystals which implies the equality of the height.

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Notes

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This trick does not work for Calabi-Yau varieties of other dimensions because there could be (extra) non-zero cohomology.

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Other Invariants

Using similar techniques one can prove some other arithmetic invariants are preserved under derived equivalence:

- Over number fields the L-series
- Over $\mathbf{Q}$ modularity
- The number of $\mathbf{F}_q$-rational points
- The Zeta function
- Being Frobenius split
- The b-number
- Others

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Two Ideas

Try to directly mimic the proof from the K3 case that liftability is preserved under derived equivalence. This is work in progress. There are lots of problems to overcome.

Use standard techniques from derived categories to produce new non-liftable Calabi-Yau threefolds.

Rough philosophy: If Y is a component of a stable moduli of sheaves on X with fixed Mukai vector, then $D(Y) \simeq D(X)$.

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Schöer's Nonliftable Threefold

Let k be algebraically closed of characteristic 3. Schöer constructs a pencil of supersingular Kummer K3 surfaces $X \to \mathbf{P}^1$.

Key reason for non-liftability: $b_3(X) = \dim H^3_{\text{ét}}(X, \mathbf{Q}_\ell) = 0$

By previous part of talk, this property will be preserved.

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Interesting Failure

Suppose Y is any fine relative moduli of sheaves of dimension 3 on $X \to \mathbf{P}^1$.

It is the case that $D(X) \simeq D(Y)$, so Y is also a non-liftable Calabi-Yau threefold.

The fibers of $X \to \mathbf{P}^1$ are supersingular K3 surfaces.

Using the fact that the Fourier-Mukai number of a supersingular K3 surface is 1, we can check that $X \simeq Y$, so we don't get any new non-liftable Calabi-Yau threefolds in this way.

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Thank you.

Questions?