

MATH 2055 — Suggested Problems

Velleman, *How To Prove It*, 3rd edition

Taylor Dupuy, University of Vermont

Fall 2026

How to use this list

These problems are **not collected and not graded**. Nothing here is turned in. They are nevertheless the course: the syllabus asks you to plan on about three hours outside class for every hour in it, and this list is what fills that time.

Quiz problems will be *similar to these but not identical*. That is deliberate. The point is not to have these particular problems memorized — it is to be able to do *things of this kind*, cold, on paper, in the time a quiz gives you. A student who has copied down twenty solutions and a student who can produce a new one look identical on Thursday and completely different on Friday.

Read the problem and attempt it before you look anything up. Ten minutes of being stuck is worth more than an hour of reading someone else's solution, and it is the only way to find out which step you actually cannot do. Look at the book's appendix, or at the worked examples, *after* you have written something down and gotten stuck — not instead of writing something down.

Work in groups. Proof writing is a language, and you learn a language by using it on someone who will tell you when you are not making sense. Say your proof out loud to another person before you believe it. If you cannot say it, you have not got it.

Each section has a list of **Problems** and then a list under **Also do**. Both lists are recommended and neither is ranked; the split is only about where to begin. Start with the first list, and do the second one too.

Sections marked “Skip” list exercises that quietly depend on material we do not cover (§4.4, §5.4, §6.2, §6.5, and Chapter 7). They are not harder; they use vocabulary you will not have. Every other exercise in every section we cover is fair game.

Those five are the whole of what this course does not cover. In particular §5.5 (images and inverse images) and Chapter 8 (infinite sets) *are* covered — they were off the list in earlier drafts of this course and are back on it — and both have full sections below. §5.5 is taught on November 18 and Chapter 8 has the last three meetings of the term.

A note on ordering. Chapters 1 and 2 are taught out of the book's order, in three units: propositional logic, then predicate logic, then sets. See the reading-order page for why. From Chapter 3 onward we read the book straight through. This list is organized in *teaching* order, so the section numbers jump around at the start and then settle down.

This list supersedes the shorter problem list on the reading-order page. Everything recommended there is recommended here, and this list adds to it; the two agree. The complete skip list for the whole course is §4.5 #25, #26, #27; §5.1 #19(b); §5.2 #22(b) and #22(g); §5.5 #5; §8.1 #2(b), #16(b), #16(c), #19, #24(b); §8.2 #3; and §8.3 #11. Nothing in §6.1 is skipped. Each entry is explained where it occurs below.

Unit I — Propositional Logic

§1.1 Deductive Reasoning and Logical Connectives

9 exercises

What it is for: Turning an English sentence into a formula built from $\wedge$, $\vee$, $\neg$, and finding the premises and the conclusion inside an argument.

Problems **1.** the basic translation drill; **3.** “Alice and Bob are not both” versus “both not” versus “neither” — four sentences, two of which are secretly the same, and this is the one that kills people; **4.** the same trap with “either both” versus “both either”; **6.** translation in the other direction, formula to English; **9.** identify premises and conclusion, then guess whether the argument is valid. Do this one together with §1.2 #7, which checks your guesses.

Also do **2.** more translation, including the first arithmetic sentence; **5.** which strings are well-formed formulas — one minute, and worth it; **7.** formula to English, harder; **8.** formula to English; (b) and (c) are two ways of saying “exactly one.”

Skip None.

Notes This section has only nine exercises; the list above is all of them. #9 is the longest thing here and the only one with real content — everything else is vocabulary.

§1.2 Truth Tables

18 exercises

What it is for: Deciding validity, equivalence, tautology and contradiction mechanically; and then learning to do the same thing by citing the ten laws instead of building a table.

Problems **1.** build two truth tables, correctly, with the rows in a systematic order; **7.** check the four arguments from §1.1 #9 — this is where you find out whether your intuition about validity was any good; **9.** tautology, contradiction, or neither; **11.** the first proof-shaped writing in the course: simplify by *citing laws*, not by building a table. Learn to name the law you used at each step; **13.** derive the second De Morgan law from the first — three lines, and the template for everything in Unit III; **18.** what can you conclude about an argument whose conclusion is a tautology? Short, and much better than it looks.

Also do **2.** two more tables; **3.** exclusive or; **5.** nor, and the fact that $\downarrow$ alone generates everything; **6.** nand, the same phenomenon (this is why NAND gates are the ones chips are built out of); **8.** which of five formulas are equivalent; **12.** more simplification by law; **14.** associativity really does let you drop parentheses; **15.** one line; **16.**, **17.** build a formula with a prescribed truth table. These two are the hardest in the section and the most useful: the method behind them (one conjunction per true row, joined by $\vee$) is disjunctive normal form, and it proves that $\wedge$, $\vee$, $\neg$ can express *any* truth table at all.

Notes #7 is four separate truth tables and takes real time — budget for it, it is not hard. #4 is subsumed by #5(c) and #6(c) if you do those.

§1.5 The Conditional and Biconditional Connectives

12 exercises

What it is for: $\rightarrow$ and $\leftrightarrow$: the vocabulary (necessary, sufficient, only if, unless), the converse, and the equivalences that Chapter 3’s proof strategies are built out of.

Problems **1.** necessary and sufficient conditions, in English; **3.** the best converse-versus-conditional problem in the chapter — seven sentences, and your job is to sort them into the two piles; **4.** truth tables for three arguments, two of which are valid; **7.** $(P \rightarrow R) \wedge (Q \rightarrow R)$ is equivalent to $(P \vee Q) \rightarrow R$. *This is proof by cases*, and you will use it from §3.5 to the end of the term; **11. do this one.** $(P \vee Q) \leftrightarrow Q$ is equivalent to $P \rightarrow Q$, and $(P \wedge Q) \leftrightarrow Q$ is equivalent to $Q \rightarrow P$. We cite it by name in Unit III, at §2.2

#13, three weeks from now.

Also do **2.** more necessary/sufficient, including “unless”; **5.** two more arguments; **6.** $P \leftrightarrow Q$ as a disjunction of the two agreeing cases; **8.** (a) is the hardest single equivalence in the section; (b) is a one-line surprise; **9.**, **10.** $\neg$ and $\rightarrow$ suffice to express everything; **12.** which of five nested conditionals are equivalent — this is where currying lives.

Skip None.

Notes That is all twelve. #4 has the longest solution in the section and is pure bookkeeping; #8(a) is the one that is genuinely hard. #6(a) and #7 are the two equivalences that turn into proof strategies later, so know them by name.

Unit II — Predicate Logic

Before this unit starts we borrow one symbol from set theory: $x \in A$ means “ x is one of the things in A .” That is all we need, because $\forall x \in A P(x)$ is an abbreviation for $\forall x(x \in A \rightarrow P(x))$. Nothing else about sets is used here.

§1.3 Variables and Sets (first part only)

2 exercises here; the rest in Unit III

What it is for: Statements with free variables — predicates. A formula with a free variable in it is not true or false; it becomes true or false when you put something in the slot.

Problems 1. analyze logical forms, now with variables in them. (a) is the same sentence as §1.1 #2(c) and you should be able to do better this time; 2. the same, with two variables.

Also do Nothing else from §1.3 yet.

Skip 3–9 *not skipped — postponed.* Every one of them is a set-builder exercise, and set-builder notation is Unit III. They are assigned there.

Notes Velleman explains free and bound variables using $\{x \mid P(x)\}$, because the braces are his example of something that binds. We meet binding through $\forall$ and $\exists$ instead — §2.1 #3 is the drill — and meet the braces as a second binder in Unit III. If you are reading §1.3 straight through and hit set-builder notation, that is why.

§2.1 Quantifiers

10 exercises

What it is for: $\forall$ and $\exists$: translating between English and quantified formulas, telling free variables from bound ones, and discovering that the order of two quantifiers is the whole ballgame.

Problems 1. translation into logic. (d) and (e) differ by one word in English and by a quantifier in logic; 3. **our free-versus-bound drill.** Four statements about $\mathbb{R}$; name the free variables in each. Renaming a bound variable changes nothing, renaming a free one changes what you said; 5. translation into idiomatic *mathematical* English — (b) says something false and famous; 7. five formulas about parents and children that differ only in where the quantifiers sit. **This is the $\forall\exists$ versus $\exists\forall$ problem** and it is worth an afternoon; 8. six statements over $\mathbb{N}$, true or false.

Also do 2. more translation, including four sentences where English “anyone” is sometimes $\forall$ and sometimes $\exists$; 4. formula to English; 6. three statements that look alike and are not; 9., 10. #8 again over $\mathbb{R}$ and over $\mathbb{Z}$. **Do both.** The answers change, and watching *which* ones change is the entire lesson about the universe of discourse.

Skip None.

Notes That is all ten. #8, #9, #10 are one problem done three times and are the best value in the section.

§2.2 Equivalences Involving Quantifiers (logic half)

What it is for: Pushing $\neg$ through a quantifier, deciding which quantifier distributes over which connective, and $\exists!$.

Problems 1(a)(b)(d). negate, then re-express as a *positive* statement. This is the single most-used skill in Chapter 3: to prove something by contradiction you must first be able to say what its negation asserts; 3. true or false over $\mathbb{N}$, including two $\exists!$ statements; 4. derive the second quantifier negation law from the first. Three lines. The propositional analogue was §1.2 #13; 6. $\exists$ distributes over $\vee$; 7. $\exists x(P(x) \rightarrow Q(x))$

is equivalent to $\forall xP(x) \rightarrow \exists xQ(x)$. **Genuinely surprising**, and it reuses #6 as a lemma. §3.3 #1 and §3.5 #30 come back and prove the two directions again by hand; **9.** is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall xP(x) \vee \forall xQ(x)$? **This is the first counterexample you construct in this course.** Constructing counterexamples is a skill separate from proving things and you will be asked for one on a quiz.

Also do 2(a)(b). more negation; **5.** negating a bounded quantifier; **15.** six statements built from $\exists$ and $\exists!$ — which are equivalent? Slow, subtle, and the best test of whether you actually understand $\exists!$.

Notes #1 and #2 have long write-ups because they have four parts each, not because any part is hard. #4 and #6 are the theorems; the rest is practice.

The set-flavored half of §2.2 — #1(c), #2(c), #8, #10, #11–#14 — is in Unit III.

Unit III — Sets

§1.3 Variables and Sets (rest)

7 exercises

What it is for: Set-builder notation, elementhood tests, truth sets — and the braces as a second thing that binds a variable.

Problems **3.** write elementhood-test definitions for four listed sets; **5.** simplify; say which variables are free and which are bound; if there are no free variables, say whether it is true or false. This is the exercise §1.3 was postponed for; **7.** list the elements of $\{x \in \mathbb{R} \mid 2x^2 + x - 1 = 0\}$ and the same set with $\mathbb{R}$ replaced by $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{N}$. The formula never changes; the set does; **9.** truth sets, ending with $5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}$ sitting inside another set-builder. **This is where you find out whether you know what the braces do.**

Also do **4.** three more elementhood tests, from a list of numbers; **6.** more free/bound, including a set-builder nested inside a set-builder; **8.** truth sets in English.

Skip None.

Notes That is the whole rest of §1.3. #5 and #6 are the same problem twice and both are worth doing; #9(c) is the one people get wrong.

§1.4 Operations on Sets

17 exercises

What it is for: $\cap$, $\cup$, $\setminus$, $\subseteq$, Δ , Venn diagrams — and the discovery that every set identity is a logical equivalence with “ $x \in -$ ” wrapped around each letter.

Problems **1.** compute with three concrete sets; decide which are disjoint and which are subsets of which; **5. the bridge problem of the whole block.** Verify the identities of #4 by writing out what $x \in -$ means for each side and citing the laws from §1.2 by name. This is the moment Unit I pays off, and it is the template for every set proof you will write for the next month; **7.** the same method on harder identities; **9.** five sets built from A , B , C ; write out membership for each and decide which are equal. **The most valuable single exercise in the section;** **10.** $(A \cup B) \setminus B \subseteq A$, but when is it equal? And what *is* $(A \cup B) \setminus B$ always equal to? A counterexample followed by a repair — exactly the habit Chapter 3 will demand; **14.** associativity of symmetric difference, by Venn diagram. §3.6 #9 cites this by name.

Also do **2.** more concrete computation; **3.** check the two pictures of $A \Delta B$ agree; **4.** two identities by Venn diagram (do this before #5); **6.** two more by Venn diagram (do this before #7); **8.** any method; **11.** is $(A \setminus B) \cup B = A$? If not, what is true? **12.** why four sets do not fit in four overlapping circles; **13.** $(A \cup B) \setminus C$ versus $A \cup (B \setminus C)$; **15.** three symmetric-difference identities; **17.** fill in the blanks to make three symmetric-difference identities true — harder than #15 and #16, and more fun.

Skip None.

Notes #9 has by far the longest write-up in the section (five sets, ten comparisons) and is worth the time. #16 is #15 again with the operations permuted; do one of the two unless you want the drill.

§2.2 Equivalences Involving Quantifiers (set half)

What it is for: Proving statements *about sets* by rewriting them in logic and citing equivalences. Set theory turns out to be logic with $\in$ in it.

Problems **11.** $A \subseteq B$ and $A \setminus B = \emptyset$ are equivalent. Write both in symbols and grind. This is the cleanest demonstration of the slogan above, and both characterizations get used constantly from here on; **13.** $A \subseteq B$ iff $A \cup B = B$; and $A \subseteq B$ iff $A \cap B = A$. **This is the one that cites §1.5 #11** — go and

find your solution to it; **14.** $A \cap B = \emptyset$ iff $A \setminus B = A$.

Also do 1(c). negate $A \cup B \subseteq C \setminus D$; **2(c).** negate a bounded-quantifier statement about sets; **8.** $(\forall x \in A P(x)) \wedge (\forall x \in B P(x))$ is equivalent to $\forall x \in (A \cup B) P(x)$; **10.** the $\exists$ version of #8, and then the trap: is $\exists x \in A P(x) \wedge \exists x \in B P(x)$ the same as $\exists x \in (A \cap B) P(x)$? (No.) **12.** $C \subseteq A \cup B$ iff $C \setminus A \subseteq B$.

Notes #11, #13 and #14 are the three standard characterizations of $\subseteq$ and disjointness. You will reach for them without thinking by November; learn them now.

§2.3 More Operations on Sets

16 exercises

What it is for: Power sets, families of sets, $\bigcup \mathcal{F}$ and $\bigcap \mathcal{F}$, indexed families — and the first appearance of sets whose elements are sets.

Problems **1.** rewrite set statements using only $\in$, $=$, the connectives and the quantifiers — no $\subseteq$, no $\mathcal{P}$, no $\cup$, no braces. **This is the exercise that makes the notation stop being magic;** **3.** what is $\mathcal{P}(\{\emptyset\})$? Half of you will say it has one element. It has two; **4.** compute $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ for a concrete three-element family; **9.** analyze $\bigcup_i (A_i \cap B_i)$ against $(\bigcup_i A_i) \cap (\bigcup_i B_i)$ and friends, then test your guess against #8's concrete family. Guess, test, revise — this is how mathematics is actually done; **11.** $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. **A standard theorem;** you will prove it again by the methods of Chapter 3 at §3.4 #7, and comparing the two proofs is instructive; **16. Russell's paradox.** There is no set of all sets. The very first thing you learned in Unit I — that $P \leftrightarrow \neg P$ is a contradiction — is what breaks naive set theory, in the last section of the block.

Also do **2.** more rewriting, this time with $\bigcup$; **5.** another concrete family; **6.** an indexed family over $I = \{2, 3, 4, 5\}$; **7.** who was alive in 1750–1759 — a genuinely nice concrete $\bigcup/\bigcap$; **8.** the concrete family that #9 needs; **10.** construct two indexed families with a prescribed failure; **12.** when is $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$? **13.** three identities by the write-out-membership method; **14.** doubly indexed families, $A_{i,j}$. **A real step up** — if you can do this one you can do anything with subscripts; **15.** $\bigcup \emptyset = \emptyset$, and why $\bigcap \emptyset$ is a problem.

Skip None.

Notes That is all sixteen. #13 and #14 are the heaviest; #14 in particular takes noticeably longer than its neighbours and is where indexed-family notation either clicks or does not. #15 and #16 are the conceptual pair at the end and neither one is computational — do them last and think about them.

Chapter 3 — Proofs

From here on we read Velleman in his own order.

§3.1 Proof Strategies

17 exercises

What it is for: Hypotheses and conclusion; what a theorem does and does not tell you; the strategy for a goal of the form $P \rightarrow Q$; and reading a proof critically.

Problems 1. what does the theorem tell you at $n = 6$, at $n = 15$, at $n = 11$? **The $n = 11$ part is the whole exercise:** the hypotheses fail, so the theorem tells you nothing; 4. fill in the missing middle of a proof; 8. the first set-flavored proof: $A \setminus B \subseteq C \cap D$ and $x \in A$; show $x \notin D \rightarrow x \in B$; 10. if $a < b$ then $(a + b)/2 < b$. Two lines, and the shape of every inequality proof you will write; 12. $0 < a < b$, $d > 0$, $ac \geq bd$; show $c > d$. Harder than its neighbours and the best one here; 16. **what is wrong with this proof?** It assumes the conclusion and derives the hypothesis. Affirming the consequent, in the wild. Then give a correct proof.

Also do 2. hypotheses and conclusion of the quadratic-formula theorem; 3. find a counterexample to a false theorem; 5., 6., 7. three short inequality proofs; 9. nested conditionals as a goal; 13., 14. two more with a linear constraint; 15. prove the first theorem of Example 3.1.1 *by citing* Example 3.1.2. The first time you use an earlier result instead of starting from scratch; 17. another incorrect theorem: this proof divides by something that can be zero.

Skip None.

Notes #16 and #17 are the two most valuable exercises in the section and they are at the end, so do not run out of time. #12 is the only one whose difficulty is out of line with its neighbours — it needs two multiplications in the right order.

§3.2 Proofs Involving Negations and Conditionals

18 exercises

What it is for: Contrapositive and contradiction: two strategies for a negative goal, and the ways they go wrong.

Problems 1. prove propositional facts *by the methods of this chapter*, not by truth table. (a) is transitivity of $\rightarrow$; 5. it cannot be that $x \in A \setminus B$ and $x \in B \setminus C$; 6. redo Example 3.2.1 by contradiction. **The point is the comparison** — the same theorem, two strategies, and you decide which reads better; 9. if $a < 1/a < b < 1/b$ then $a < -1$. The hardest computation in the section by some margin; 12. **what is wrong with this proof?** It negates “ $x \neq 3$ and $y \neq 8$ ” as “ $x = 3$ and $y = 8$.” It is not. This is the single most common error in the course; 13. another bad contradiction proof: “ $x \notin B$ and $B \subseteq C$ ” does not give $x \notin C$.

Also do 2. two more propositional proofs; 3., 4. short set proofs with disjointness; 7. Example 3.2.5 by contradiction; 8., 10., 11. three algebra-flavored ones; 14. modus tollens by truth table — one minute, do it; 16. check #1 by truth table, to confirm that the proofs you wrote were about something real; 18. can the proof of Example 3.2.2 be modified? Explain. A thinking problem, not a writing problem.

Skip None.

Notes #12 and #13 are the whole reason this section exists. If you only do two problems here, do those two.

§3.3 Proofs Involving Quantifiers

26 exercises

What it is for: The four rules — arbitrary element for a $\forall$ goal, produce a witness for an $\exists$ goal, instantiate a $\forall$ given, name a witness from an $\exists$ given — and the discipline of keeping track of which variables are arbitrary and which are chosen.

Problems **3.** if $A \subseteq B \setminus C$ then A and C are disjoint; **7.** for every real $x > 2$ there is a real y with $y + 1/y = x$. **Your first real “produce a witness” problem** — the scratch work is a quadratic formula and the proof does not mention it; **9.** if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$ then $\bigcap \mathcal{F} \subseteq A$. Short, and the definition of $\bigcap$ finally becomes usable; **12.** if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$; **17.** if every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$. The best quantifier-juggling problem in the section; **18.** if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$, and if $ac \mid bc$ with $c \neq 0$ then $a \mid b$. These two divisibility facts are used without comment from §4.5 through §6.4 — this is where they get proved; **20.** what is wrong with this proof? It negates $\forall x P(x)$ as $\forall x \neg P(x)$. Compare §2.2 #4, which is the fact it violates.

Also do **1.** the easy direction of §2.2 #7, now proved by hand rather than by equivalences; **2.** A and $B \setminus C$ disjoint gives $A \cap B \subseteq C$; **4.** if $A \subseteq \mathcal{P}(A)$ then $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$; **8.** $A \in \mathcal{F}$ gives $A \subseteq \bigcup \mathcal{F}$; **10.** the universal property of $\bigcap$; **11.** if $\emptyset \in \mathcal{F}$ then $\bigcap \mathcal{F} = \emptyset$; **13.** the $\bigcap$ analogue of #12 — note the inclusion reverses; **19.** for all real x, y there is a z with $x + z = y - z$; and what happens over $\mathbb{Z}$; **21.** an incorrect theorem about $\forall x \in A$ versus $\forall x \in B$; **22.** what is wrong with this proof? It writes “let $x = y/(y^2 + 1)$ ” where y is not yet in scope. Variable scope, and the hardest “what is wrong” in the chapter.

Skip None.

Notes #18 is the one to be sure of — everything about divisibility later leans on it. #22 has a much longer write-up than anything around it because the answer requires explaining what scope means. #5, #6, #14, #15, #16, #23, #24, #25, #26 are all fine and all optional; #26 is a reflection question about the symmetry between the rules, and it is the one that makes the four strategies feel like two.

§3.4 Proofs Involving Conjunctions and Biconditionals

27 exercises

What it is for: Splitting a conjunctive goal, proving an “iff” in two directions or as a chain of equivalences, and the first genuinely mathematical content: even, odd, and divisibility.

Problems **1.** $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$, now proved rather than tabled; **6.** $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ by a chain of equivalences, not by two inclusions. Learn both styles and learn when the chain is honest; **7.** $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ — §2.3 #11 again by Chapter 3 methods. Compare your two proofs; **9.** odd times odd is odd. The first number-theory proof in the book and the model for all of them: unpack the definition, do algebra, repack the definition; **11.** n^3 is even iff n is even. Both directions, and the backwards one wants the contrapositive. This is the cubic cousin of Theorem 3.4.3, which is what makes the irrationality of $\sqrt{2}$ work in §6.4; **13.** $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \leftrightarrow x = 1]$. An “iff” with a quantifier inside it, which is where students stop being able to see the shape of the goal; **27.** $15 \mid n$ iff $3 \mid n$ and $5 \mid n$; and it is false that $60 \mid n$ iff $6 \mid n$ and $10 \mid n$. **The standard coprimality caution**, and the reason you cannot split a divisibility test any way you like.

Also do **2., 3., 4., 5.** four short subset proofs — warm-ups; **8.** $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$; **10.** odd minus odd is even; **12.** a putative theorem about $n^2 - m^2$: find the error, then decide whether the statement is true; **16.** $\bigcup \mathcal{F}$ and $\bigcap \mathcal{G}$ disjoint; **24., 25.** indexed families and $\setminus, \cap$: prove one inclusion, then find a counterexample to the other. **The prove-then-refute pattern** is a quiz favorite.

Skip None.

Notes #9, #10, #11 are the parity block; do all three in one sitting. #21 and #23 have much longer solutions than anything else here — #23(c) asks you to guess the theorem before proving it, which is worth doing once but not under time pressure. #17, #18, #19, #20, #22 are all about families of sets and all

reasonable; pick a couple if you want more.

§3.5 Proofs Involving Disjunctions

33 exercises

What it is for: Proof by cases, in both roles: a disjunctive given splits into cases, a disjunctive goal lets you assume the negation of one side.

Problems 1. $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. The first cases proof; 5. $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ give $A \subseteq B$. **Two cases, and picking the right thing to case on is the whole problem;** 11. $|x - 3| > 3$ implies $x^2 > 6x$, by splitting on the sign of $x - 3$; 12. $|2x - 6| > x$ iff $|x - 4| > 2$. An iff *and* cases; 13. **the triangle inequality**, in four graded steps: $|a| \leq b$ iff $-b \leq a \leq b$; then $-|x| \leq x \leq |x|$; then $|x + y| \leq |x| + |y|$; then the reverse form. **This is the natural source of the triangle inequality** and §4.3 #6 and §5.1 #19(c) both send you back here; 20. $A \triangle B$ and C disjoint iff $A \cap C = B \cap C$; 27. is this cases proof correct? **It proves $x < 6$ in one case and $0 < x$ in the other and then claims both.** The most instructive wrong proof in the chapter.

Also do 2., 3., 4. three short subset proofs by cases; 6. if $A \triangle B \subseteq A$ then $B \subseteq A$; 7. $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$; 8. $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$; 14. $x^2 + x$ is even, for every integer x — two cases, and §6.1 #9(a) proves it again by induction; 15. the remainder of x^4 on division by 8 is 0 or 1; 16. $\bigcup(\mathcal{F} \cup \mathcal{G}) = (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$, then guess and prove the $\bigcap$ version; 31. is this proof correct? It does the case split *inside* the “let x be arbitrary” and concludes a statement about all of A from a fact about one x . Compare #27.

Skip None.

Notes This is the longest exercise set in the book. #17 has by far the longest solution in the section — roughly three times its neighbours — and #16 is not far behind; both are optional. #30 is the converse of #1 and Velleman himself warns in §3.3 #1 that it is “quite a bit harder”; save it for when you want a challenge. #19 and #21–#26 are the symmetric-difference block and are good extra practice if $\triangle$ still feels slippery.

§3.6 Existence and Uniqueness Proofs

13 exercises

What it is for: $\exists!$: produce a witness, then show any two witnesses are equal. Two halves, and people forget the second one.

Problems 1. for every real x there is a unique real y with $x^2y = x - y$. **The template:** existence by algebra, uniqueness by “suppose y_1 and y_2 both work”; 3. the same with a hypothesis on x , so the existence half needs care; 4. a unique y such that *for every* z , $zy = z/x$. Nested quantifiers inside $\exists!$; 6. there is a unique $A \in \mathcal{P}(U)$ with $A \cup B = B$ for all B , and a unique one with $A \cup B = A$. **This says $\emptyset$ and U are the identity and absorbing elements for $\cup$, and that they are the only ones;** 7. the same story for $\cap$, with the roles swapped; 12. design a proof strategy for “there are exactly two values of x ,” then use it on $x^3 = x^2$. **You invent the strategy yourself**, which is the point.

Also do 2. a unique x working for every y — note the quantifier order; 5. $\bigcup^! \mathcal{F}$, and it equals $\bigcup \mathcal{F}$ exactly when $\mathcal{F}$ is pairwise disjoint; 8. unique relative complement: for each A there is a unique B with $C \setminus A = C \cap B$ for all C ; 9. **symmetric difference has a unique identity, unique inverses, and unique solutions to $A \triangle C = B$.** If that sounds like a group, it is one — this is the best preview of MATH 3551 in the book. It cites §1.4 #14 for associativity; 10., 11. two harder uniqueness problems about families of sets; 13. a unique c such that $x^2 + 3x + c = 0$ has a unique solution — $\exists!$ nested inside $\exists!$, and part (b) shows the two do not commute.

Skip None.

Notes That is all thirteen. #5 and #9 have the longest write-ups and #9 is the one worth the time. #6,

#7, #8 together are the algebra of sets stated as uniqueness theorems; do them as a block.

§3.7 More Examples of Proofs

10 exercises

What it is for: Everything at once, on real mathematics: ε - δ limits, and proofs that pick their own strategy.

Problems **1.** there is a unique smallest set A with $\mathcal{F} \subseteq \mathcal{P}(A)$. Existence, uniqueness, and a minimality condition in one problem; **4.** three statements about $\triangle$ and $\subseteq$ are equivalent. **Prove a three-way equivalence as a cycle** $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$, not as six implications. That trick is worth knowing; **8.** if $\lim_{x \rightarrow c} f(x) = L$ and $L > 0$ then f is positive near c . **The ε - δ problem to be able to do** — the whole game is choosing $\varepsilon = L$, and if you have seen why, you have understood the definition; **9.** if $\lim_{x \rightarrow c} f(x) = L$ then $\lim_{x \rightarrow c} 7f(x) = 7L$. The other ε - δ pattern: given ε , produce the right δ ; **10. there are irrationals a, b with a^b rational.** A famous non-constructive proof — at the end you know such a and b exist and you do not know which pair works. Read the proof and decide whether it is legitimate. (It quotes Theorem 6.4.5, which we prove in December.)

Also do **2.** a unique smallest m with a stated property; **3.** what can you say about $\mathcal{P}(A \setminus B) \setminus (\mathcal{P}(A) \setminus \mathcal{P}(B))$? It is not empty — try examples first; **5.** power sets and indexed unions; **6.** a long multi-part problem about $\bigcup$ and $\bigcap$ over a family of index sets. **Noticeably harder than everything else in the book so far** — its solution runs several times the length of its neighbours. Attempt it when you have an afternoon, not the night before a quiz; **7.** another limit computation.

Skip None.

Notes That is all ten. §3.7 is where the course stops being about logic and starts being about mathematics. #8 and #9 are the two to have solid before analysis next year.

Chapter 4 — Relations

§4.1 and §4.2 are taught *thin* — just enough to say what a relation is and to have inverse and composition in hand — and §4.4 (ordering relations) is not taught at all. Everything in this chapter is pointed at §4.5.

§4.1 Ordered Pairs and Cartesian Products

15 exercises; taught thin

What it is for: (a, b) and $A \times B$: the machinery that lets a two-variable statement have a truth set.

Problems **1.** truth sets of two-variable statements; **3.** truth sets that are subsets of $\mathbb{R}^2$ — draw the pictures; **4.** compute everything in Theorem 4.1.3 for concrete A, B, C, D and check each part; **6. what is wrong with this proof?** It considers two cases where there are four. Missing cases is the characteristic Chapter 4 error; **8.** is $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$? Prove it or refute it; **15. is this proof correct?** “Let a be an arbitrary element of A ” — what if A is empty? The empty-set edge case, which will bite you again in §4.3 #20.

Also do **2.** more truth sets; **5.** prove parts 2 and 3 of Theorem 4.1.3; **7.** counting: $|A \times B| = mn$; **10.** $(A \setminus C) \times (B \setminus D) \subseteq (A \times B) \setminus (C \times D)$; **12.** disjointness of products.

Skip None.

Notes #9, #11, #13, #14 are correct and fine but go further into the algebra of products than this course needs; do them only if you want more §3.4-style practice. #4 and #5 together are Theorem 4.1.3, which is the only thing from this section we actually use later.

§4.2 Relations

15 exercises; taught thin

What it is for: A relation is a set of ordered pairs. Domain, range, inverse R^{-1} , composition $S \circ R$ — and nothing else, yet.

Problems **1.** domains and ranges; **5.** compute $S \circ R$ and $S \circ S^{-1}$ for small concrete relations. **Do this by hand until composition stops feeling backwards** — $S \circ R$ means R first; **7. finish Theorem 4.2.5.** Parts 3, 4 and 5 give you $(R^{-1})^{-1} = R$, associativity of $\circ$, and $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. That last one is the socks-and-shoes rule; §5.3 #7 uses it and so will MATH 3551. **If you do one problem in this section, do this;** **8.** what does “an enemy of my enemy is my friend” say about E and F ? Translating an English slogan into $E \circ E \subseteq F$ is exactly the skill; **11.** $S \circ R = \emptyset$ iff $\text{Ran}(R)$ and $\text{Dom}(S)$ are disjoint.

Also do **2.** more domains and ranges; **3., 4.** describe composites of the example relations in words; **6.** more concrete composition; **9.** $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$, with equality under a hypothesis; **10.** three claims about R^{-1} and $\cup$ — prove or refute; **15.** why $S \circ R$ does not depend on which intermediate set you use.

Skip None.

Notes #12, #13, #14 are a long block on how $\circ$ interacts with $\cap, \cup, \setminus$. They are good practice but not on the path to §4.5; skip them without guilt if time is short. #12(b) is another “what is wrong with this proof” and is the pick of the three.

§4.3 More About Relations

24 exercises

What it is for: Reflexive, symmetric, transitive — the three properties, restated in terms of i_A , R^{-1} and $R \circ R$ so they can be manipulated instead of just checked.

Problems **2.** draw the directed graph and read off the three properties. Do this before anything abstract; **5.** compute $S \circ R$ from a picture; **7.** R is reflexive on A iff $i_A \subseteq R$ (Theorem 4.3.4, part 1); **8.** R

is transitive iff $R \circ R \subseteq R$ (part 3). Together with part 2, **R is symmetric iff** $R = R^{-1}$, these three reformulations are what make the rest of the chapter possible; **10.** $i_D \subseteq S^{-1} \circ S$ and $i_R \subseteq S \circ S^{-1}$; **12.** the three properties pass to R^{-1} ; **13.** do the three properties survive $R_1 \cup R_2$? **Two yes and one no**, and the no is the interesting one — transitivity is not preserved by union, which is exactly why transitive closure has to be defined the hard way; **22. is this proof correct?** It “proves” that symmetric plus transitive implies reflexive. It does not, and finding the gap tells you why the definition of an equivalence relation lists all three conditions instead of two.

Also do **1.** draw a relation between letters and words; **3.** the directed graph of i_A ; **4.** read relations off directed graphs and classify them; **6.** $D_r \circ D_s = D_{r+s}$ — uses the triangle inequality from §3.5 #13(c); **9.** $R \circ i_A = R$ and $i_B \circ R = R$; **11.** reflexive implies $R \subseteq R \circ R$; **14.** the $\cap$ version of #13 (all three survive); **15.** the $\setminus$ version of #13; **17.** $R \circ S$ is symmetric iff $R \circ S = S \circ R$. The best short proof here; **24.** why the book says “reflexive *on* A ” rather than just “reflexive.” Two lines, and it settles a question you have probably already had.

Skip None.

Notes #16, #18 are fine extra practice. #19, #20, #21, #23 lift relations up to $\mathcal{P}(A)$ and are a clear step up in difficulty from the rest of the section — #20’s parenthetical (“why did the empty set have to be excluded?”) is the same empty-set trap as §4.1 #15, and #23 is a research-flavored problem contributed by a colleague of Velleman’s. All four are optional.

§4.5 Equivalence Relations

27 exercises

What it is for: Equivalence relations, equivalence classes, partitions, and A/R — and congruence mod m , which is where we get modular arithmetic without Chapter 7. **This is the destination of the whole chapter.**

Problems **3.** which relations on English words are equivalence relations, and what are the classes? Concrete, and the “at least one letter in common” one fails transitivity in a way you should be able to exhibit; **6.** prove carefully that the same-birthday classes are exactly the sets P_d — and notice which assumption about birthdays you had to make. **This is where “every element lies in exactly one class” actually gets proved;** **10.** an equivalence relation determines a partition determines the same equivalence relation. **One half of the fundamental correspondence;** **11.** finish the proof that $\equiv_m$ is an equivalence relation, then find all the classes for $\equiv_2$ and $\equiv_3$ and say how many there are for $\equiv_m$; **12.** every square is 0 or 1 mod 4. Short, useful, and the standard tool for showing some equations have no integer solutions; **13. if** $a' \equiv a$ **and** $b' \equiv b$ **then** $a' + b' \equiv a + b$ **and** $a'b' \equiv ab \pmod{m}$. **If you do one problem in this section, do this one.** It is precisely the statement that $+$ and $\cdot$ are well defined on $\mathbb{Z}/\equiv_m$; it is what §5.1 #21 and #22 build on; and it is the direct prerequisite for quotient groups and quotient rings in MATH 3551; **16.** the union of a partition of A and a partition of a disjoint B is a partition of $A \cup B$; **24. a relation descends to the quotient exactly when it is compatible with the equivalence relation.** The general well-definedness theorem; §5.1 #21 is its version for functions.

Also do **1.** all partitions of $\{1, 2, 3\}$; **2.** all equivalence relations on $\{1, 2, 3\}$ — do #1 first and then match them up; **4.** which relations on $\mathbb{R}$ are equivalence relations? (b) is $\mathbb{R}/\mathbb{Q}$, which is stranger than it looks; **5.** the same for lines in the plane; the “same slope” classes are the directions; **7.** similar triangles; **8.** finish Lemma 4.5.7; **9.** if $A/R = A/S$ then $R = S$ — the other half of the correspondence in #10; **14.** restricting an equivalence relation to a subset, and $[x]_S = [x]_R \cap B$; **20.** $R \cap S$ is an equivalence relation and $[x]_{R \cap S} = [x]_R \cap [x]_S$; **22.** list the elements of $\mathcal{F} \otimes \mathcal{F}$ for the sign partition of $\mathbb{R}$ and draw them in the plane — nine regions, and a good picture.

Skip **25.** preorders, and part (c) asks you to show T is a *partial order* — §4.4; **26.** preorders again, and asks whether the induced order is total — §4.4; **27.** makes the set of partitions into a *partial order* under refinement, and part (c) asks for a greatest lower bound — §4.4.

Notes The three skipped above — 25, 26 and 27 — are all of them, and 27 is the last exercise in the section. §4.5 in the 3rd edition has exactly 27 exercises. If you meet a reference to an “exercise 30” of §4.5 anywhere, it is a 2nd-edition number and there is no such exercise in our book.

#25(a) and #25(b) are perfectly doable on their own if you want them — it is only #25(c) that needs §4.4. #17 has a much longer solution than its neighbours; #18, #21, #23 are the products-and-refinements block and are optional. #19 needs #18 first.

Chapter 5 — Functions

§5.1 Functions

22 exercises

What it is for: A function is a relation in which every input has exactly one output. Then: composition, restriction, and **when a function on a quotient is well defined**.

Problems **1.** is it a function? Three cases, and each fails or succeeds for a different reason; **3.** evaluate; (c) is the floor function defined as a set of ordered pairs, which is worth staring at; **7. restriction** $f \upharpoonright C$. Prove it is a function, and prove $g = f \upharpoonright C$ iff $g \subseteq f$. Restriction is used in §5.2 #13 and §5.3 #14 and #16; **10.** if $f \neq g$ are both functions $A \rightarrow B$ then $f \cup g$ is not a function. Short, and it is really a statement about what “exactly one” means; **20. every function induces an equivalence relation** ($x \sim y$ iff $f(x) = f(y)$), **and every equivalence relation arises this way**, from the map $x \mapsto [x]_R$. The two halves of one idea; **21. if f is compatible with R then there is a unique $h : A/R \rightarrow B$ with $h([x]_R) = f(x)$. This is the theorem the course is built toward.** It is the function version of §4.5 #24; §5.2 #19 and §5.3 #13 finish the story; and it is exactly the universal property of a quotient that MATH 3551 uses for quotient groups; **22.** the concrete instance: $h([x]) = [x^2]$ is well defined mod 5, and $h([x]) = [2^x]$ is not. **Do #21 and #22 together** — #22 is what makes #21 stop being abstract.

Also do **2.** three more “is it a function”; **4.** evaluate three more; **5.** describe $L \circ H$ and $H \circ L$ in words; **6.** composite formulas; **8.** $g \subseteq f$ makes g a function on a subset; **11.** i_A is the only relation that is both an equivalence relation and a function; **12.** when is $f \cup g$ a function? The answer is stated with restrictions — #7 first; **13.** if $S \circ R$ is a function then S is, but R need not be; **14.** pulling a relation back along f preserves all three properties; **18.** “eventually equal” is an equivalence relation on functions $\mathbb{R} \rightarrow \mathbb{R}$.

Skip **19(b).** asks you to show a relation is a **preorder but not a partial order** — §4.4, via §4.5 #25 which we also skip.

Notes **Parts (a) and (c) of #19 are fine and worth doing**, and #19(a) is where the big-oh notation $f \in O(g)$ gets defined. Several later exercises (§6.3 #13, #21; §6.4 #15, #16) refer to that definition, so if you want those, do #19(a) first. #19(c) uses the triangle inequality from §3.5 #13(c).

#12 and #19 have the longest solutions in the section. #15, #16, #17 are optional; #15 is the honest converse of #14 and two of its three parts are false.

§5.2 One-to-One and Onto

23 exercises

What it is for: Injective and surjective, how they interact with composition, and the cancellation properties that characterize them.

Problems **2.** which of the earlier examples are one-to-one? Onto? Do #1, #3, #4 too — they are one minute each; **5.** $f(x) = x/(x-1)$ on $\mathbb{R} \setminus \{1\}$ is a bijection and $f \circ f = i_A$. **Harder than its neighbours** — the algebra is fiddly and the payoff (an involution) is worth it; **6.** $f(x) = ax + b$ is a bijection. The one everyone should be able to do from memory; **10. if $g \circ f$ is onto then g is onto; if $g \circ f$ is one-to-one then f is one-to-one.** Note that the conclusions land on *different* functions. Used constantly from here on; **11.** the contrapositive-flavored companion to #10; **13.** restriction preserves one-to-one, and reflects onto; and neither converse holds; **18. the quotient map $A \rightarrow A/R$ is always onto, and is one-to-one exactly when $R = i_A$; 19. the induced $h : A/R \rightarrow B$ from §5.1 #21 is one-to-one exactly when R is the whole relation “ $f(x) = f(y)$.”** This plus §5.3 #13 is the first isomorphism theorem in embryo.

Also do **1., 3., 4.** the quick classification exercises; **7.** $f(x) = 1/x - x$ on $\mathbb{R}^+$ is a bijection onto $\mathbb{R}$; part (c) changes one sign and everything breaks; **8.** a function into $\mathcal{P}(\mathbb{R})$; **9.** $\bigcup : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathbb{R})$ — is it one-to-one? Onto? **12.** the fibers of f as a function $B \rightarrow \mathcal{P}(A)$; **14.** constant functions; **16.** the converse direction of §5.1 #13; **20. f is onto iff it can be cancelled on the right:** $g \circ f = h \circ f$ implies $g = h$. #21

is the dual, for one-to-one and left cancellation. **These two are “epimorphism” and “monomorphism”** and are the form these ideas take in MATH 3551.

Skip 22. parts (b) and (g) are about **preorders and partial orders** — §4.4, via §4.5 #25.

Notes Parts (a), (c), (d), (e), (f) of #22 do not need §4.4 and are all correct exercises — but #22 as a whole has the longest solution of anything in Chapter 5 by a factor of about four, so it is not a good use of your time either way.

#23 (why “ f maps onto B ” and not just “ f is onto”) is two lines and worth reading even if you do not write it up: whether a function is onto depends on the codomain you declared, which is not true of one-to-one.

§5.3 Inverses of Functions

18 exercises

What it is for: When is f^{-1} a function? The answer — exactly when f is one-to-one and onto — plus the one-sided-inverse characterization and the factorization of an arbitrary function through its quotient.

Problems 3. show a given $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bijection and find a formula for f^{-1} . **The computation to be fluent at;** 5. the same for $f(x) = 10^{2-x}$, where the codomain matters; 9. **prove part 2 of Theorem 5.3.3:** the inverse of a one-to-one onto function is itself one-to-one and onto. Standard; 11. **if f is one-to-one and $f \circ g = i_B$ then $g = f^{-1}$;** if f is onto and $g \circ f = i_A$ then $g = f^{-1}$; and if only one of the two composites is the identity, then f is onto but not one-to-one. **The one-sided inverse theorem**, and part (c) is the one that surprises people; 13. **every onto $f : A \rightarrow B$ factors as a bijection $A/R \rightarrow B$ composed with the quotient map**, where R is “ $f(x) = f(y)$.” **This is the payoff of §5.1 #20–21 and §5.2 #19** — read those three and this one as a single argument. In MATH 3551 it is called the first isomorphism theorem; 14. if $f \circ g = i_B$ then g is the inverse of f restricted to $\text{Ran}(g)$ — the general form of what #11(c) hints at; 18. if g is a bijection then any $f : A \rightarrow C$ factors as $g \circ h$. Two lines once you see it.

Also do 1., 2. read off two inverses from earlier examples; 4. $f(x) = 2x^3 - 3$; 6. f on $\mathbb{R} \setminus \{2\}$ — you must find the right codomain B yourself; 7. $(f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1}$, verified by hand on a concrete pair. This is Theorem 4.2.5 part 5 (§4.2 #7) cashed in. Socks and shoes: you take them off in the opposite order; 8. the other half of Theorem 5.3.2, twice, the second time by a trick; 10. an alternative proof of Theorem 5.3.5; 12. if f is one-to-one, f^{-1} is a function on the range; 16. $f(x) = 4x - x^2$: find the range, then cut the domain down until it is invertible. **This is what “restrict to $[0, \infty)$ ” in precalculus was really doing;** 17. conjugacy of functions — $f \sim g$ iff $g = h \circ f \circ h^{-1}$ for some bijection h — is an equivalence relation, and fixed points are an invariant of it.

Skip None.

Notes #6, #16 and #17 have the longest write-ups. #15 is a short application of #14 and needs it first.

§5.5 Images and Inverse Images: A Research Project

7 research questions

What it is for: The image $f(X)$ of a subset of the domain, the inverse image $f^{-1}(Y)$ of a subset of the codomain — and the fact that $f^{-1}(Y)$ means something for *every* function, including ones that have no inverse.

This section is not shaped like the others. Velleman does not state the theorems and ask you to apply them; he asks the questions and leaves the theorems for you to find. So the items below are numbered *research questions*, not exercises, and each one is literally of the form “will it always be true that...?” The honest answer to most of them is “no, but yes if f is one-to-one” or “no, but yes if f is onto,” and **your job is to work out which, prove the half that is true, and build a counterexample for the half that is not.** Read §5.5 up through Theorem 5.5.2 first: the text takes one such question — is $f(W \cap X) = f(W) \cap f(X)$? — and works it out in front of you, including how it went looking for a counterexample once the proof stalled. That worked example is the model for all seven.

Velleman means this section to be *done*, not watched. There are no answers in the back of the book for any of it. Give it a real sitting before you look at anything.

Problems **1(a).** is $f(W \cup X) = f(W) \cup f(X)$? **Start here.** This is the one that is true with no hypothesis at all, and proving it will show you why union behaves and intersection does not; **1(b).** is $f(W \setminus X) = f(W) \setminus f(X)$? Compare your answer with Theorem 5.5.2 — **the containment goes the other way** from the one you just proved for $\cap$, which is worth pausing over; **2(a).** is $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$? **Unconditionally yes, and the proof is a four-step chain of “iff.”** Write it as a chain. This is the first sign of the moral of the section: **inverse images are better behaved than images**; **3.** is $f^{-1}(f(X)) = X$? Always one containment; the other needs a hypothesis. The counterexample is already in the text ($f(x) = x^2$); **4.** is $f(f^{-1}(Y)) = Y$? The mirror image of #3, and it needs the *other* hypothesis. **This is the one that gets used later** — see the note below; **7.** the inverse image of Y under f and the image of Y under the function f^{-1} are the same set, when f is a bijection so that both phrases mean something. **This is the exercise that makes the notation $f^{-1}(Y)$ safe** and is why nobody has to say which of the two things they meant. Short; do it.

Also do **1(c).** is $W \subseteq X \leftrightarrow f(W) \subseteq f(X)$? One arrow is free; **2(b).**, **2(c).** inverse image commutes with $\cup$ and with $\setminus$ as well, by the same chain-of-iffs method as #2(a) — three near-identical proofs, so do them together and notice you are writing the same thing three times; **2(d).** is $Y \subseteq Z \leftrightarrow f^{-1}(Y) \subseteq f^{-1}(Z)$? The reverse arrow needs f onto. **The longest of the seven**; **6.** what can you prove about images and inverse images under $g \circ f$? Deliberately open-ended. Two things are true and both are clean: $(g \circ f)(X) = g(f(X))$ and $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$ — **note the order reverses in the second one**, exactly as in §5.3 #7.

Skip **5.** proves that “ C is closed under f ,” $f(C) \subseteq C$, and $C \subseteq f^{-1}(C)$ are equivalent. **“Closed under f ” is §5.4**, which we do not cover. The proof itself is three short implications and needs nothing but the two definitions in this section, so if you read §5.4’s first page you can do it — but it is not assumed anywhere else on this list.

Notes **#4 is the natural source of a fact used in Chapter 8.** The proof that $A \sim B$ implies $\mathcal{P}(A) \sim \mathcal{P}(B)$ (§8.1 #5) is exactly #4 applied to the map $X \mapsto f(X)$, and Theorem 5.5.2 itself gets used again in §8.2 #15(d). If you skip this section you will meet both of them again as gaps.

One counterexample does most of the work. Take $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $f = \{(1, 4), (2, 5), (3, 5)\}$ — the function from Figure 5.6. With suitable W, X, Y, Z it settles the false halves of #1(b), #1(c) and #2(d). Once you have found it once, do not go looking for a new one each time.

Difficulty. #2(a), #2(b), #2(c) and #7 are a few lines each and are the place to start if the section looks impossible. #2(d) and #1 are the long ones. #6 has no fixed length, because you decide what to prove.

Chapter 6 — Mathematical Induction

§6.1 Proof by Mathematical Induction

20 exercises

What it is for: Base case, inductive hypothesis, inductive step — and the fact that all three are required.

Problems 1. $0 + 1 + \cdots + n = n(n+1)/2$. **The first one; write out every step by name** (“base case,” “let n be arbitrary and suppose,” “then”) and keep that skeleton for the rest of the chapter; 3. the sum of cubes is the square of the sum; 4. **find** the formula for $1 + 3 + 5 + \cdots + (2n-1)$ and then prove it. Guessing and proving are two different jobs and both are graded; 7. find and prove a formula for $3^0 + 3^1 + \cdots + 3^n$. §6.3 #5 generalizes this to arbitrary ratio; 9. $2 \mid (n^2 + n)$ and $6 \mid (n^3 - n)$. **The template for every divisibility induction:** add and subtract the right multiple in the inductive step; 15. $2^n > n^3$ for $n \geq 10$. **An induction whose base case is not 0**, and the inductive step needs a genuine inequality argument rather than algebra; 16. **every natural number is either even or odd and not both** — and then every integer is. This is the fact §3.4 used without proof; this is where it gets proved; 18. **what is wrong with this proof?** It has an inductive step and no base case, and the “theorem” is false. **The single most valuable exercise in the section.** Then find the correct formula.

Also do 2. sum of squares; 5. $\sum n(n+1)$; 6. $\sum n(n+1)(n+2)$ — guess it from #1 and #5; 10. $64 \mid (9^n - 8n - 1)$; 11. $9 \mid (4^n + 6n - 1)$; 12. $7^n - 5^n$ is even, and a harder divisibility; 13. $(a-b) \mid (a^n - b^n)$ for all integers a, b . **The hint tells you to fix a and b first and induct only on n** — learning what to induct on is half the skill; 14. $(a+b) \mid (a^{2n+1} + b^{2n+1})$; 17. $\sum (n+1)2^n = n2^{n+1}$; 19. signs of powers of a negative number.

Skip None.

Notes Nothing in §6.1 is skipped, and that includes #7. §6.1 #7 is “find a formula for $3^0 + 3^1 + \cdots + 3^n$ ” and needs nothing but induction — it is on the list above. All twenty exercises here were checked one by one against the not-covered list (§4.4, §5.4, §6.2, §6.5, Chapter 7) and none of them touches it.

#20 has by far the longest solution in the section — four parts, one of which needs n th roots, and the last is the power-mean inequality. Optional, and out of line with everything around it.

§6.3 Recursion

22 exercises

What it is for: Recursive definitions and $\sum$ notation — and induction proofs where the object itself is defined by induction.

Problems 1. find and prove a formula for $\sum_{i=1}^n \frac{1}{i(i+1)}$. Telescoping, and the answer is prettier than you expect; 5. **the geometric series** $\sum_{i=0}^n r^i = \frac{r^{n+1}-1}{r-1}$. **This is the natural source** — §6.1 #7 and Example 6.1.1 are both special cases, and you will use this formula in every course you take after this one; 6. $\sum_{i=1}^n 1/i^2 \leq 2 - 1/n$. **An induction proving an inequality, where the inductive step needs an estimate you have to find;** 7. $\sum (a_i + b_i) = \sum a_i + \sum b_i$ and $c \sum a_i = \sum c a_i$. Linearity of $\sum$. **You use these silently forever afterwards; prove them properly once;** 8. **the harmonic numbers.** $H_n - H_m \geq (n-m)/n$, then $H_{2^n} \geq 1 + n/2$, then $\sum 1/i$ diverges. **The best problem in the section** — an induction that proves a real theorem about a real series; 12. $2^n > n$, then $n! \geq (2n)^2$ for $n \geq 9$, then $n! \leq 2^{n^2}$. Three growth rates, three inductions; 15. prove $a_n = 2^n - n - 1$ for a recursively defined sequence. **The standard form of a recursion problem:** the formula is given, you verify it; 16. the same, except **you have to find the formula first.** Compute a_0, a_1, a_2, a_3 and look.

Also do 2., 3., 4. three more $\sum$ identities by induction — pure drill, do two of them; 9. another harmonic-number inequality (needs #8); 10., 11. find and prove two more summation formulas; 13. $n! \geq k^n$ eventually, for a fixed k ; 14. $(a^m)^n = a^{mn}$ — the exponent laws, from the recursive definition of a power. Short, and it shows what these definitions are actually for; 17. another find-the-formula recursion; 20.

$0 < a_n < 1$ for a recursive sequence — **an induction whose statement is an inequality rather than an identity**, which is a different feel.

Skip None outright, but three warnings:

Notes #18 is the binomial theorem and Pascal's triangle. It is a genuinely important result and its solution is *four times* the length of anything else in the section — treat it as a weekend project, not a homework problem. Its part (c) hint refers you to §6.2 exercises 11 and 12, which **we do not cover**; ignore the hint and prove it directly. #19 likewise offers three routes, one of which goes through §6.2 — use one of the other two. #21 parts (d) and (f) use the big-oh notation from §5.1 #19(a); do that part first or skip #21.

#7 has the second-longest solution here and is still worth every line.

§6.4 Strong Induction

21 exercises

What it is for: Assuming the statement for *all* smaller values, not just the previous one; the well-ordering principle; the division algorithm; and the irrationality of $\sqrt{2}$.

Problems 1. **derive strong induction from ordinary induction.** Two short parts, and after this you will never wonder whether strong induction is a separate axiom; 3. $\sqrt{2}$ is irrational, by well-ordering: take the smallest denominator and produce a smaller one. Compare it with the parity proof (Theorem 6.4.5) — same theorem, two completely different arguments; 4. $\sqrt{3}$ is irrational, and $\sqrt[3]{2}$ is. **Now do it yourself** on a case the book did not do; 5. **the Martian bead problem:** every $n \geq 12$ is a non-negative combination of 3 and 7. **The standard strong-induction template** — you reach back three steps, not one, and the base case is therefore three cases wide; 6. if $x + 1/x$ is an integer then so is $x^n + 1/x^n$. **The cleanest example of needing two previous cases;** 7. four identities about Fibonacci numbers. Do at least (a) and (b); 12. a recursively defined sequence turns out to be the Fibonacci sequence; 14. **the division algorithm:** for $m > 0$ and any integer n there are *unique* q and r with $n = qm + r$ and $0 \leq r < m$; and every integer is exactly one of 0, 1, 2 mod 3. **This is the natural source of the fact that makes $\mathbb{Z}/m$ have exactly m classes** (§4.5 #11(b)), and it is used constantly in MATH 3551. **Star of the section.**

Also do 2. rewrite Theorem 6.4.5 as a strong induction; 10. find and prove a formula for the Lucas numbers (use #9); 11. another linear recurrence solved by the #9 method; 13. the number of subsets of $\{1, \dots, n\}$ with no two consecutive elements is F_{n+2} . **The nicest combinatorial application in the chapter;** 17., 18. two more find-the-formula recursions; 19. there are no positive integers a, b, c, d with $a^2 + b^2 = 3(c^2 + d^2)$, by infinite descent. **Uses #14(c) and the well-ordering principle together** and is the best hard problem here; 20(a). the golden ratio from a rectangle.

Skip None outright, but three warnings:

Notes #15 and #16 use the big-oh notation defined in §5.1 #19(a); do that part first or skip these. #20(c) sends you to Example 6.2.3 for the angles of a polygon, and **§6.2 is not covered** — the fact itself (interior angles sum to $(n - 2) \cdot 180^\circ$) is standard, so the exercise survives, you just will not have seen the example.

One thing that looks like it should be skipped and is not: several exercises here talk about the *smallest* element of a set of natural numbers. That is the well-ordering principle, which we cover in this very section — it is not the order theory of §4.4. Do them.

Difficulty warning. #8 has by far the longest solution of any exercise in the book — five parts on Fibonacci identities, several times the length of #7. #9 (the general Fibonacci sequence and Binet's formula), #20 and #21 are also well out of line with their neighbours. #9 is beautiful and is the machinery behind #10 and #11, so if you want one of the four, take #9.

Chapter 8 — Infinite Sets

This is the last three class days of the term. §8.1 and §8.2 are the chapter; **§8.3 is a stretch goal and not a promise.** If we arrive at December 11 with the Cantor–Schröder–Bernstein theorem still in front of us, we will do it, and if we do not, nothing earlier on this list depends on it. Plan your last week accordingly: get §8.1 and §8.2 solid first, and treat §8.3 as the reward.

§8.1 Equinumerous Sets

30 exercises

What it is for: Measure the size of a set by what it can be put in one-to-one correspondence with, not by counting. This gives finite vs. infinite, denumerable vs. uncountable, and — for the first time — an honest definition of $|A|$.

Problems 1. $\mathbb{N}$ and the even integers are denumerable. **The first one, and both parts are a single explicit formula plus two short verifications.** Everything in the chapter looks like this; 4. if $A \sim B$ and $A \times C \sim B \times D$, must $C \sim D$? And the union version. **Both answers are no**, so this is counterexample practice, and the counterexamples are small; 6. if $I_n \sim I_m$ then $n = m$; hence a finite set is equinumerous with exactly one I_n . **This is the exercise that makes $|A|$ mean anything at all.** Every later statement with a $|\cdot|$ in it is standing on it. Induction on n ; 8. a subset of a finite set is finite, with $|B| \leq |A|$, and $<$ if the containment is proper. **The most-used fact in the section** — #10, #15, #20 and §8.2 #15 all call it; 11. if $|A| > |B|$ then f is not one-to-one. **This is the pigeonhole principle**, and part (c) — for finite sets of equal size, one-to-one and onto are the same condition — is the fact that fails for infinite sets and thereby makes the whole chapter interesting; 13. $f(m, n) = 2^{m-1}(2n - 1)$ is a bijection $\mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$. **Every positive integer is a power of two times an odd number, exactly once.** The cleanest proof that $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$, which is the engine of §8.2; 17. a subset of a countable set is countable. **Two lines using statement 3 of Theorem 8.1.5**, and you will use it every day after this; 22(a). $|A \times B| = |A| \cdot |B|$. Induction on $|B|$. Part (b) is the same fact with a dinner menu attached and takes ten seconds.

Also do 3. $[0, 1] \sim [0, 2]$, $(-\pi/2, \pi/2) \sim \mathbb{R}$, $(0, 1) \sim \mathbb{R}$, $(0, 1) \sim (0, 1)$. **Do (a), (b), (c); see the note on (d) below;** 5. $A \sim B$ implies $\mathcal{P}(A) \sim \mathcal{P}(B)$ — **this is §5.5 #4 cashed in;** 7. if A is finite, $A \sim B$ iff B is finite and $|A| = |B|$; 9. if $B \subsetneq A$ and $B \sim A$ then A is infinite. **Two lines, and it is the property that actually distinguishes infinite sets** — a finite set is never equinumerous with a proper subset of itself, and $\mathbb{Z}^+ \sim \mathbb{Z}$ shows an infinite one can be; 10. an onto $f : I_n \rightarrow B$ forces B finite with $|B| \leq n$; 14. finish part 2 of Theorem 8.1.2: $f \cup g$ is a bijection when the domains and the codomains are disjoint. The first half is **§5.1 #12 again**, in the case where the domains do not overlap; 18. if A is infinite and $B \subseteq A$ is finite, then $A \setminus B$ is infinite; 20. $|A \setminus B| = |A| - |B|$ for $B \subseteq A$ finite. Two lines from Theorem 8.1.7, and it is the fact Chapter 6 kept using without proof; 26. $|A \cup B| = |A| + |B| - |A \cap B|$, and the three-set version. **Inclusion–exclusion, finally proved;** 29. more than 10,000 customers means two share a PIN. **#11 in one line**, and #30 (Alice and Bob both missed half the classes) is the same trick in disguise.

Skip 2(b). uses a notation defined in §5.4 #21(b) — closures, not covered. **#2(a) ($\mathbb{Q} \times \mathbb{Q}$ is denumerable) is fine and worth doing;** 16(b)., 16(c). build a bijection $\mathbb{Z}^+ \rightarrow \mathbb{Q}^+$ out of **prime factorizations**, and the hint sends you to §7.2 #19. **Chapter 7 is not covered.** #16(a) stands alone and is a one-line warm-up; 19. extends a partial order on a denumerable set to a total order. **Two strikes:** partial and total orders are §4.4, and the hint’s recursion runs through §6.2 #2; 24(b). counts the **total orders** on an n -element set — §4.4 again. **#24(a) ($|F| = n!$ for the bijections $I_n \rightarrow A$) and #24(c) are fine**, and (a) is the cleanest proof of $n!$ you will see.

Notes #3(d) is the difficulty outlier of the section, and it is a trap. Finding an explicit bijection $(0, 1] \rightarrow (0, 1)$ by hand is genuinely hard — you have to shift a sequence and leave everything else alone. **If §8.3 happens, it does this in three lines** (Cantor–Schröder–Bernstein, and then §8.3 #9 recovers the explicit formula). Do (a), (b), (c) now; leave (d) until you have decided whether §8.3 is happening for you.

#23 defines the notation AB for the set of all functions $A \rightarrow B$, which §8.2 #6 and §8.3 #5 and #14 all refer back to. If you want any of those, do at least #23(a) first. #23 itself is long (part (b) especially).

Difficulty. Against a section median of about nine lines of solution, #24 (50), #27 (42), #12 (37), #23 (36), #16 (33), #6 (30) and #21 (25) are the long ones. **#27 is the general inclusion–exclusion principle** and is a real weekend problem; #26 is the honest version of it for two and three sets and is what you actually need. **#12 proves $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$ a second time**, by the diagonal-counting formula from Figure 8.2; it is the picture everyone remembers, but #13 is the proof everyone can write. Do #13; read #12.

At the short end, #9, #18 and #20 are two lines each and #7, #17 and #4 are three. Start there.

#15 completes the proof of Theorem 8.1.5 (an infinite subset of $\mathbb{Z}^+$ is denumerable, by listing it in increasing order). It is the only part of that theorem the text leaves to you, and it leans on #8 and #10 — do those first.

§8.2 Countable and Uncountable Sets

18 exercises

What it is for: Countability survives products, unions of countably many countable sets, and finite sequences — so it is very hard to escape. Then Cantor’s diagonal argument escapes it, and $\mathcal{P}(\mathbb{Z}^+)$ and $\mathbb{R}$ turn out not to be countable at all.

Problems 1. the irrationals are uncountable, and $\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$. **The first thing diagonalization buys you:** there are, in a precise sense, vastly more irrational numbers than rational ones, which is the opposite of what constructing them one at a time suggests. Part (a) is a two-line proof by contradiction from Theorem 8.2.1; 4. the *finite* subsets of $\mathbb{Z}^+$ form a denumerable set. **Put this next to Cantor’s theorem and stare:** $\mathcal{P}(\mathbb{Z}^+)$ is uncountable, but the finite sets in it are only countably many, so essentially all subsets of $\mathbb{Z}^+$ are infinite; 5. **Cantor’s theorem in general:** $A \not\sim \mathcal{P}(A)$ for *every* set A , finite or not. Imitate Theorem 8.2.5 — the diagonal set is $D = \{a \in A \mid a \notin f(a)\}$ and the proof is four lines. **This is the source of §8.3 #7** and hence of the whole tower of infinities; 8. $\mathcal{P}(A \cup B) \sim \mathcal{P}(A) \times \mathcal{P}(B)$ for disjoint A and B . Short, and the bijection $(X, Y) \mapsto X \cup Y$ is the obvious one for once; 12. **the algebraic numbers are countable.** Therefore transcendental numbers exist, and in fact almost every real number is one — **without exhibiting a single one.** This is the most famous application of the chapter and the argument is three steps you already have; 17. the set of grammatical English sentences is denumerable (every sentence is a finite sequence of words, and there are finitely many words); 18. **therefore there are real numbers that cannot be defined by any English phrase. Do #17 and #18 as one problem** — #17 is only worth doing for the sake of #18, and #18 is the best sentence in the chapter.

Also do 2. verify that $F(f, a)$ (append a to the sequence f) is a bijection $S_n \times A \rightarrow S_{n+1}$, completing Theorem 8.2.4. Bookkeeping, but do it once; 7. a denumerable set can be partitioned into denumerably many denumerable pieces. $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$ **doing real work**; 9. increasing unions: if $A_1 \subseteq A_2 \subseteq \dots$ exhaust the set, an uncountable subset must already be uncountable inside some A_n ; 10. if every finite subsum of $A \subseteq \mathbb{R}^+$ stays below a fixed bound, then A is countable. **The one that feels like analysis** and is the reason a convergent sum can only have countably many nonzero terms; 11. an equivalence relation on $\mathbb{R}$ with all classes equinumerous forces either uncountably many classes or uncountable classes. Uses §4.5; 13. **any countable family of functions $\mathbb{Z}^+ \rightarrow \mathbb{R}$ is dominated by a single g** , i.e. contained in $O(g)$. This is the payoff for having done §5.1 #19(a), where big-oh was defined — if you skipped it, do it now; 14. a pairwise disjoint family of subsets of $\mathbb{Z}^+$ is countable. **Read it next to #15, which says “almost disjoint” fails spectacularly**; 16. there is an $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ hitting every prescribed value somewhere along every arithmetic progression. A nice use of listing $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$.

Skip 3. gives a second proof of Theorem 8.2.4 by sending a sequence to $p_1^{g(f(1))} \dots p_n^{g(f(n))}$. **The step where this is one-to-one is unique prime factorization**, which is §7.2, and we do not prove it. The exercise is readable and the idea is lovely; you just cannot honestly close it with what we have done. **Theorem 8.2.4 itself is proved in the text and needs nothing from Chapter 7**, so nothing is lost.

Notes Read the scratch work before Theorem 8.2.5. Velleman spends two pages explaining how anyone would ever think of the set $D = \{n \in \mathbb{Z}^+ \mid n \notin f(n)\}$, and that reasoning — not the four-line proof — is the thing worth having. Then read the proof, then go back and read the scratch work again, exactly as he tells you to.

#5 is the source of a theorem used later. $A \not\prec \mathcal{P}(A)$ is what §8.3 #7 upgrades to $A \prec \mathcal{P}(A)$, and §8.3 #8 then iterates it into infinitely many distinct sizes of infinity. If §8.3 happens, #5 is the prerequisite.

Difficulty. #6 is the outlier at about 47 lines against a median near nine — it is four separate exponent laws for function sets, and it needs the AB notation from §8.1 #23, so it costs two exercises rather than one. #15 (23 lines) is the hardest *good* problem here: an uncountable family of infinite subsets of $\mathbb{Z}^+$ that pairwise meet in only finitely many points. **Its part (d) uses Theorem 5.5.2 from §5.5** — one more reason §5.5 was worth the day we spent on it. #9 (20) and #16 (19) follow. At the short end #2, #5, #7, #11 and #13 are five lines or fewer.

#4 is needed by #15, so take them in that order if you want #15.

§8.3 The Cantor–Schröder–Bernstein Theorem

14 exercises

What it is for: Write $A \lesssim B$ when some one-to-one $f : A \rightarrow B$ exists. The theorem says $A \lesssim B$ and $B \lesssim A$ together give $A \sim B$ — so “no bigger than” behaves the way the notation promises. Consequences: $\mathbb{R} \sim \mathcal{P}(\mathbb{Z}^+)$, and infinitely many distinct sizes of infinity.

Stretch goal. This is scheduled for the last day and it is the first thing that gets cut if we are behind. **Nothing else on this list depends on it**, and no quiz will assume it. It is here because it is the best theorem in the book and because §8.1 and §8.2 leave two questions visibly open — is $\lesssim$ antisymmetric up to $\sim$, and is $\mathbb{R}$ the same size as $\mathcal{P}(\mathbb{Z}^+)$ — that this section answers.

Problems 1. $\lesssim$ is reflexive and transitive. **Three lines, and do it before anything else in the section** — transitivity is used silently in every later proof here; 3. if $A \subseteq B \subseteq C$ and $A \sim C$ then $B \sim C$. **The sandwich corollary, and the form in which the theorem is actually used in practice.** Two applications of #1 and one of Theorem 8.3.2; 7. $A \prec \mathcal{P}(A)$ for every set A . **The power set is always strictly bigger**, which combined with #8 is the sentence “infinity comes in infinitely many sizes.” Uses §8.2 #5; 8. let $A_1 = \mathbb{Z}^+$ and $A_{n+1} = \mathcal{P}(A_n)$; show $A_n \prec A_m$ whenever $n < m$. **The punchline of the chapter.** Part (b) is deliberately open — yes, there is a set bigger than all of them, and finding it is the point; 9. run the construction in the proof of Theorem 8.3.2 by hand on $f(x) = x/2$ and the inclusion, and get an explicit bijection $(0, 1] \rightarrow (0, 1)$. **The one exercise that makes the proof stop being magic**, and it is also the answer to §8.1 #3(d), which was miserable to do directly. **If you do one exercise in §8.3, do this one;** 13. a pairwise disjoint family of nondegenerate intervals is countable. **Each one contains a rational** (Lemma 8.3.4) and no two can contain the same one. A one-idea proof and a genuinely useful fact.

Also do 2. $\prec$ is irreflexive and transitive — the companion to #1, and (b) is fussier than it looks; 4. $\lesssim$ respects products, disjoint unions and power sets; 5. $\lesssim$ respects function sets. **Needs the AB notation from §8.1 #23**, and part (b) asks whether the hypothesis $A \neq \emptyset$ was really needed — a good habit to be asked about; 6. if $A \lesssim B$ and B is finite then A is finite with $|A| \leq |B|$. Ties the new relation back to counting; 10. the equivalence relations on $\mathbb{Z}^+$ are equinumerous with $\mathcal{P}(\mathbb{Z}^+)$. Uses partitions from §4.5 and is the best “count the structures” problem in the chapter; 12. $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$. **The plane has exactly as many points as the line**, which is the result that convinced Cantor himself that something was wrong with his intuition. Part (a) does the work; (b) is Theorem 8.3.3 plus (a); 14. $\mathbb{R}^{\mathbb{R}} \sim \mathcal{P}(\mathbb{R})$ and $\mathbb{Q}^{\mathbb{R}} \sim \mathbb{R}$. Same notation caveat as #5.

Skip 11. counts the **total orders** on $\mathbb{Z}^+$ — §4.4, which we do not cover. It is a direct imitation of #10, so if you have done #10 and know what a total order is, it costs you nothing; it is simply not assumed.

Notes One aside in the text is not an exercise and not a problem. On the first page Velleman

remarks that $\lesssim$ fails to be a partial order and is “a preorder, in the terminology of exercise 25 of Section 4.5.” **We skip §4.5 #25** (it needs §4.4). Ignore the parenthesis: the point being made is just that $A \lesssim B$ and $B \lesssim A$ do not force $A = B$, which is exactly what makes Theorem 8.3.2 worth proving.

Similarly #7’s hint routes you through §6.2 #11 and §6.3 #12(a) to observe $2^n > n$. **§6.2 is not covered**, §6.3 is, and in any case the hint is an aside about the finite case — the exercise itself only needs §8.2 #5.

Difficulty. Median about ten lines. #14 (21), #4 (20) and #5 (16) are the long ones, and **#14(c) is for readers who have had calculus** — a continuous function is determined by its values on $\mathbb{Q}$, so there are only $|\mathbb{R}|$ of them. #1, #2, #3 and #7 are two to five lines each and are where to start.

Read Theorem 8.3.3 ($\mathbb{R} \sim \mathcal{P}(\mathbb{Z}^+)$) even if you prove nothing here. It is the reason the two uncountable sets you have met are the same set in disguise, and the paragraph after it — Cantor’s continuum hypothesis, Hilbert’s first problem, and Gödel and Cohen showing it can be neither proved nor disproved — is the natural place for this course to stop.