Arithmetic Kolchin Irreducibility

Taylor Dupuy
(with James Freitag and Lance E. Miller)

Kolchin

$$X/\mathbf{C} \text{ irreducible} \implies J_\infty(X) \text{ irreducible}$$

(singular)

Let A be a $\mathbf{C}$-algebra

X variety over $\mathbf{C}$

$$\begin{aligned} J_n(X), J_\infty(X) &= \text{new varieties over } \mathbf{C} \\ &= \text{higher order tangent spaces} \end{aligned}$$

$$J_n(X)(A) = X(A[T]/(t^{n+1}))$$

$$J_\infty(X)(A) = X(A[[T]])$$

Gillet, Mustata, de Fernex, Loeser-Sebag, Kolchin, Nicaise-Sebag, Ishii-Kollar, (Chambert-Loir)-Nicaise-Sebag

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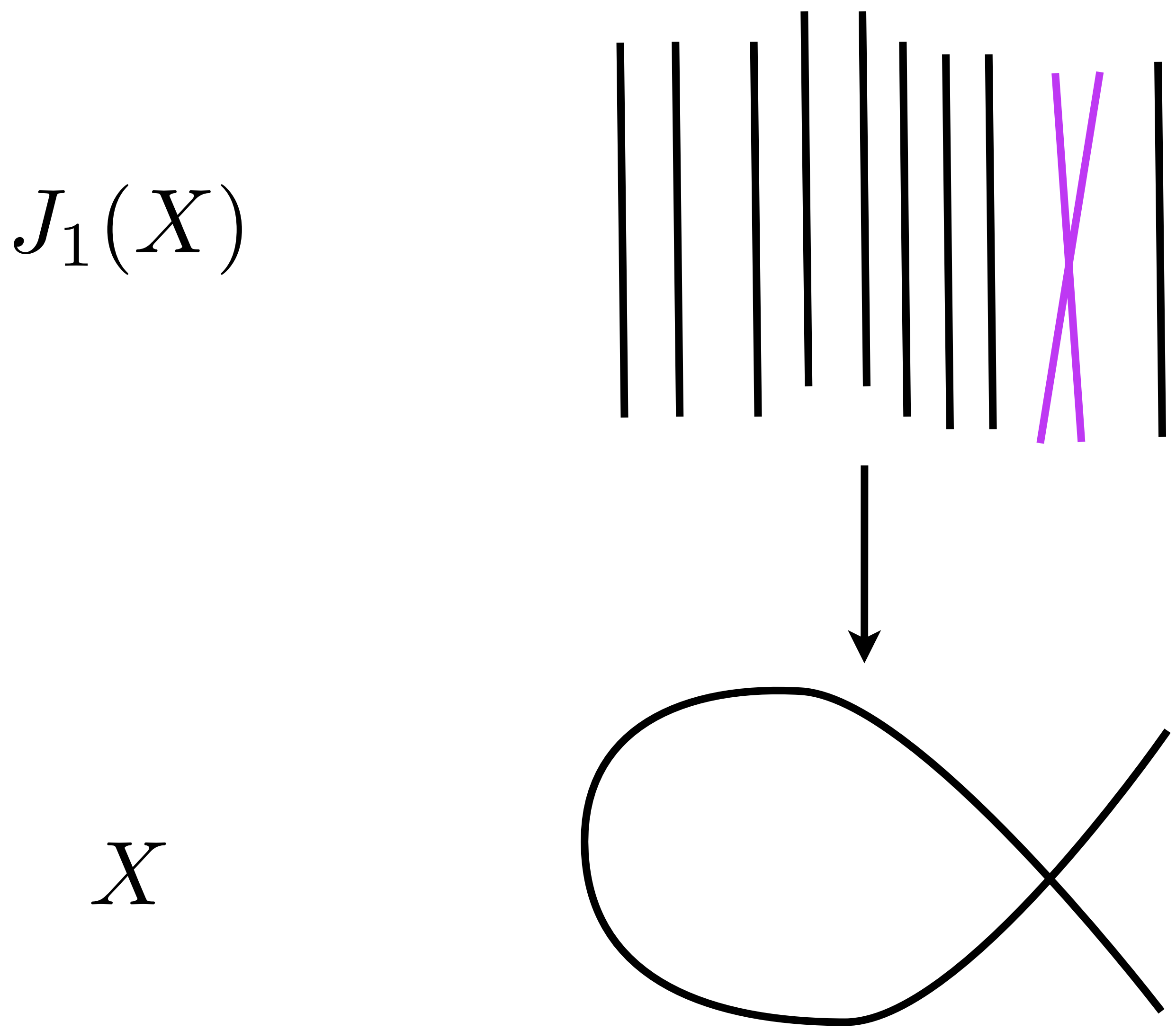
Example 1.

$$J_1(X)$$

$$X$$

Example 1.



Example 2. $X : x^4 + y^4 + z^4 = 0$

$$\text{expected dimension of } J^3(X) = 2 \cdot 4 = 8$$

$$\text{dimension above } (0,0,0) = 9$$

proof:

$$J_3(X) = \text{plug in } \mathbf{C}[t]/(t^4) \text{ valued points}$$

$$\begin{aligned} x &= x_0 + x_1 t + x_2 t^2 + x_3 t^3 \mod t^4 \\ y &= y_0 + y_1 t + y_2 t^2 + y_3 t^3 \mod t^4 \\ z &= z_0 + z_1 t + z_2 t^3 + z_3 t^3 \mod t^4 \end{aligned}$$

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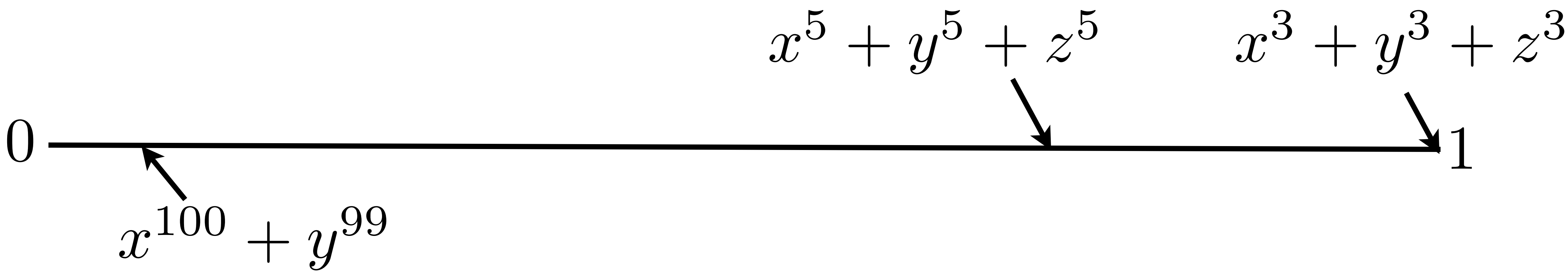
$$(x_1 t + x_2 t^2 + x_3 t^3)^4 + (y_1 t + y_2 t^2 + y_3 t^3)^4 + (z_1 t + z_2 t^3 + z_3 t^3)^4 \equiv 0$$

Mustata:

$$\mathrm{lct}(X, D) = \dim(X) - \sup_{r \geq 0} \frac{\dim J_r(D)}{r+1}$$

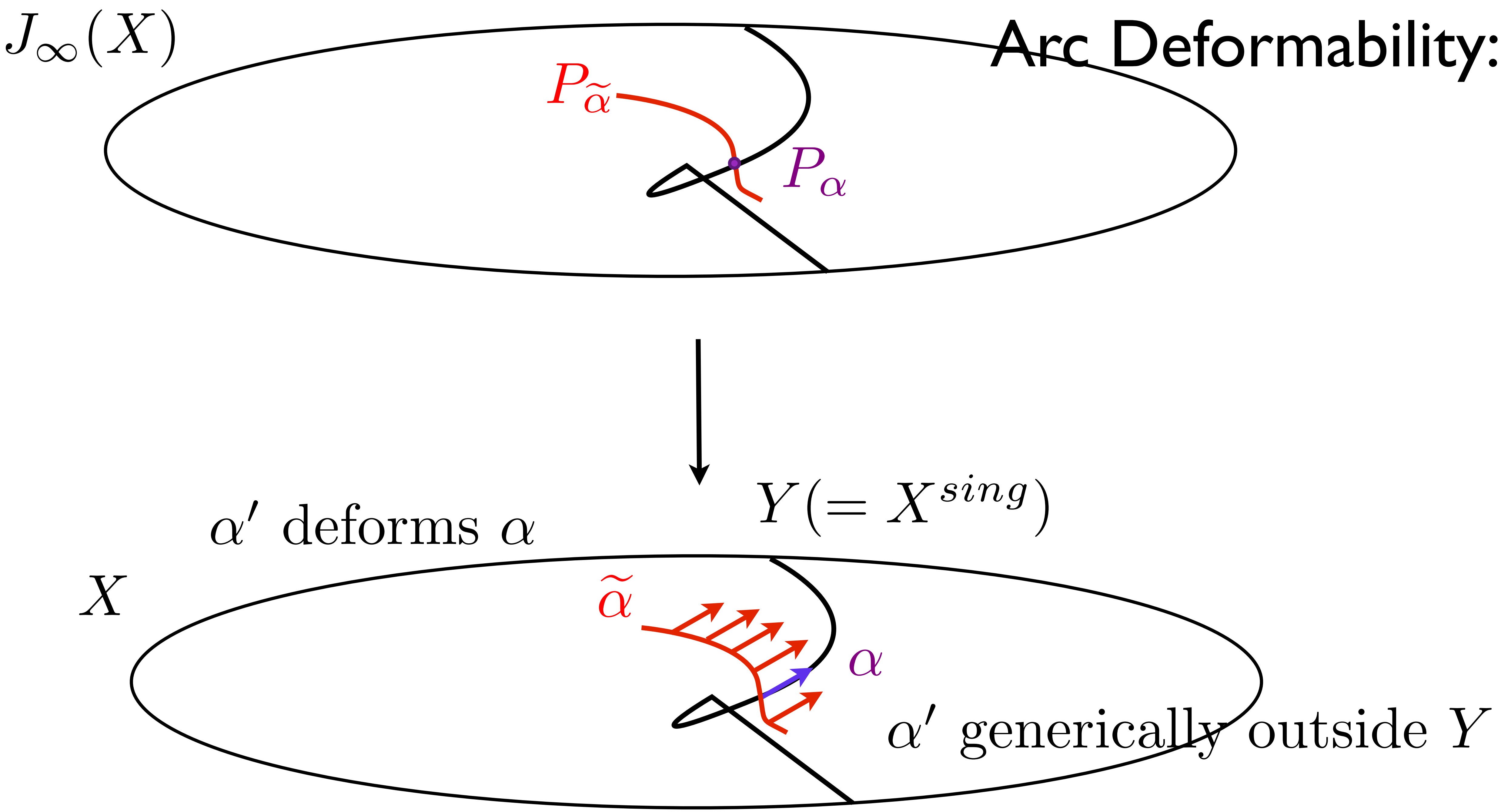


Proof of Kolchin Irreducibility

- Step 1: Deformations = Irreducibility.
- Step 2: Smooth case.
- Step 3: Reduction to Smooth Case

Step 1: Deforming Arcs = Irreducibility

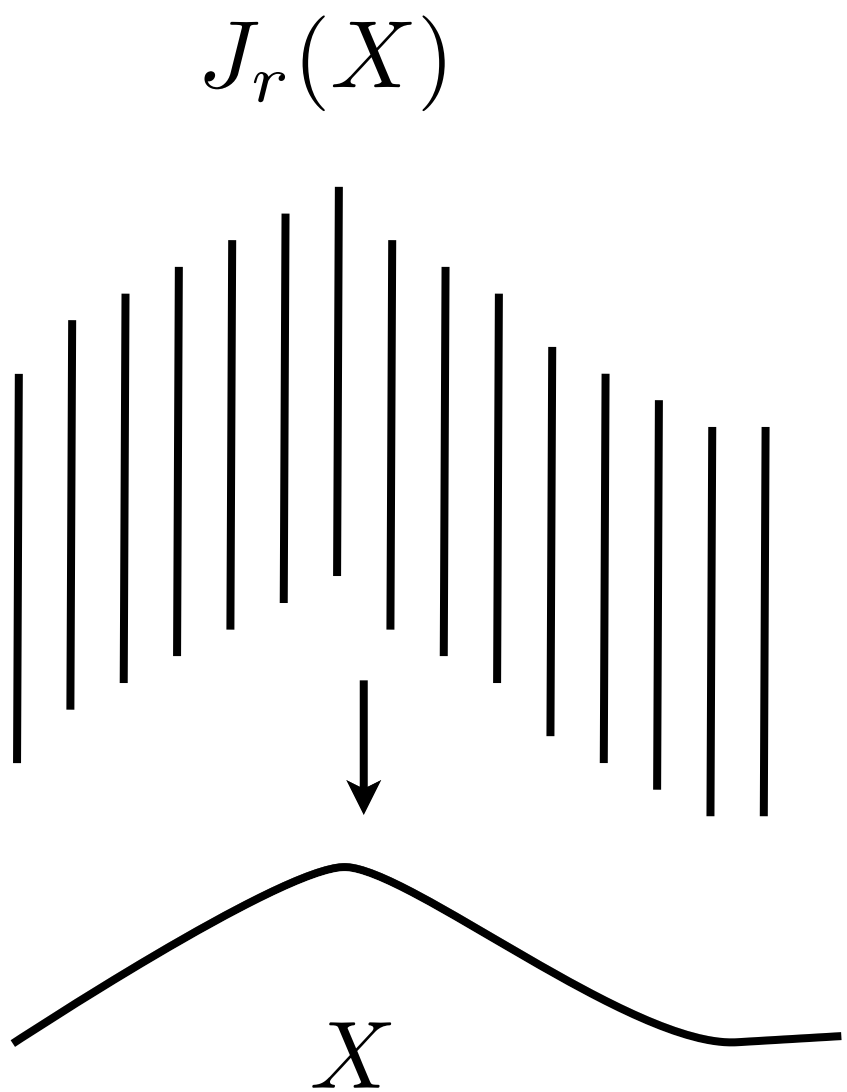
Arc Deformability:



Step 2: Smooth Case (Classical)

Theorem.

$X/\mathbf{C}$ smooth, irreducible $\implies$ $J_r(X)$ irreducible



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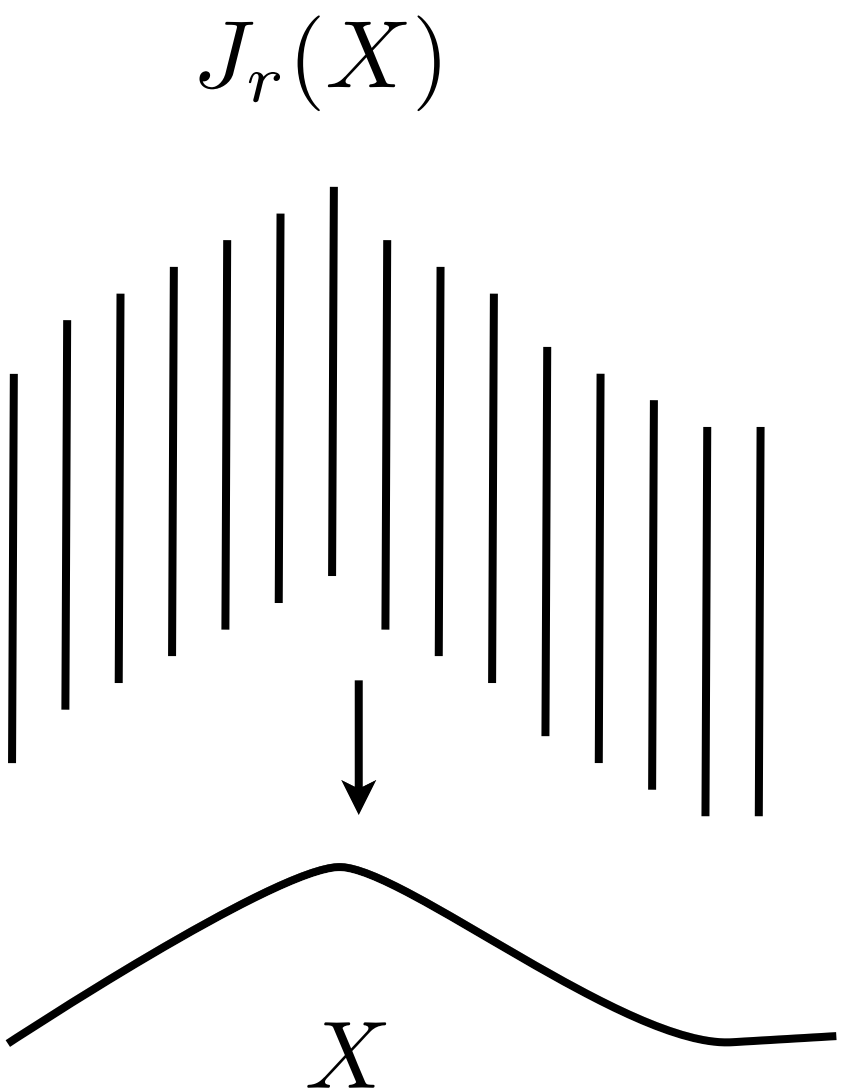

proof assuming lemma:

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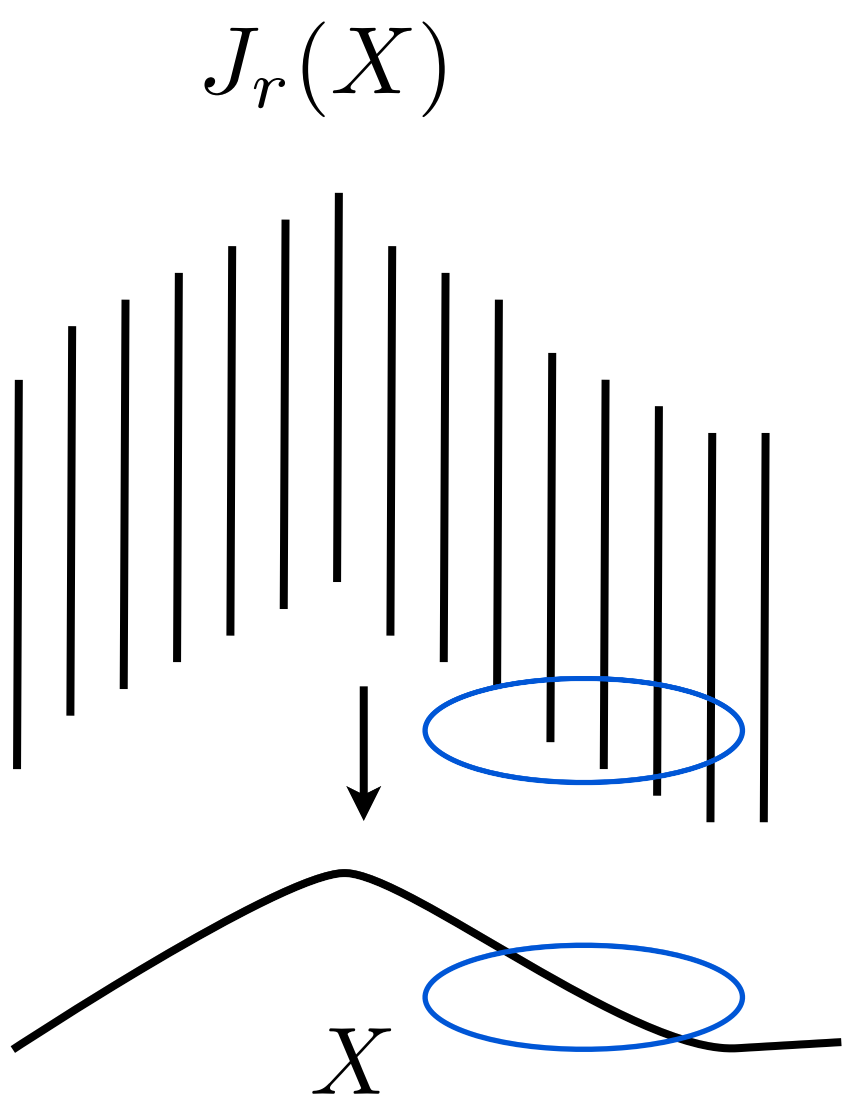

proof assuming lemma:

$$\pi_r^{-1}(U) \cong U \times \mathbf{A}^{(r+1)\dim(X)}$$

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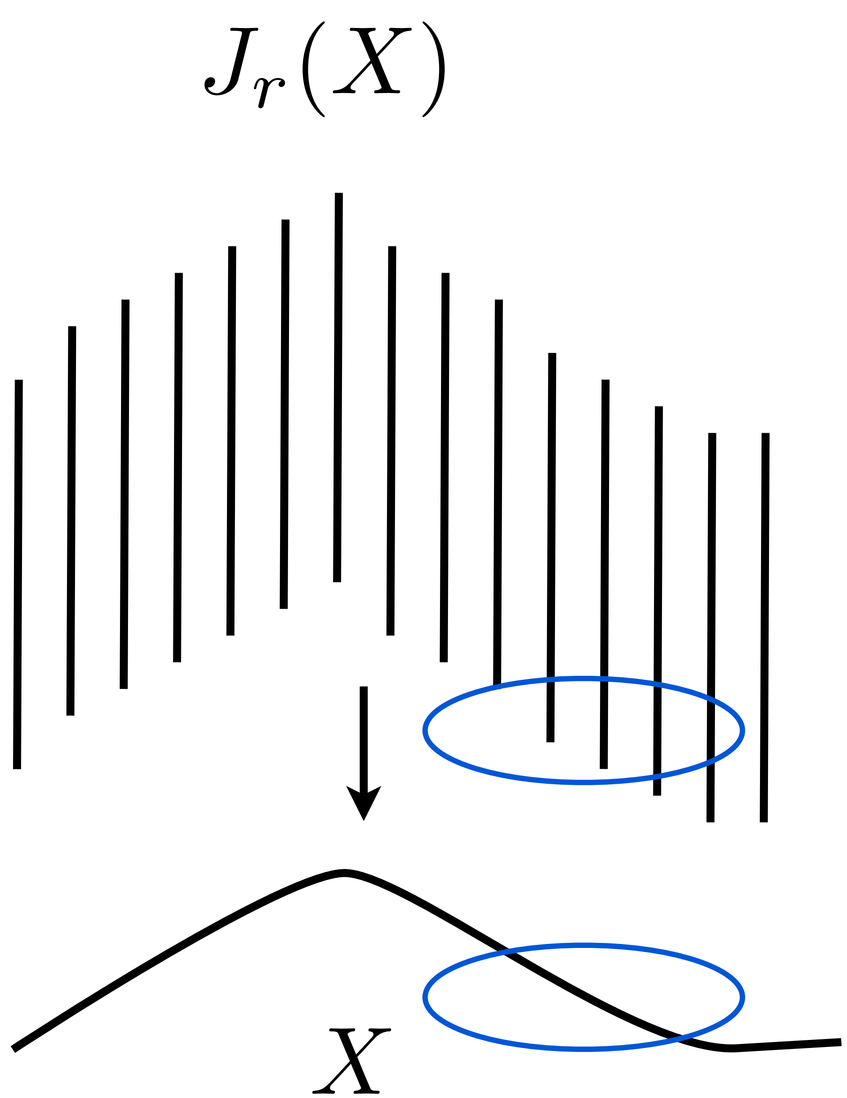

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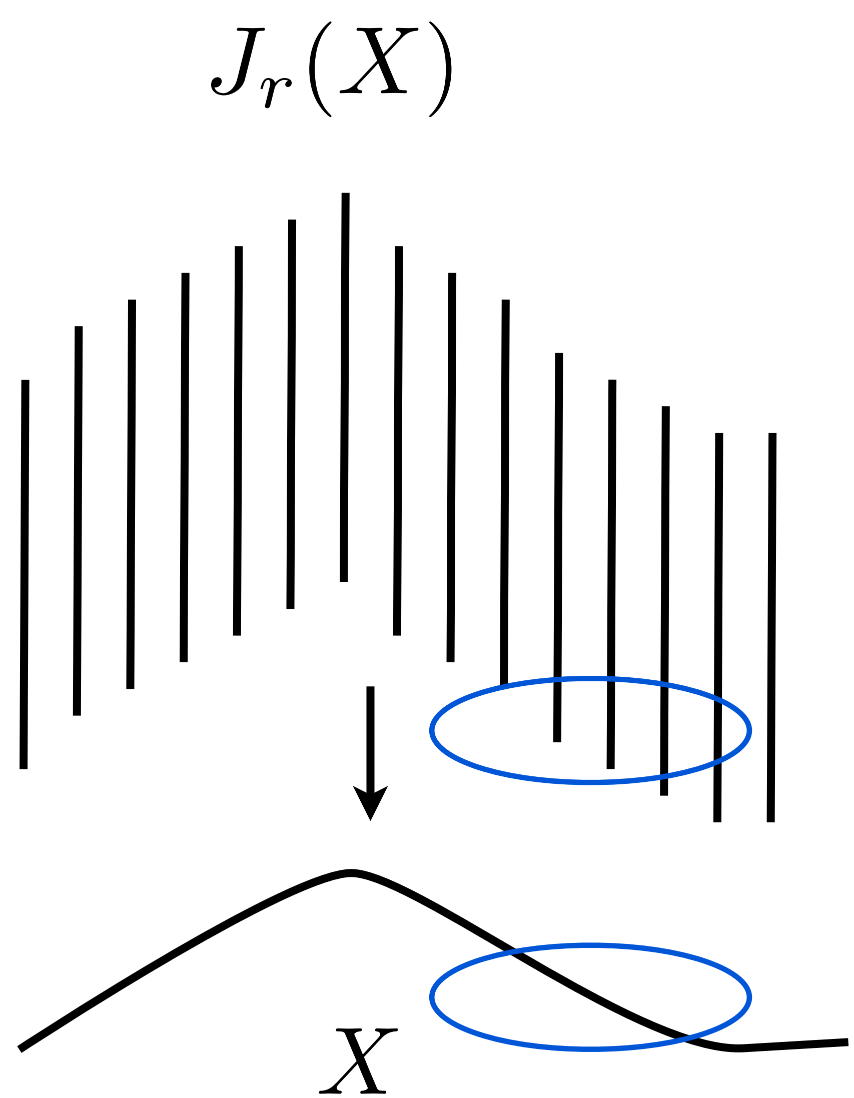

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domain

Step 3: Reduction to Smooth Case (classical)

$$J_\infty(\mathrm{Sm}(X)) \subseteq J_\infty(X)$$

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↖ irreducible

$X/\mathbf{C}$

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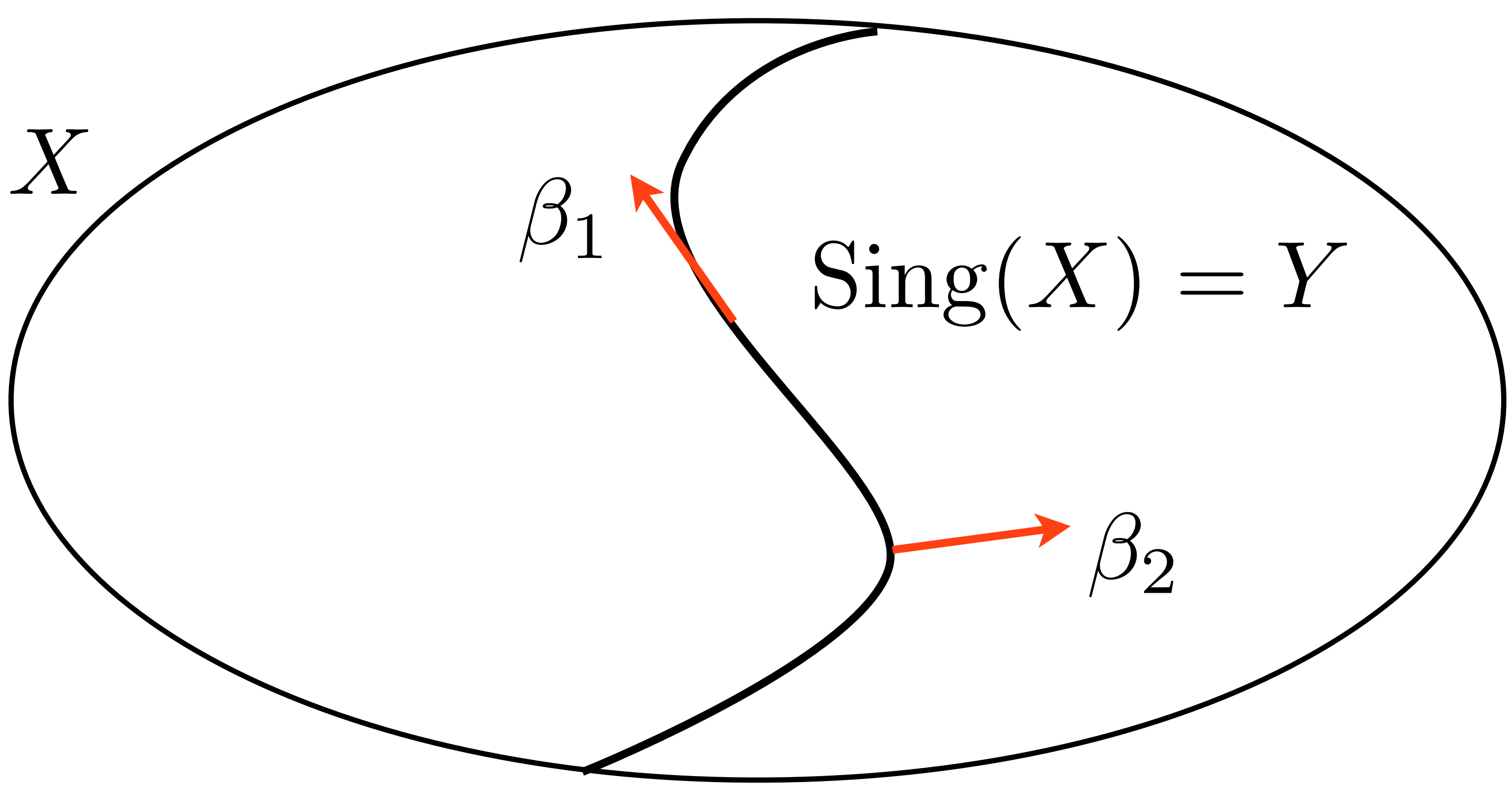

$\beta_1 \in J_\infty(Y)$

$\beta_2 \in \pi^{-1}(Y)$

Step 3: Reduction to Smooth Case (classical)

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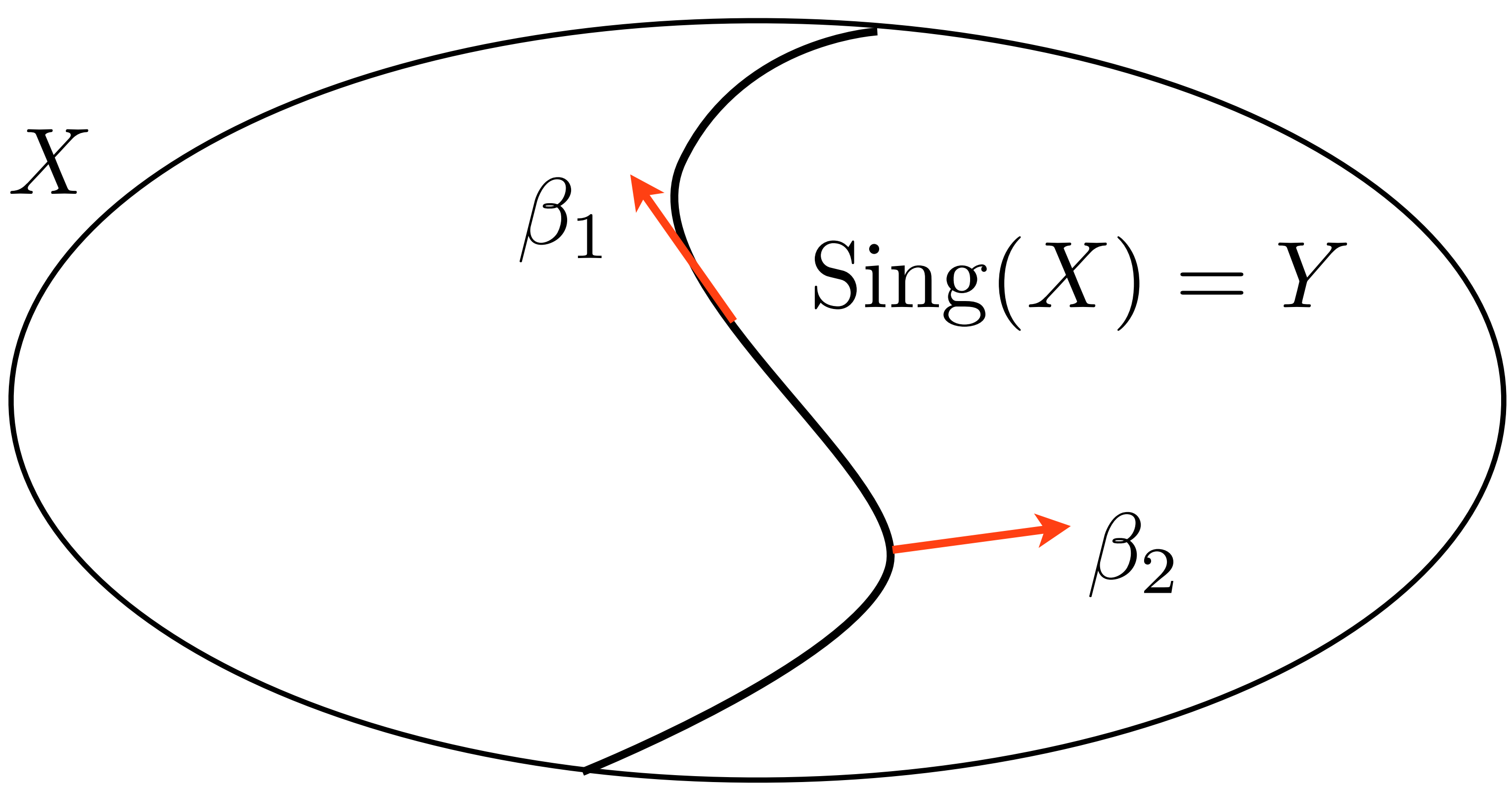
irreducible



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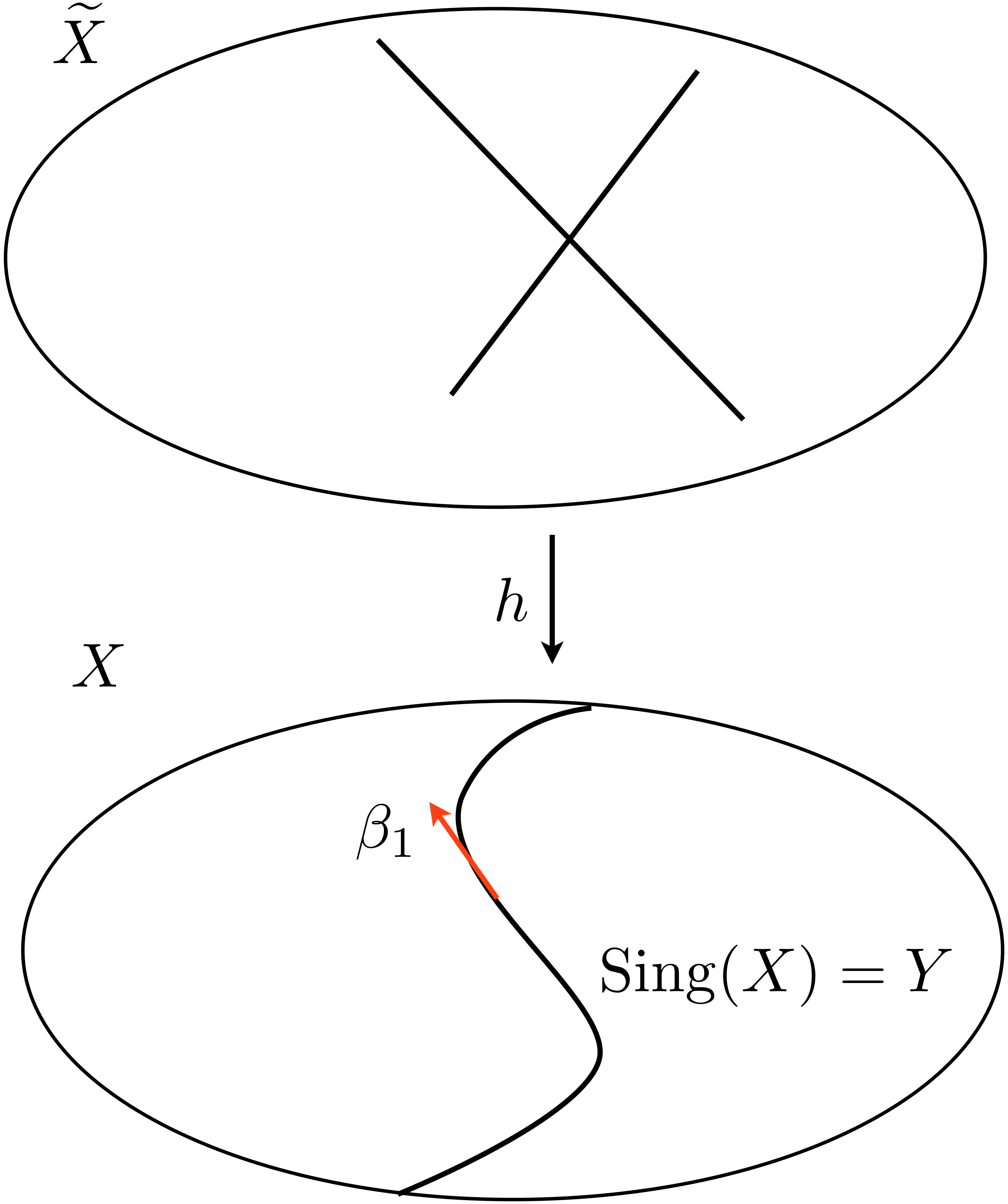

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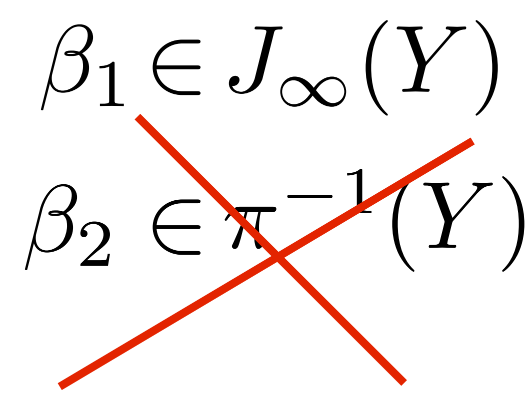

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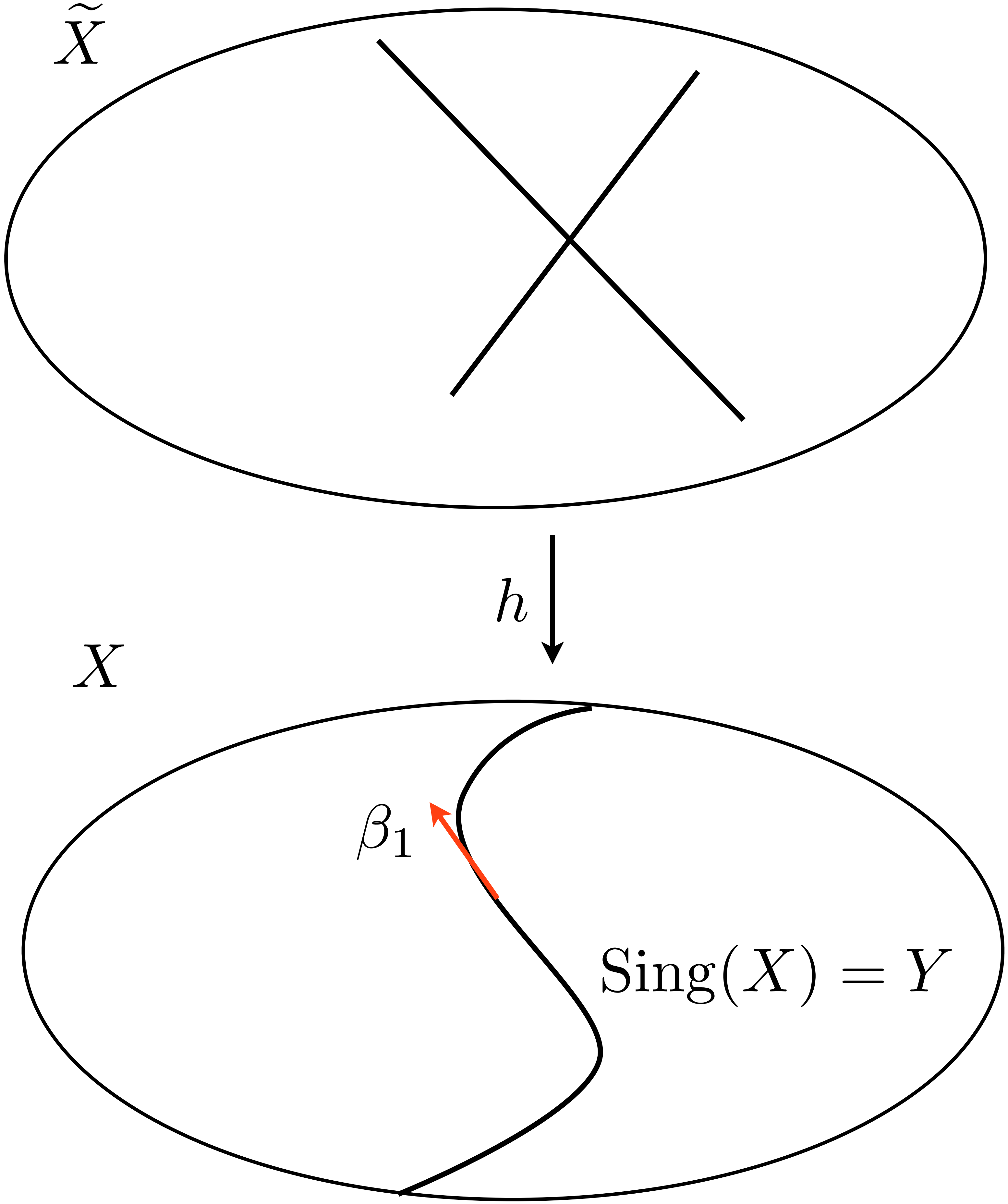

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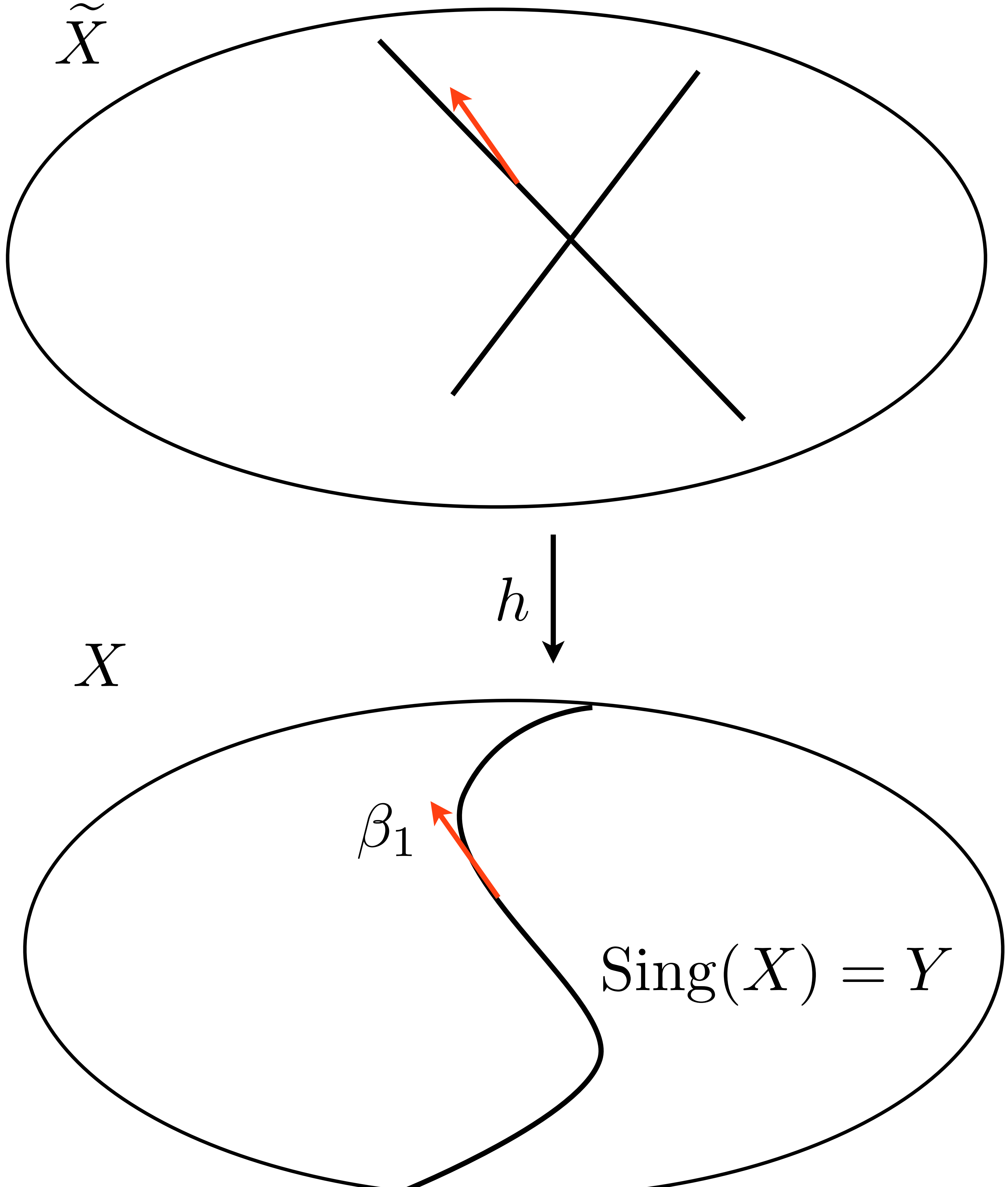
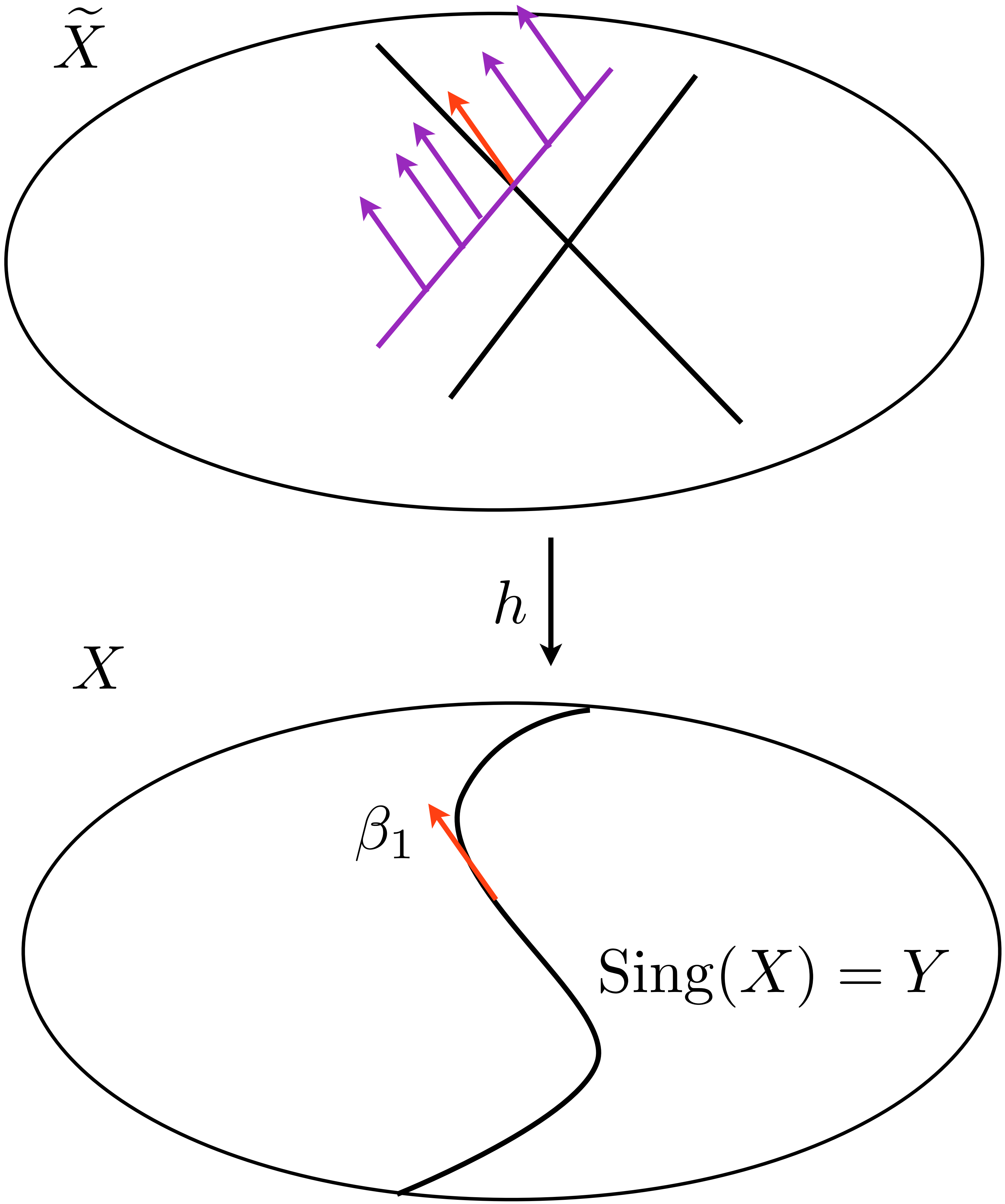


Step 3: Reduction to Smooth Case (classical)

$X/\mathbf{C}$



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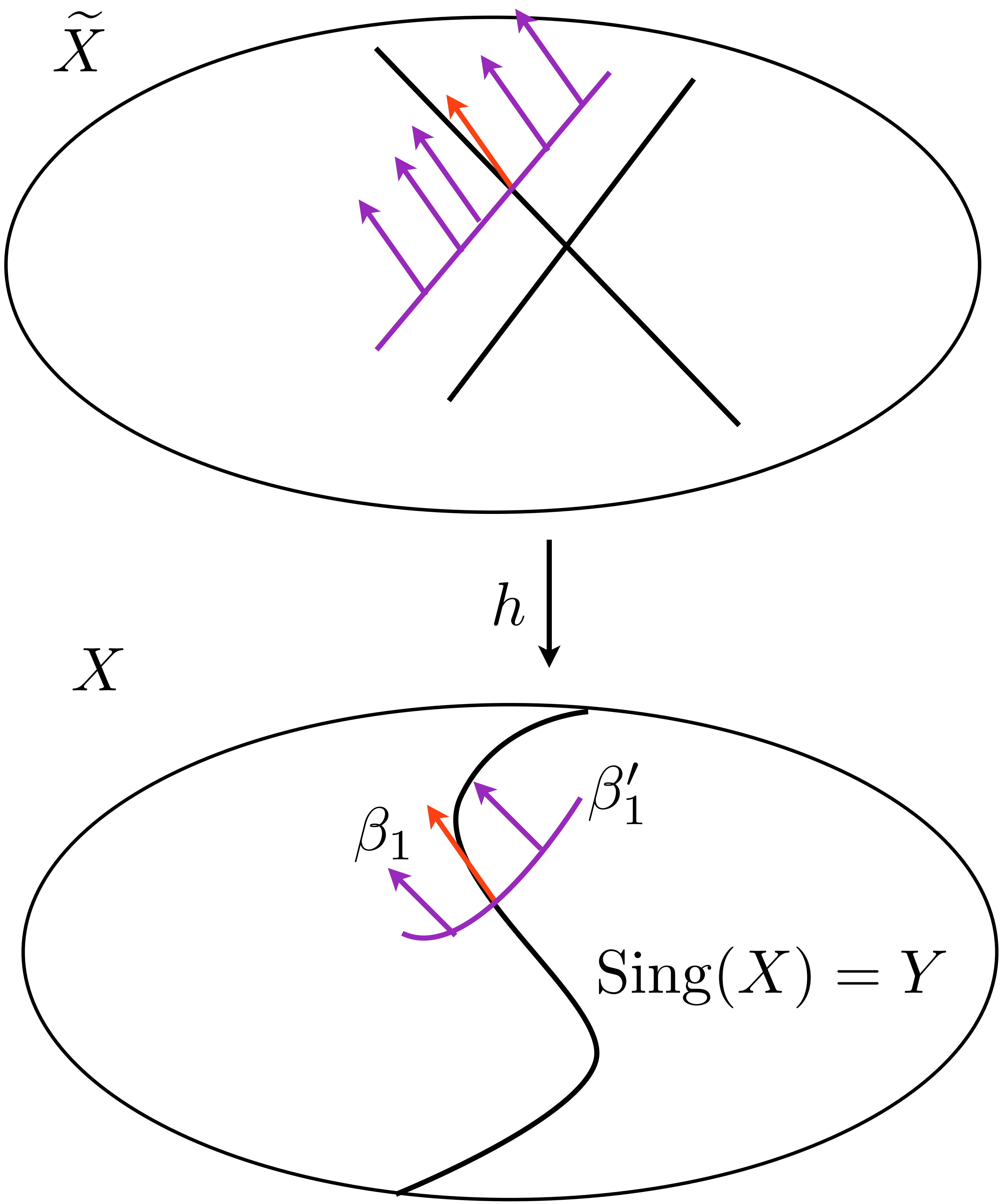

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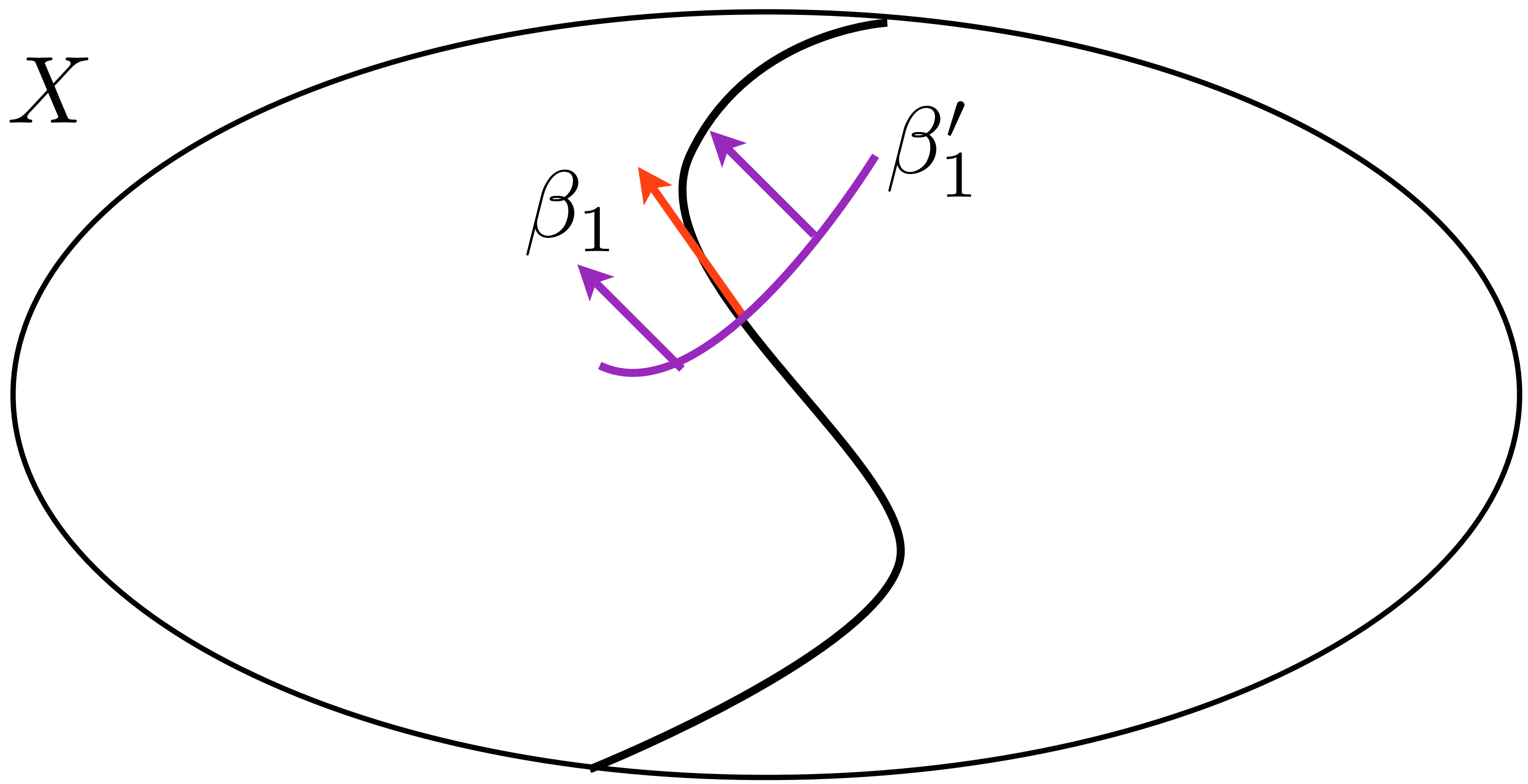

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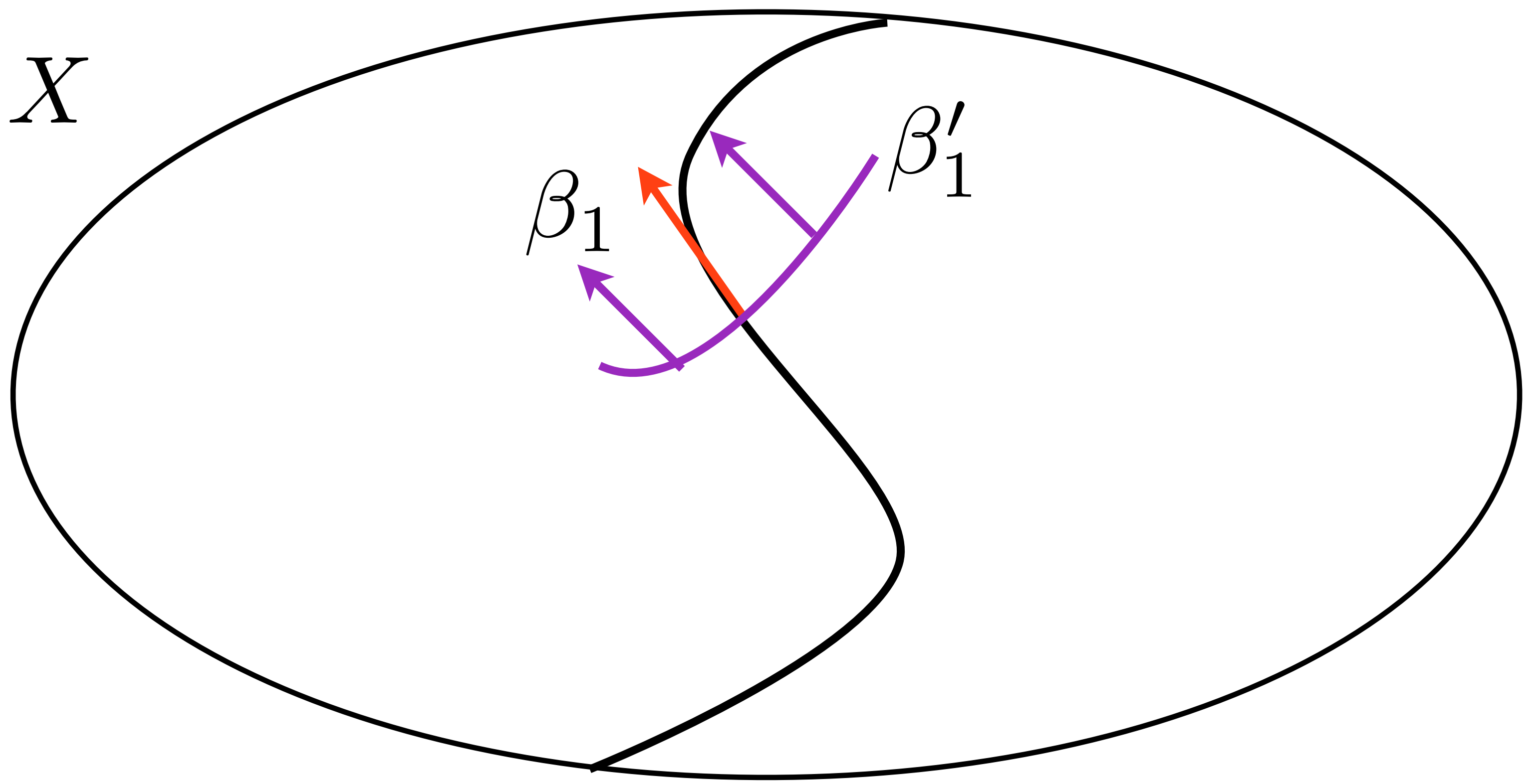

$$\beta_1 \in J_\infty(Y)$$

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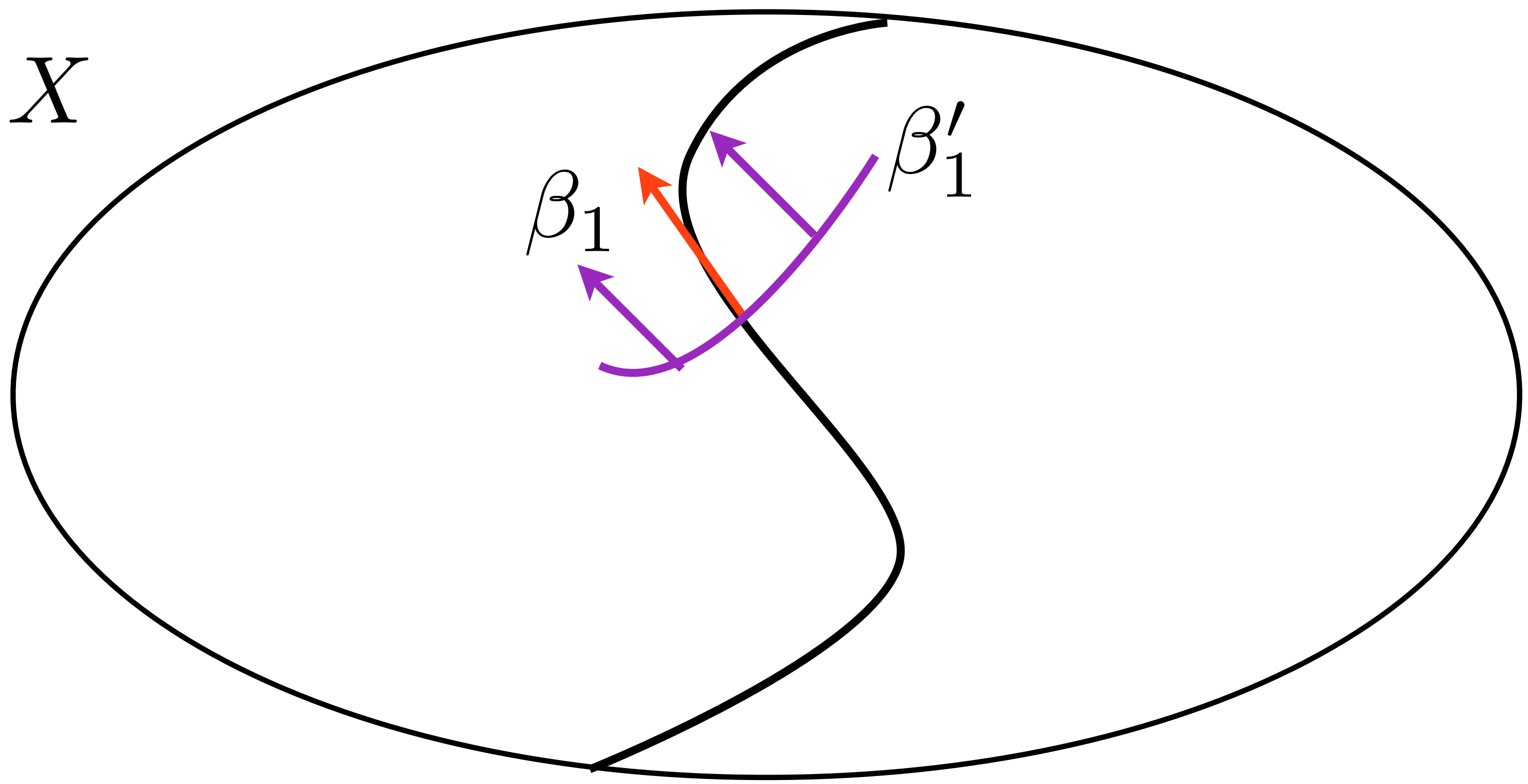

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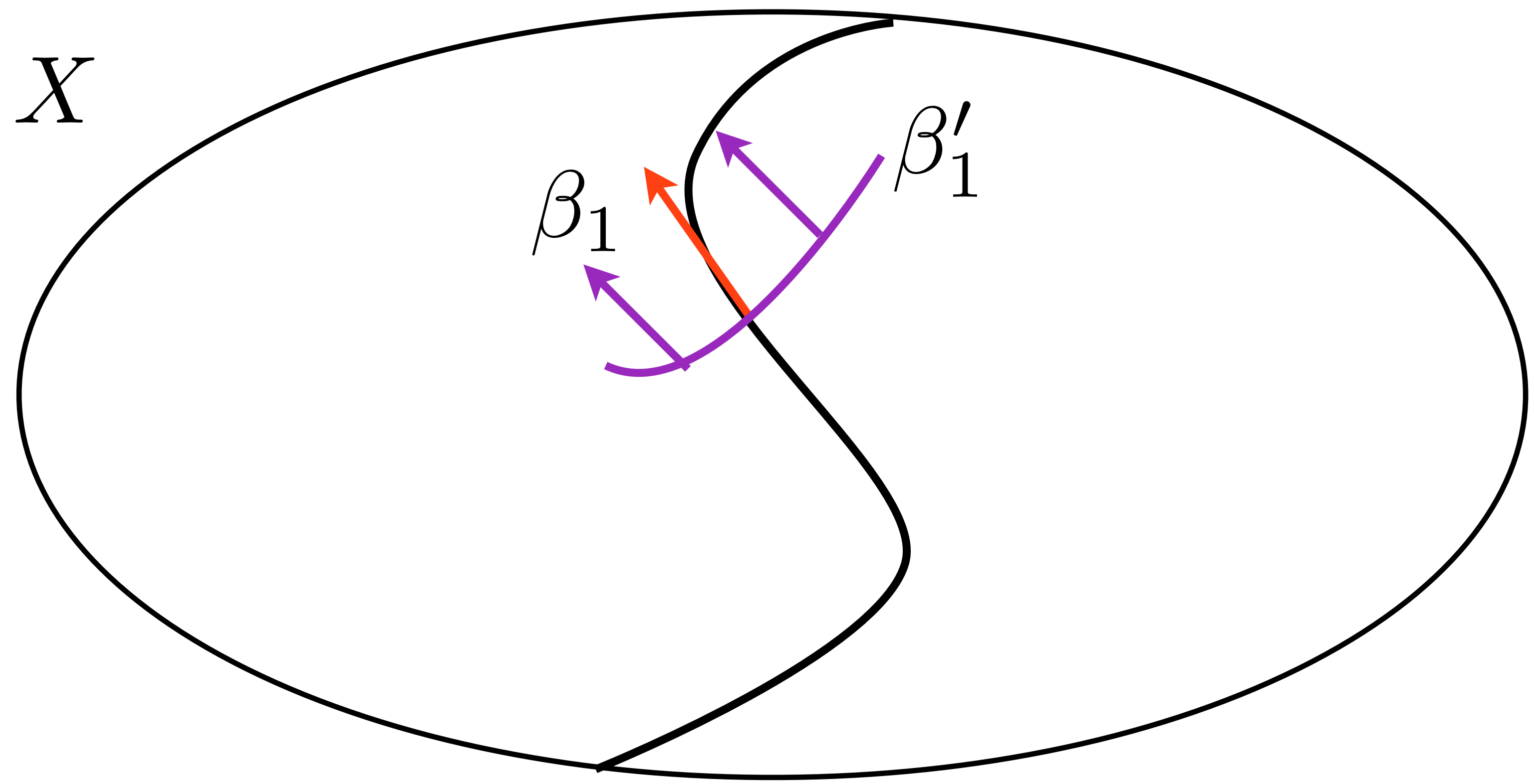

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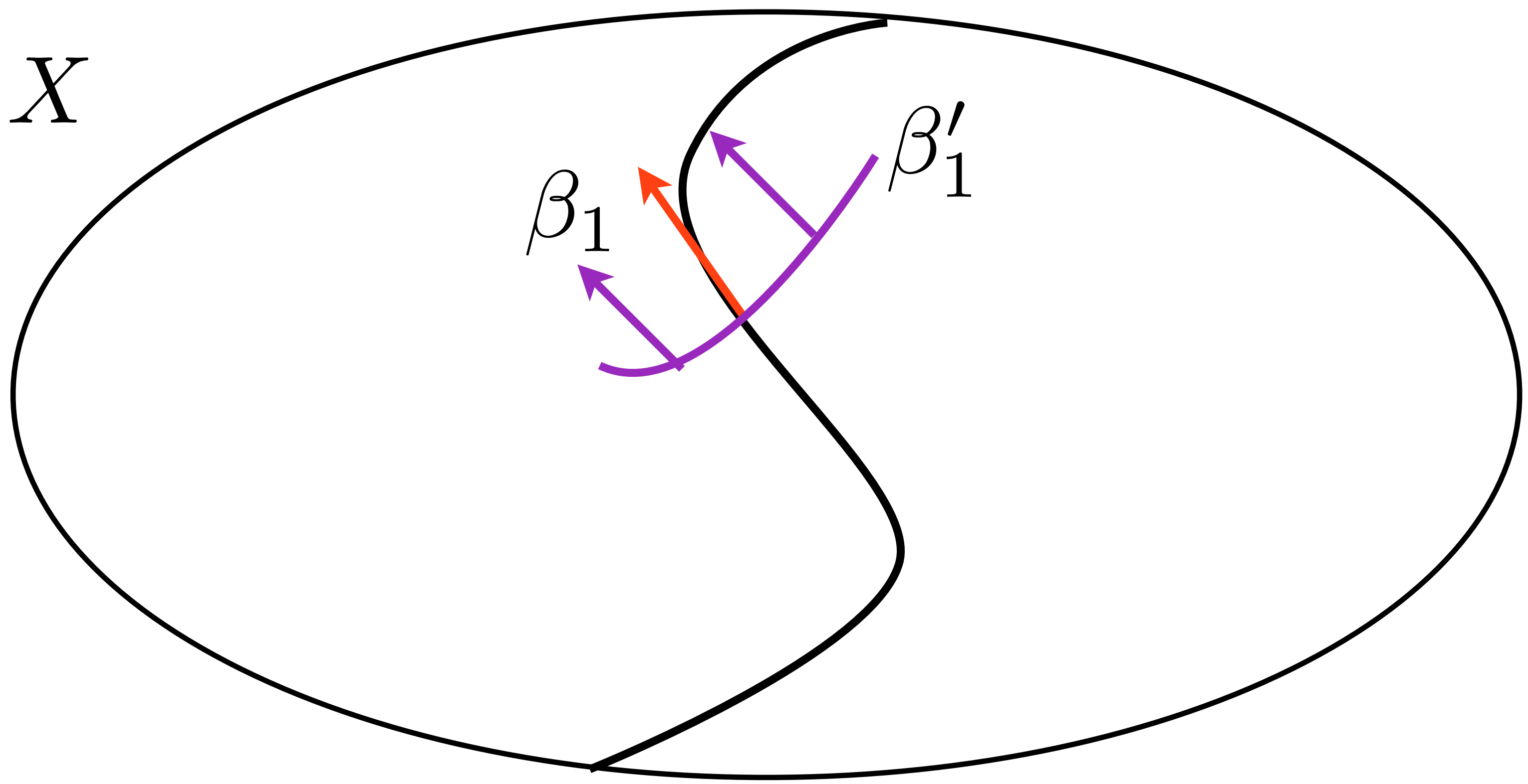

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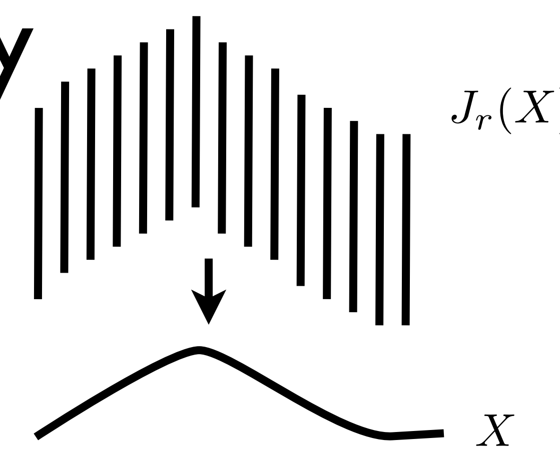
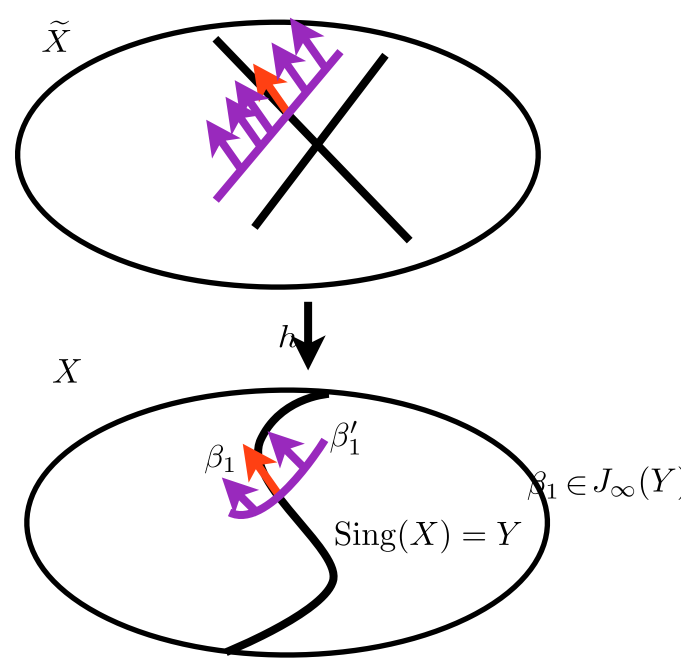
Recap of Classical

- Step 1: Deformations = Irreducibility
- Step 2: Smooth case
- Step 3: Reduction to Smooth Case





Arithmetic Jet Spaces

1. work over $\widehat{\mathbf{Z}}_p^{\mathrm{ur}}$
2. replace power series with Witt vectors

$$J_{p,r}(X)(A) = X(W_{p.r}(A))$$

Theorem (Buium)

$$\widehat{X} \text{ smooth and integral } \implies \widehat{J}_{p,\infty}(\widehat{X}) \text{ integral}$$

Theorems (Dupuy-Frietag-Miller)

$$\begin{array}{l} X \text{ smooth and affine} \\ \quad \widehat{X} \text{ integral} \end{array} \implies J_{p,\infty}(X) \text{ irreducible}$$

$Y \to X$ (weak) affine smoothening

$$\widehat{Y} \text{ integral } \implies J_{p,\infty}(X) \text{ (weakly) irreducible}$$

Example of a conditional result:

S1

X smooth and $\widehat{X}$ integral $\implies$ $J_{p,\infty}(X)$ irreducible.

S2

$Y \to X$ (weak) smoothening

$\widehat{Y}$ integral $\implies$ $J_{p,\infty}(X)$ (weakly) irreducible

S1 $\implies$ S2

Step 2: Smooth Case

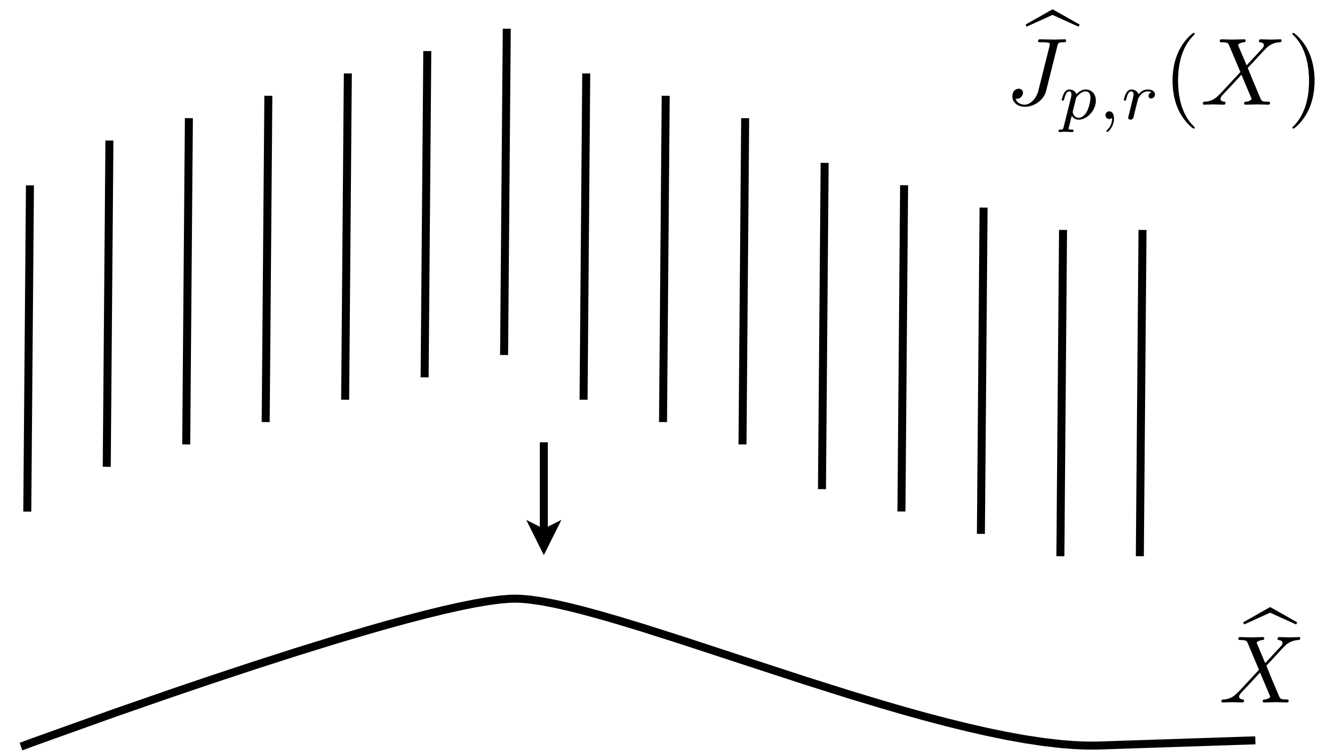
Theorem. (Buium)

X/R smooth

$R = W_{p,\infty}(\mathbf{F}_p^{alg})$

$\widehat{J}_{p,r}(X) \to \widehat{X}$

an affine bundle



Corollary.

smooth

$\widehat{X}$ irreducible $\Longrightarrow$ $\widehat{J}_{p,r}(X)$ irreducible

Smoothenings

Alterations?? (Introduces Ramification)

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

Neron Smoothenings (Sebag-Loeser,Nicaise-(Chambert-Loir)):

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Neron Smoothenings (Sebag-Loeser,Nicaise-(Chambert-Loir)):

$$\exists h : Y \to X$$

- Y smooth, $\widehat{Y}$ irreducible.
- $Y(W_{p,\infty}(\mathbf{F}_p^{alg})) \to X(W_{p,\infty}(\mathbf{F}_p^{alg}))$ surjective

THANK YOU