# Weak Kolchin Irreducibility for Arithmetic Jet Spaces

Taylor Dupuy
(with James Freitag and Lance E. Miller)

## Kolchin (1970s)

$$X/\mathbf{C} \text{ irreducible} \implies J_\infty(X) \text{ irreducible}$$

(singular)

## Claim:

$$X/W_{p,\infty}(\mathbf{F}_p^{alg})$$

$$\widehat{X} \text{ irreducible} \implies J_{p,\infty}(X) \text{ weakly irreducible}$$

$+\varepsilon$

$$\exists Z \subset J_{p,\infty}(X)$$

- $Z$ is closed irreducible subset.
- $Z$ contains an open.
- $Z$ has all of the closed points.

**Claim:**

$X/W_{p,\infty}(\mathbf{F}_p^{alg})$

$\widehat{X}$ irreducible $+\varepsilon$ $\Longrightarrow$ $J_{p,\infty}(X)$ weakly irreducible

$$\exists h : Y \to X$$

$X = \mathrm{Spec}(A)$

- $Y$ smooth, $\widehat{Y}$ irreducible.
- $Y(W_{p,\infty}(\mathbf{F}_p^{alg})) \to X(W_{p,\infty}(\mathbf{F}_p^{alg}))$ surjective

$$\exists h : Y \to X$$

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## Includes

- $X/R$ generically smooth.
- $X = \text{Spec } R[x,y]/(y^2 - x^2(x-1))$

## Excludes

- $X = \text{Spec } R[x,y]/(y^2 - x^2(x+p)$
- $X = \text{Spec } R[x,y,z]/(x^p = zy^p)$
- $X = \text{Spec } R[x.y]/(xy - p)$

# Background

- Let $D_1 : \mathsf{CRing} \to \mathsf{CRing}$ be the functor

$$A \mapsto A[t]/(t^2).$$

- A **derivation** $A \to A$ is the same as a section of

$$D_1(A) \to A.$$

| Functor | Operation |
|---|---|
| $D_1$ | Derivation |
| $W_{p,1}$ | $p$-Derivation |
| $A \mapsto A \oplus A$ | Ring Endo |
| $W$ big witt | $\lambda$-rings |

- (Borger-Weiland 00s, Tall-Wraith 70s)

  When $\mathcal{R}$ is an affine ring scheme

  $$\mathcal{R} = \mathrm{Spec}(Q)$$

  there exists a left adjoint

  $$\mathsf{CRing}(Q \odot A, B) = \mathsf{CRing}(A, \mathcal{R}(B)).$$

- The bifunctor $\odot$ is called the **composition product**

- For $X$ a scheme define **functor of jets**

  $$J_Q(X) := X(\mathcal{R}(-)) : \mathsf{CRing} \to \mathsf{Set}$$

- If $\mathcal{R}$ a comonad, then call it a **functor of arcs**.

- When functor representable, we call it a **jet** or **arc space**.

# Jet Functor

$$J_Q(X)(A) := X(\mathcal{R}(A))$$

- There exists a relative version of this construction as well.
- For a fixed action $Q \odot C \to C$ on a base we let

$$J_Q(X/C, \rho)$$

denote the relativized version.

## Prolongations

$$\begin{array}{ccc} Q \odot A & \longrightarrow & B \\ \uparrow & & \uparrow \\ Q \odot C & \longrightarrow & C \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{s_1} & \mathcal{R}(B) \\ \text{alg map} \uparrow & & \uparrow \mathcal{R}(\text{ alg map }) \\ C & \xrightarrow{s_0} & \mathcal{R}(C) \end{array}$$

$$
\begin{array}{ccc}
A & \xrightarrow{s_1} & \mathcal{R}(B) \\
\text{alg map} \uparrow & & \uparrow \mathcal{R}(\text{ alg map }) \\
C & \xrightarrow{s_0} & \mathcal{R}(C)
\end{array}
\qquad
\begin{array}{ccc}
\text{Spec } \mathcal{R}(B) & \xrightarrow{\text{Spec}(s_1)} & X \\
\downarrow & & \uparrow \\
\text{Spec } \mathcal{R}(C) & \xrightarrow{\text{Spec}(s_0)} & \text{Spec}(C)
\end{array}
$$

prolongation pt

**Relative Jet Functors:** $J_Q(X/C, \rho)$

$$J_Q(X/C, \rho)(B) = \{P \in X(\mathcal{R}(B)) : \text{ prolongation pt } \}$$

$$J_Q(X/C, \rho)$$

# Notations.

- $J_n(X/C, D) =$ nth order classical jet spaces
- $J_\infty(X/C, D) =$ classical arc spaces
- $J_{p,r}(X/C, \rho) = J_{p,r}(X)$ truncated $p$-jet spaces
- $J_{p,\infty}(X/C, \rho) = J_{p,\infty}(X)$ $p$-arc spaces
- $\widehat{J}_{p,r}(X)$ and $\widehat{J}_{p,\infty}(X)$, Buium's $p$-formally completed version

# Example.

$J_1(X/C, D)$ = classical first order tangent space

$$J_1(X/C) = \begin{cases} T_{X/C}, & D = \text{ trivial} \\ \text{twisted } T_{X/C}, & D = \text{ not trivial} \end{cases}$$

- Let $X/\mathbf{C}[[t]]$ be defined by $xy = t$.

  Consider $\mathbf{C}[[t]]$ as having a trivial derivation.

  The equations for $J_1(X/\mathbf{C}[[t]]) \subset \operatorname{Spec} \mathbf{C}[[t]][x, y, x'y']$ are $xy = t$ and $x'y + y'x = 0$.

- Let $X/\mathbf{C}[[t]]$ be defined by $xy = t$.

  Consider $\mathbf{C}[[t]]$ with its nontrivial derivation $D = d/dt$.

  The equations of $J_\infty(X/\mathbf{C}[[t]])$ are then

$$
\begin{aligned}
xy - t &= 0 \\
\dot{x}y + x\dot{y} - 1 &= 0 \\
\ddot{x}y + 2\dot{x}\dot{y} + x\ddot{y} &= 0 \\
&\vdots
\end{aligned}
$$

- Let $f(x, y) = xy - t$.

  Then we look to satisfy the equation

$$0 = (x_0 + x_1\varepsilon + \cdots)(y_0 + y_1\varepsilon + \cdots) - \exp(t)$$

  where $\exp(t) = t + \varepsilon$ given the system of equations

$$\begin{aligned} x_0 y_0 - t &= 0 \\ x_0 y_1 + y_0 x_1 - 1 &= 0 \\ x_0 y_2 + 2x_1 y_1 + x_2 y_0 &= 0 \\ &\vdots \end{aligned}$$

- Special fiber of $J_{p,\infty}(X)$ example:

  Let $X = \mathrm{Spec} R[x,y]/(xy-p)$.

  Let

  $x = (x_0, x_1, \ldots)$

  $y = (y_0, y_1, \ldots)$

  $p = (0, 1, 0, \ldots)$

  Multiplication by $p$ acts by translating to the right and $p$th powering

$$\begin{aligned} x_0 y_0 &= 0 \\ x_0^p y_1 + y_0^p x_1 &= 1 \\ m_2 &= 0 \\ \vdots & \quad \vdots \end{aligned}$$

  and one can trivially see that $\mathrm{Gr}_\infty(X) = V(x_0) \cup V(y_0)$.

- Suppose now we are working ever $R$.

  Then $p$ is not $p \cdot 1$ in a ring where we replace everything by the witt vectors

  $\exp_p(p) = (p, 1 - p^{p-1}, \ldots)$

  which this means that $m_i(x,y) = \exp_p(p)_i$ whose reduction modulo $p$ recover the previous ones.

  It is nontrivial to see that this scheme is irreducible.

# example:

$$y^2 = x^2(x+1)$$

$$\frac{\partial f}{\partial p} + (3x^{2p} + 2x^p)\dot{x} + p(3x^p + 1)\dot{x}^2 + p^2\dot{x}^4 = 2y^p\dot{y} + p\dot{y}^2$$

$$\frac{\partial f}{\partial p} = \frac{f(x^p, y^p) - f(x,y)^p}{p}$$

$$\pi_1^{-1}(0,0)$$

$$\dot{y}^2 = \dot{x}^2(1 + p\dot{x})$$

# For the rest of the talk assume

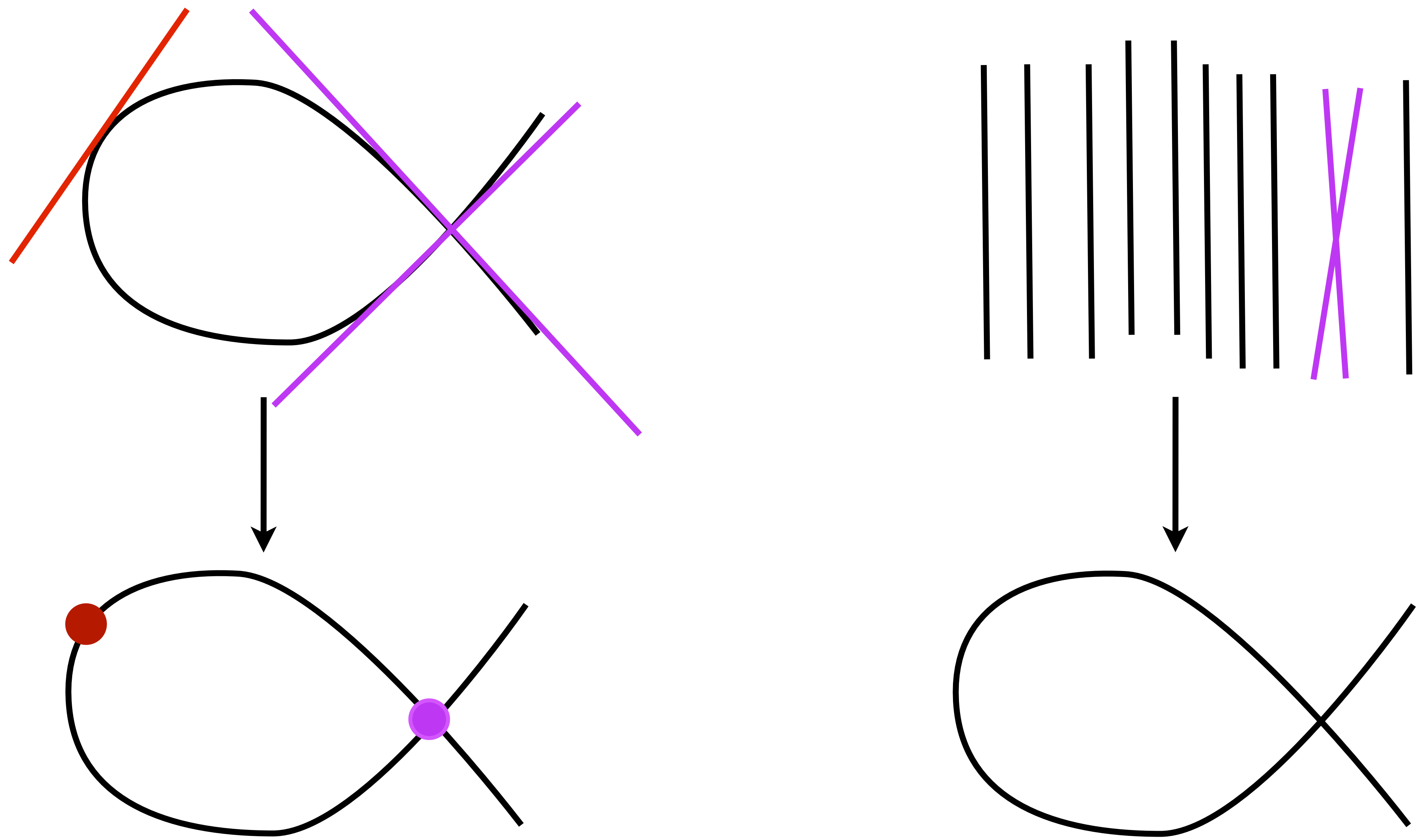
$X$ is affine.

(this deals with representability issues)

Moosa-Scanlon, Bhatt-Lurie, Borger

# Classical Jet Spaces and Singularities

# Why do we care about jet spaces?



# Why do we care about jet spaces?

## Example.

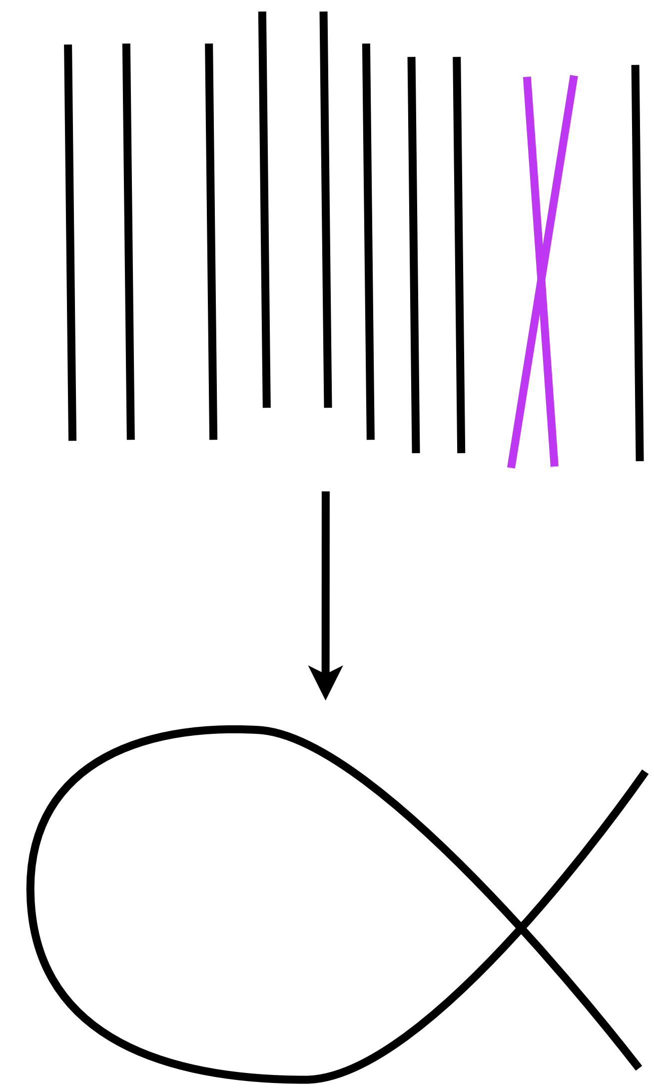
$$X : x^4 + y^4 + z^4 = 0$$

$$x = x_0 + x_1 t + x_2 t^2 + x_3 t^3 \mod t^4$$

$$y = y_0 + y_1 t + y_2 t^2 + y_3 t^3 \mod t^4$$

$$z = z_0 + z_1 t + z_2 t^3 + z_3 t^3 \mod t^4$$

$$x_0 = y_0 = z_0 = 0$$



$$
\begin{aligned}
x &= x_0 + x_1 t + x_2 t^2 + x_3 t^3 \mod t^4 \\
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z &= z_0 + z_1 t + z_2 t^3 + z_3 t^3 \mod t^4
\end{aligned}
$$

$$x_0 = y_0 = z_0 = 0$$

$$X : x^4 + y^4 + z^4 = 0$$

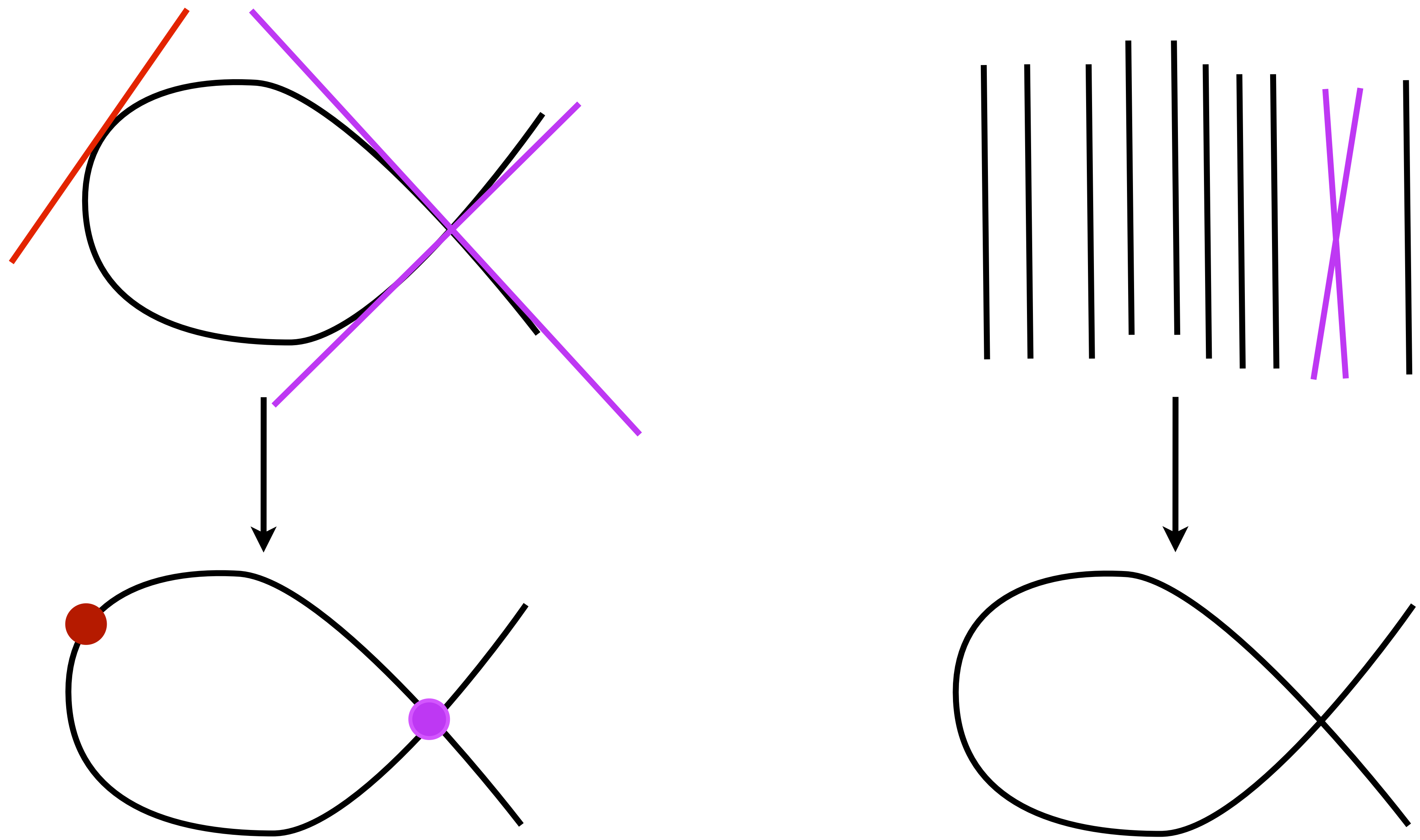
$$(x_1 t + x_2 t^2 + x_3 t^3)^4 + (y_1 t + y_2 t^2 + y_3 t^3)^4 + (z_1 t + z_2 t^3 + z_3 t^3)^4 \equiv 0$$

$$\dim \pi_4^{-1}(0,0,0) = 9$$

$$(4+1)\dim(X) = 5 \cdot 2 = 10$$

$$4 \dim(X) = 8$$

# Why do we care about jet spaces?



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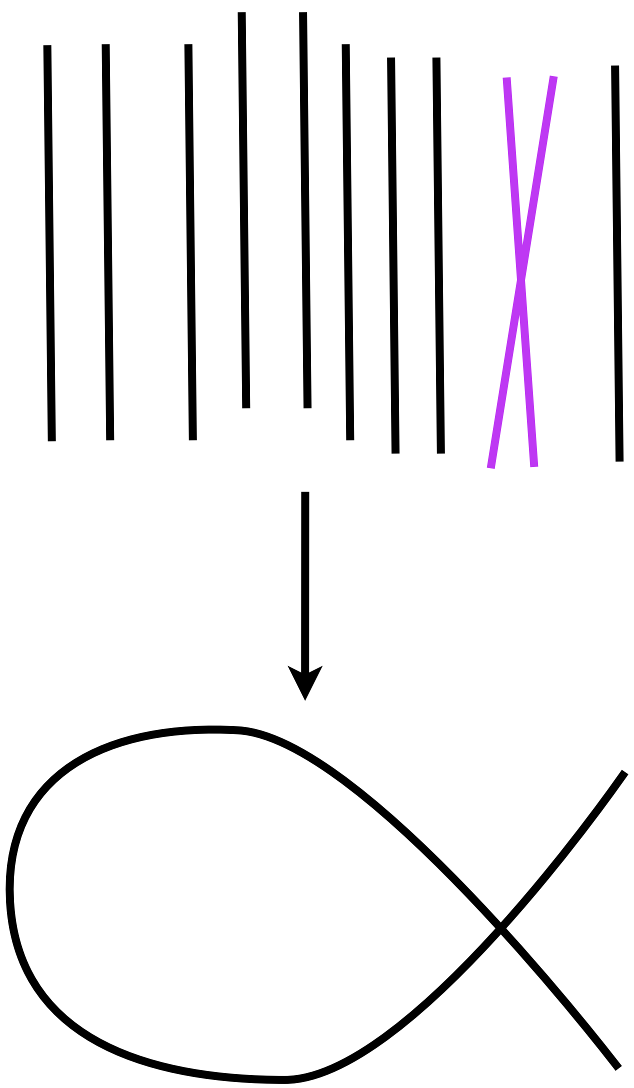
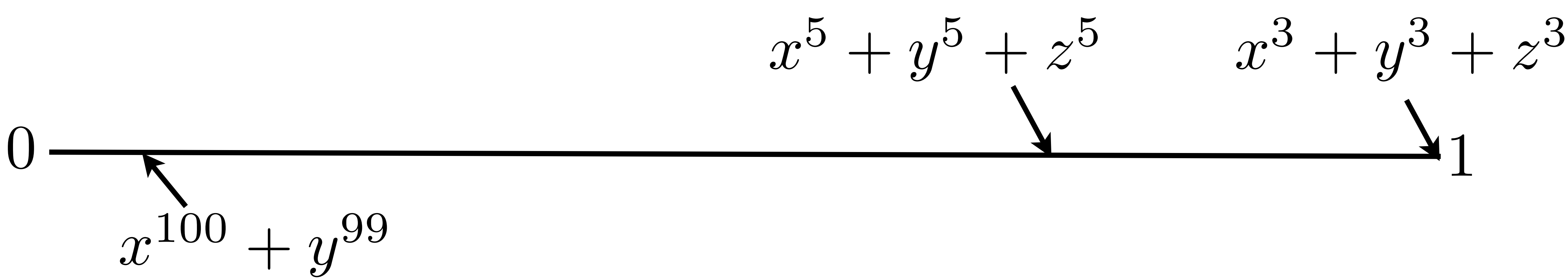


**Mustata:**

$$\operatorname{lct}(X, D) = \dim(X) - \sup_{r \geq 0} \frac{\dim J_r(D)}{r+1}$$



# Why do we care about jet spaces?

**Kolchin (1970s)**

$$X/\mathbf{C} \text{ irreducible} \implies J_\infty(X) \text{ irreducible}$$

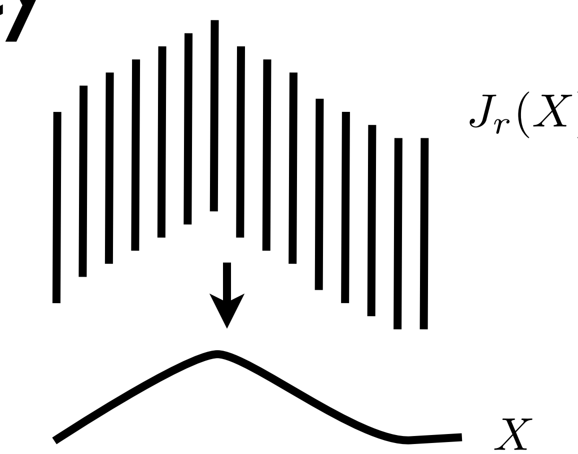
(singular)

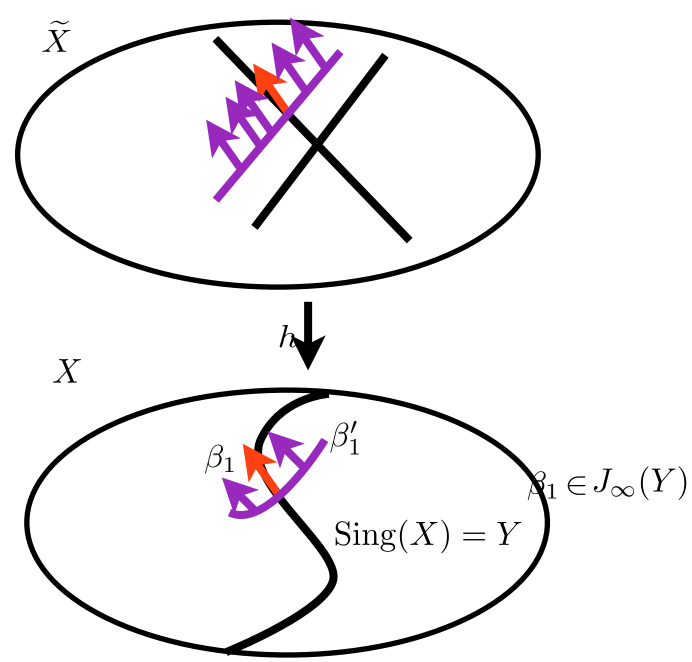
$$\lim_n J_n(X)$$

Gillet, Mustata, de Fernex, Loeser-Sebag, Kolchin, Nicaise-Sebag, Ishii-Kollar, (Chambert-Loir)-Nicaise-Sebag

# Proof of Kolchin Irreducibility

- Step 1: Deformations = Irreducibility (general).
- Step 2: Smooth case.
- Step 3: Reduction to Smooth Case



# Arc Deformations and Irreducibility

# Step 1: Deforming Arcs = Irreducibility

**Kolchin (1970s)**

$$X/\mathbf{C} \text{ irreducible} \implies J_\infty(X) \text{ irreducible}$$

(singular)

**Claim:**

Arc deformability $\longleftrightarrow$ Irreducibility

# Step 1: Deforming Arcs = Irreducibility

Arcs:

$$P \in J_Q(X)(A) \quad \leftrightarrow \quad \alpha \in X(\mathcal{R}(A))$$

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{P} & J(X) \\ \downarrow & \searrow & \downarrow \pi \\ \mathrm{Spec}(\mathcal{R}(A)) & \xrightarrow{\alpha} & X \end{array}$$

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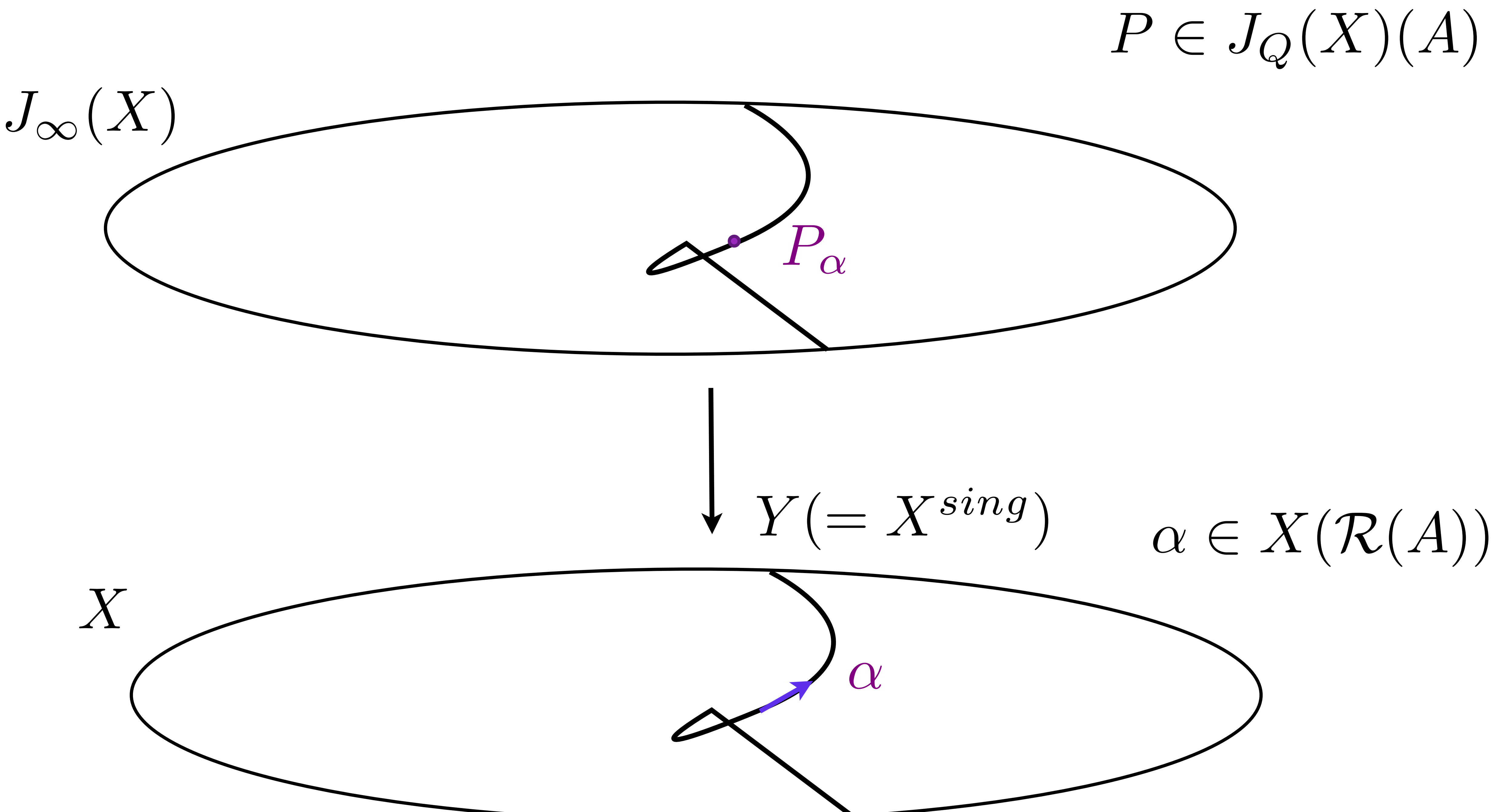

# Step 1: Deforming Arcs = Irreducibility

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$$P \in J_Q(X)(A) \;\leftrightarrow\; \alpha \in X(\mathcal{R}(A))$$

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{\quad P \quad} & J(X) \\ \downarrow & \searrow & \downarrow \pi \\ \mathrm{Spec}(\mathcal{R}(A)) & \xrightarrow{\quad \alpha \quad} & X \end{array}$$

## Deformations:

$$\alpha' \in X(\mathcal{R}(A'))$$

$$\eta_\alpha = \text{ generic of } \alpha(\mathrm{Spec}(\mathcal{R}(A)))$$

$$\overline{\{\eta_{\alpha'}\}} \ni \eta_\alpha$$

# Step 1: Deforming Arcs = Irreducibility

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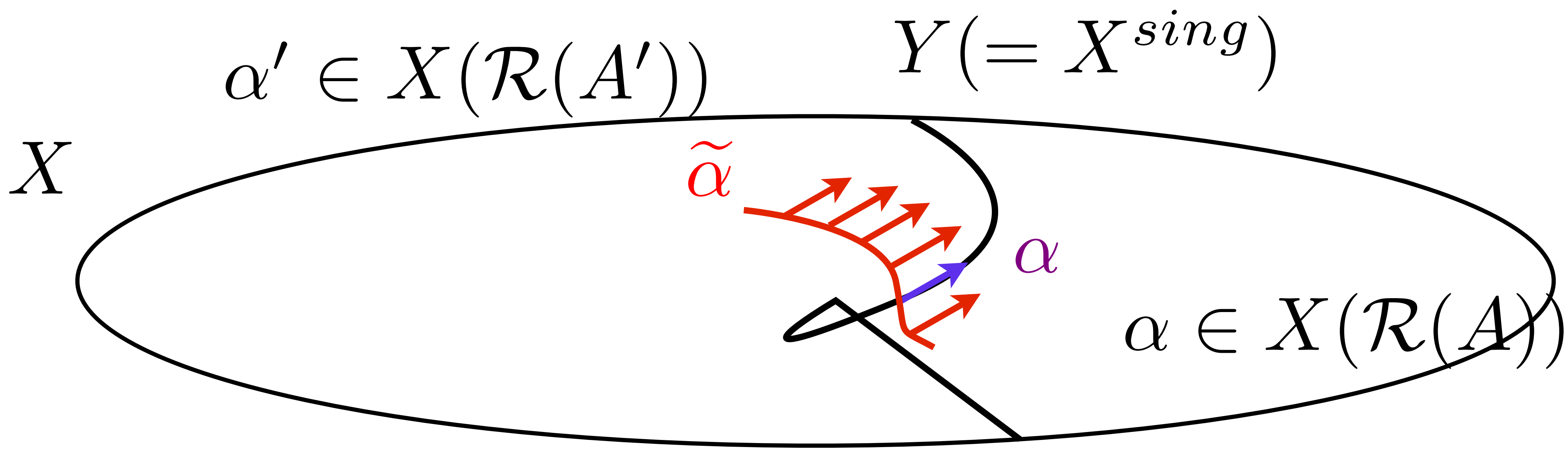

# Step 1: Deforming Arcs = Irreducibility

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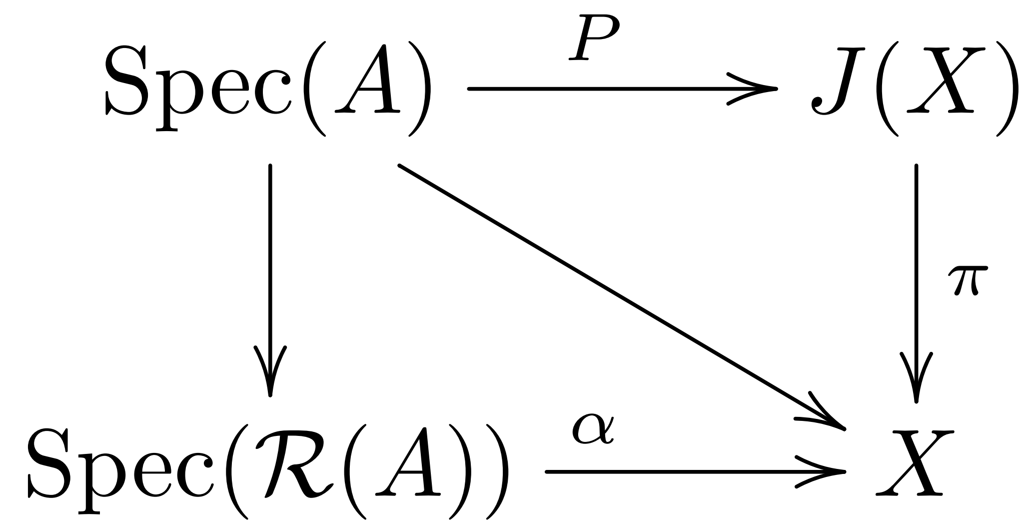


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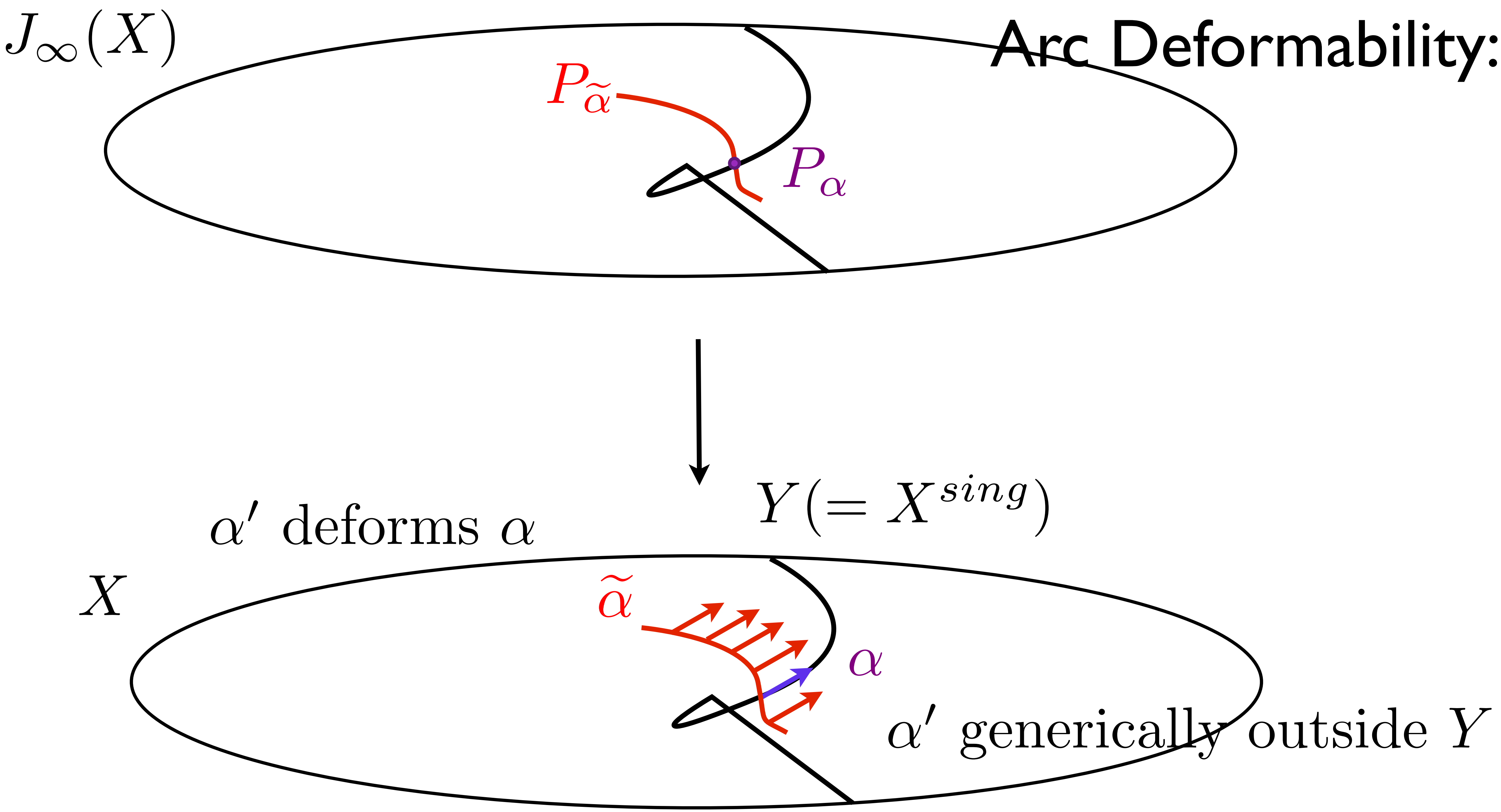
Arc Deformability:

$$\forall \alpha \in X(\mathcal{R}(A))), \forall Y \subsetneq X, \exists \alpha' \in X(\mathcal{R}(A'))$$

$$\alpha' \text{ deforms } \alpha$$

$$\alpha' \text{ generically outside } Y$$

# Step 1: Deforming Arcs = Irreducibility



# Step 1: Deforming Arcs = Irreducibility

$$\forall \alpha \in X(\mathcal{R}(A))), \forall Y \subsetneq X, \exists \alpha' \in X(\mathcal{R}(A'))$$

$$\alpha' \text{ deforms } \alpha$$

$$\alpha' \text{ generically outside } Y$$

# Step 1: Deforming Arcs = Irreducibility

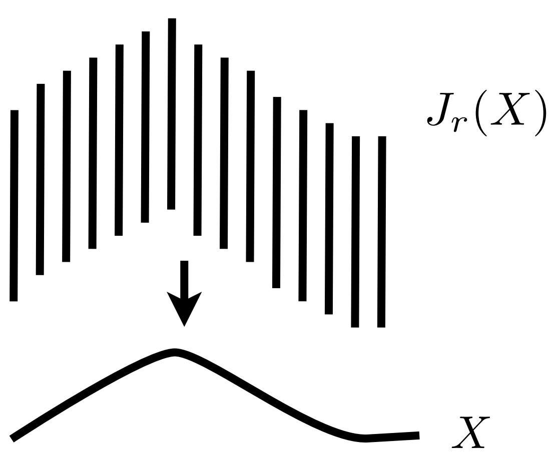
**Deformation Idea**

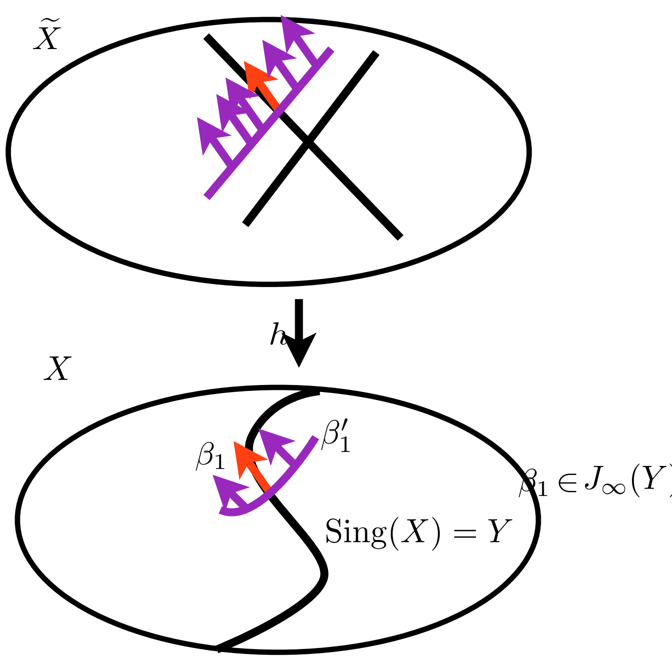
Arc deformability $\longleftrightarrow$ Irreducibility

Simple Case:

- $\pi^{-1}(\mathrm{Sm}(X))$ nonempty.
- $A$ a domain $\implies \mathcal{R}(A)$ a domain.

# Classical Kolchin Irreducibility



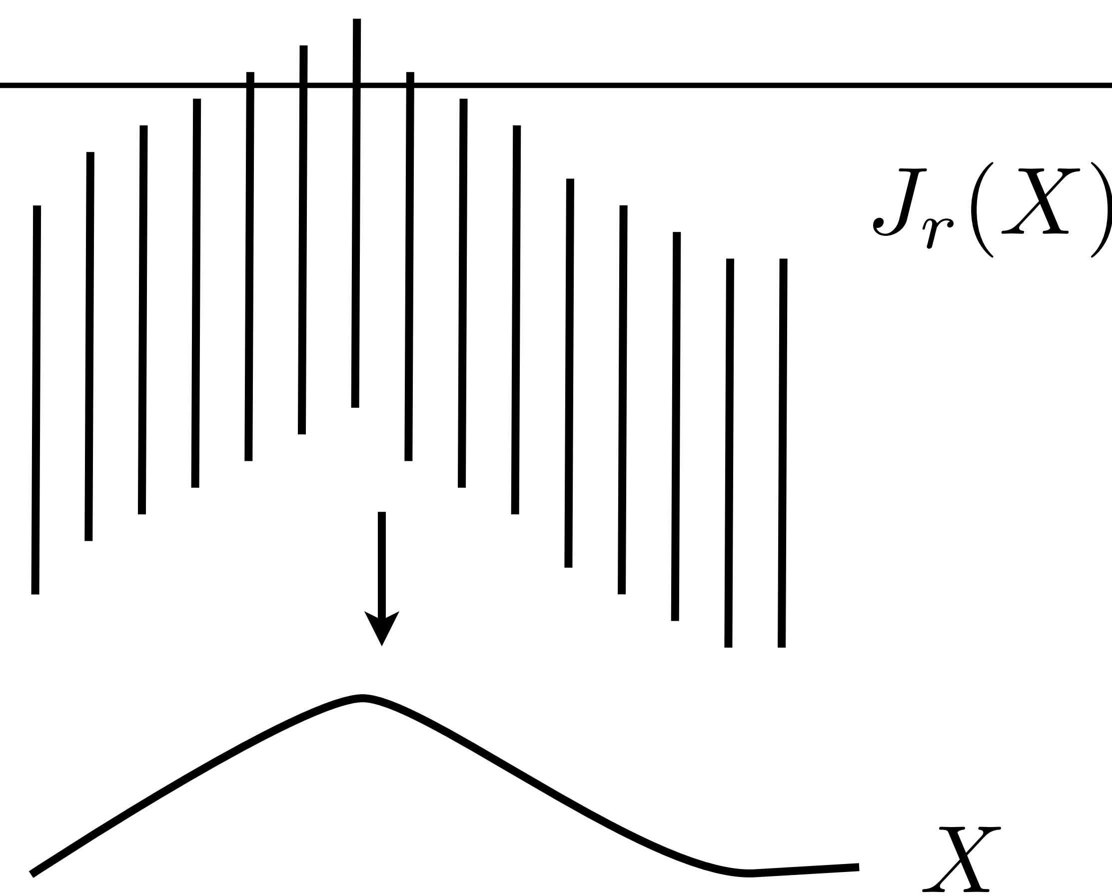
# Step 2: Smooth Case (Classical)

**Theorem.**

$X/\mathbf{C}$ smooth, irreducible $\implies$ $J_r(X)$ irreducible

**Lemma.**

$X/\mathbf{C}$ smooth, irreducible $\implies$ $J_r(X)$ affine bundle



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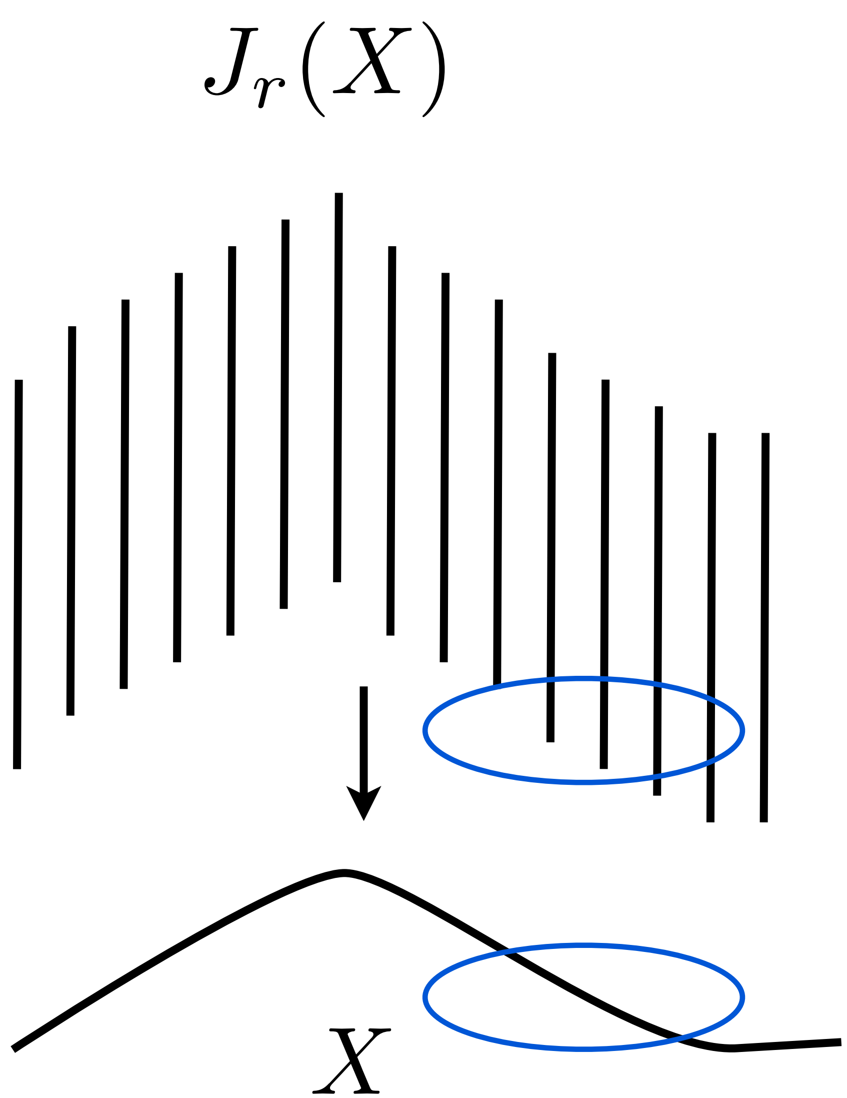
**Theorem.**

$X/\mathbf{C}$ smooth, irreducible $\implies$ $J_r(X)$ irreducible




proof assuming lemma:

$$\pi_r^{-1}(U) \cong U \times \mathbf{A}^{(r+1)\dim(X)}$$

$$\mathcal{O}(\pi_r^{-1}(U)) \cong \mathcal{O}(U)[\text{ variables }]$$

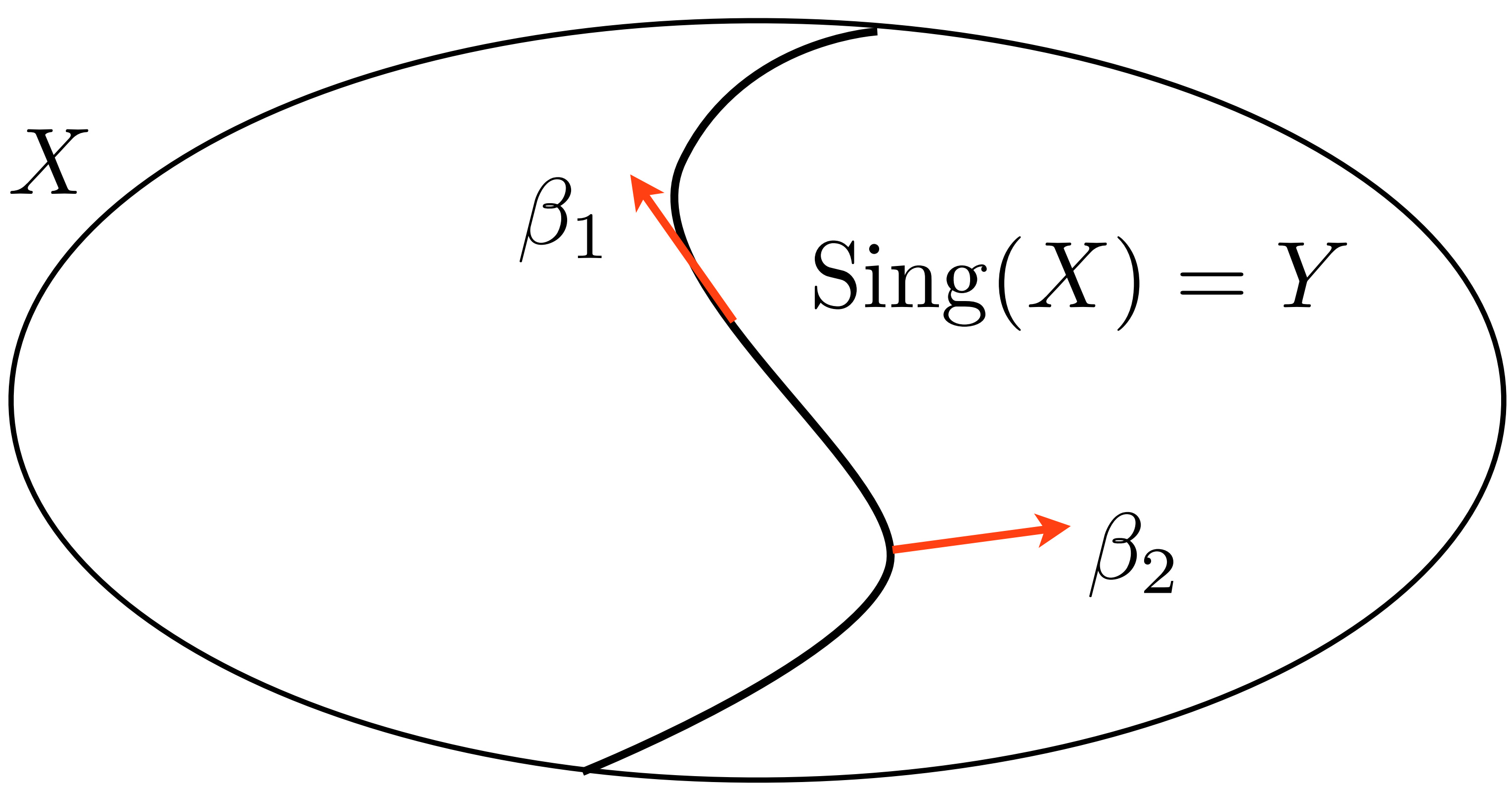
domain

$X/\mathbf{C}$

# Step 3: Reduction to Smooth Case (classical)

$$\overline{J_\infty(\mathrm{Sm}(X))} = J_\infty(X)$$

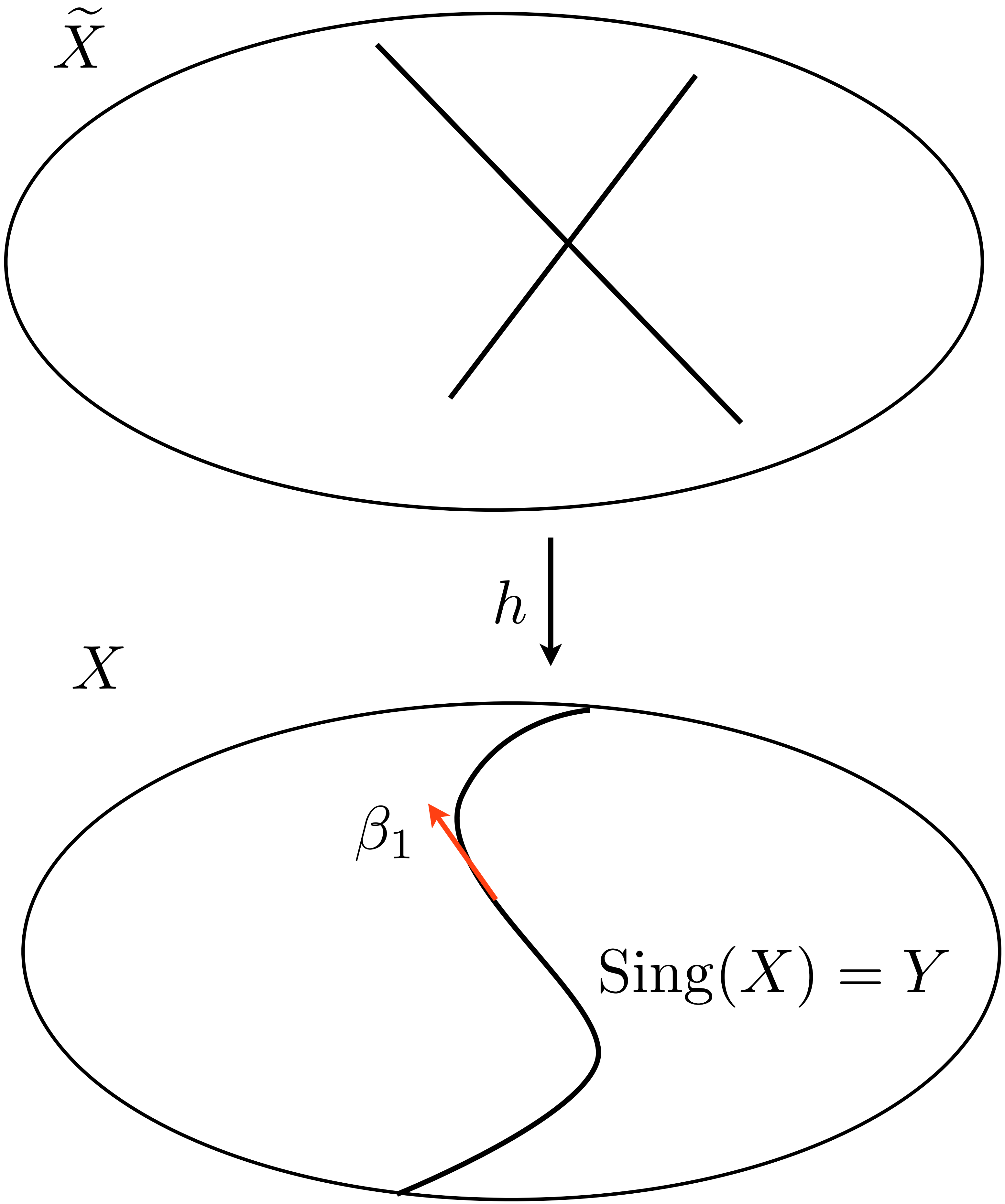
irreducible




$\beta_1 \in J_\infty(Y)$

$\beta_2 \in \pi^{-1}(Y)$

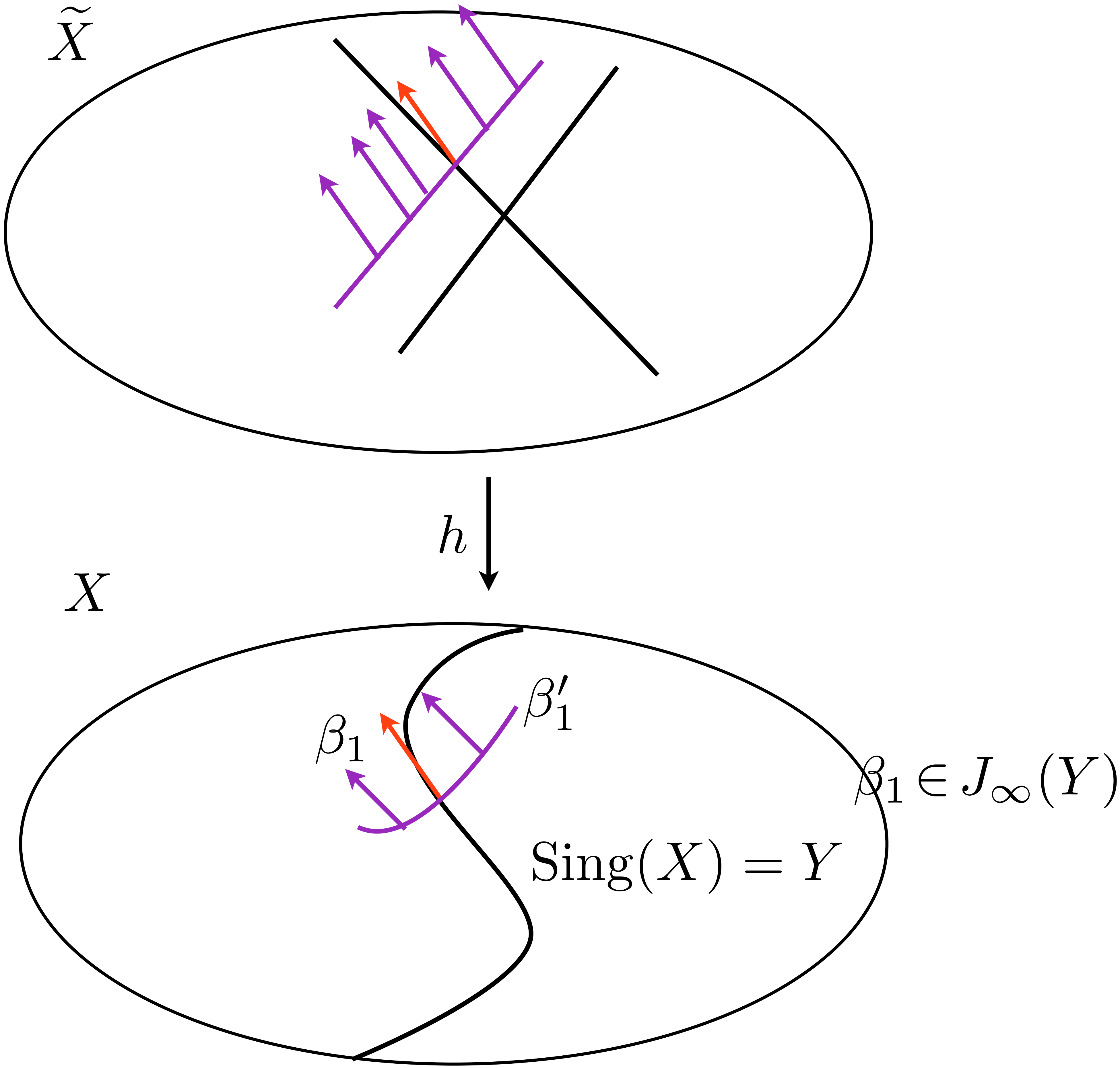
# Step 3: Reduction to Smooth Case (classical) $X/\mathbf{C}$



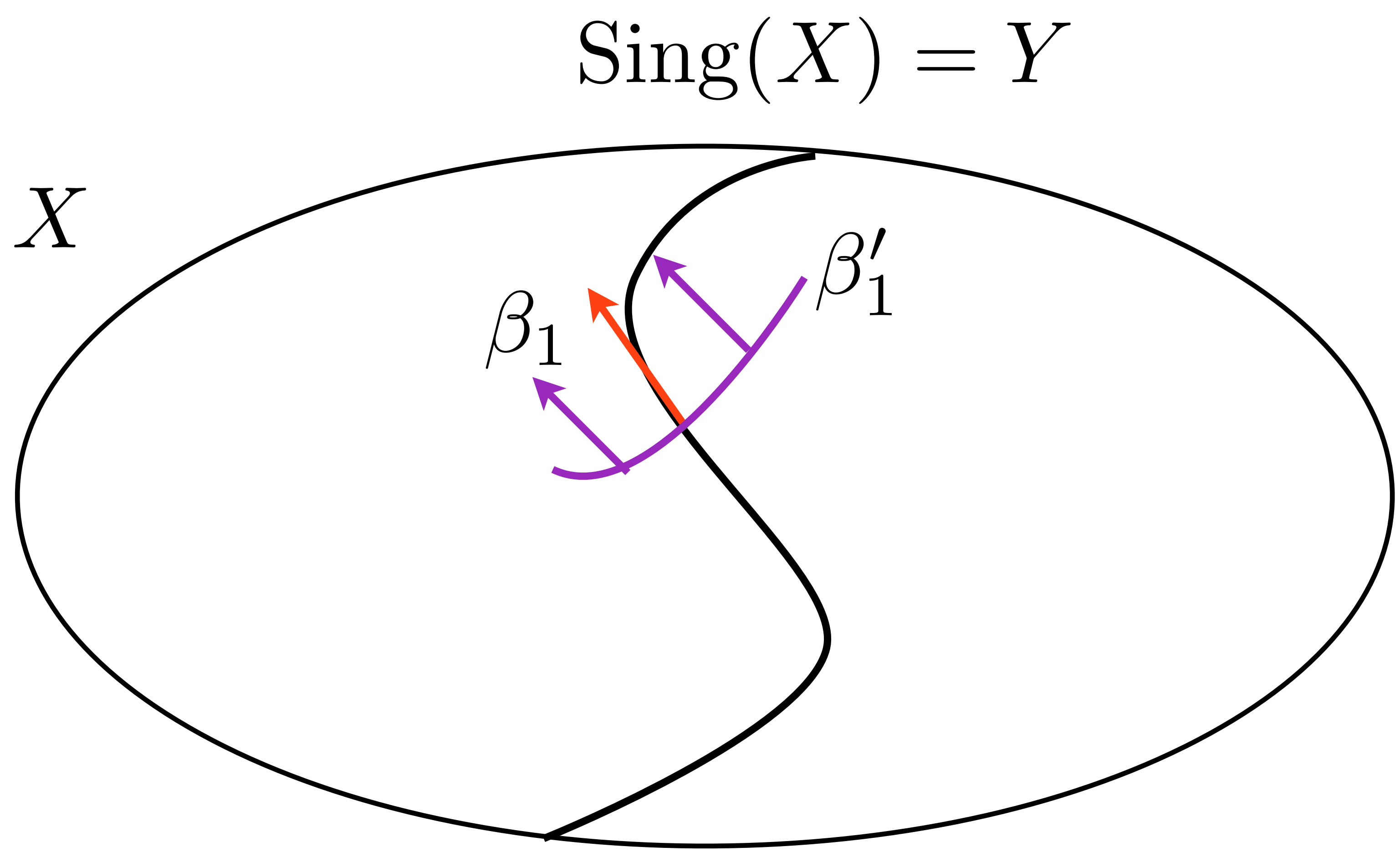

$$\overline{J_\infty(\mathrm{Sm}(X))} \subseteq J_\infty(X)$$

$$\beta_1 \in J_\infty(Y)$$

# Step 3: Reduction to Smooth Case (classical) $X/\mathbf{C}$



# Step 3: Reduction to Smooth Case (classical)

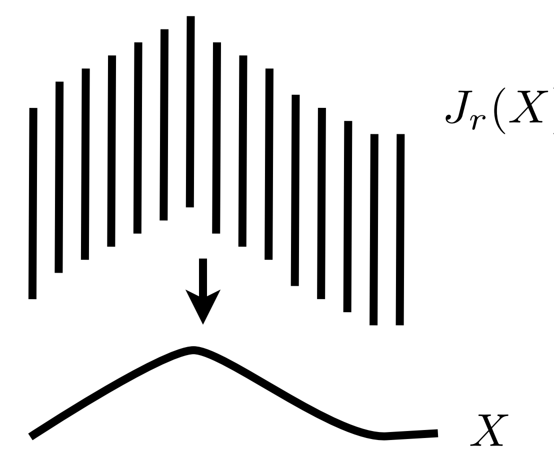


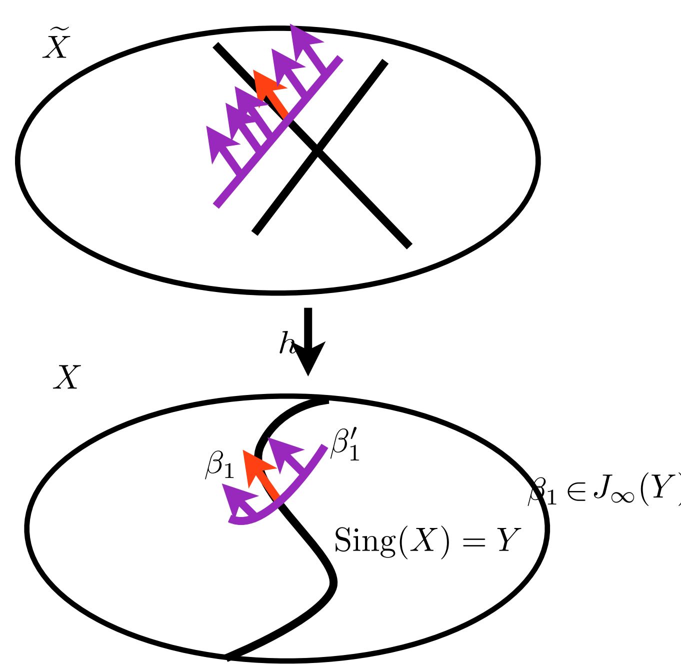

$\beta_1 \in J_\infty(Y)$

$$\beta_1 \in \overline{J_\infty(\mathrm{Sm}(X))} = J_\infty(X)$$

# Recap of Classical

- Step 1: Deformations = Irreducibility (general).
- Step 2: Smooth case (classical).
- Step 3: Reduction to Smooth Case (classical)



# Step 2: Smooth Case (formal arithmetic)

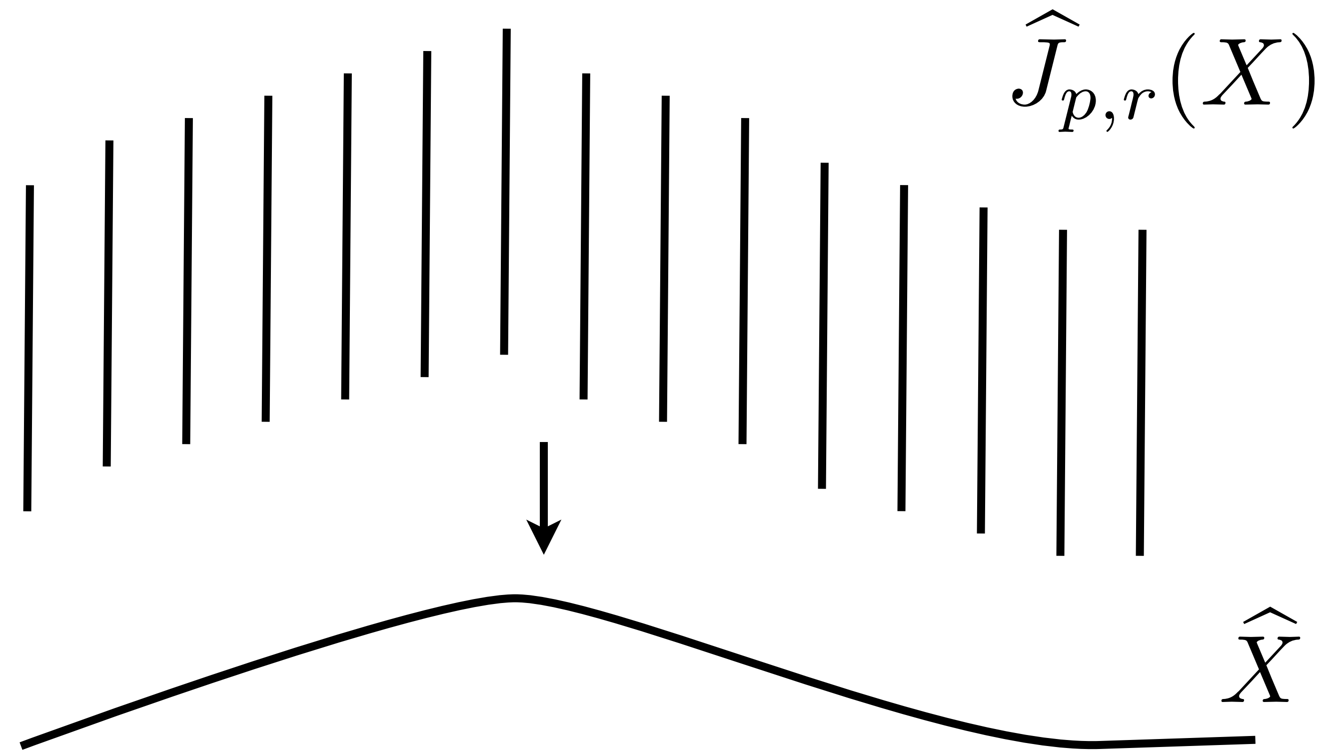
**Theorem. (Buium)**

$X/R$ smooth

$R = W_{p,\infty}(\mathbf{F}_p^{alg})$

$\widehat{J}_{p,r}(X) \to \widehat{X}$

an affine bundle



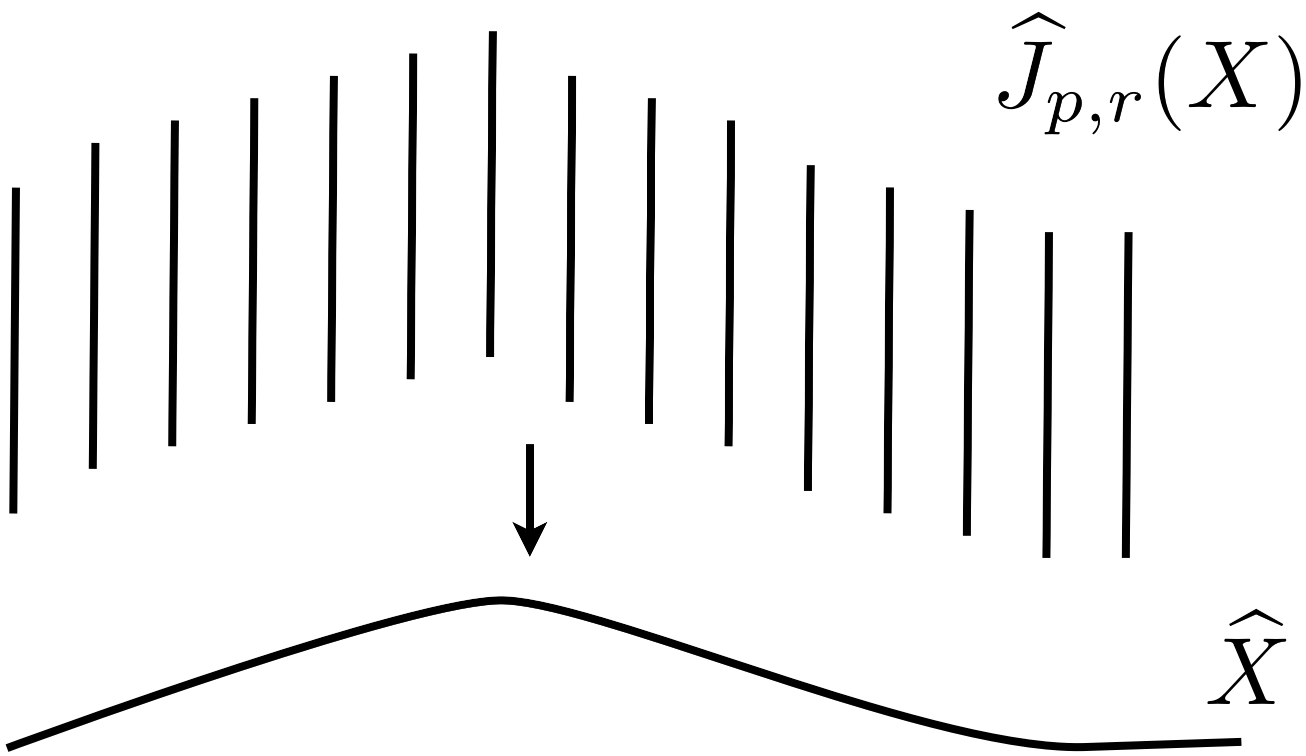

**Corollary.**

$\widehat{X}$ irreducible $\Longrightarrow \widehat{J}_{p,r}(X)$ irreducible

# Step 2: Smooth Case (formal arithmetic)

**Corollary.**

$$\widehat{X} \text{ irreducible} \Longrightarrow \widehat{J}_{p,r}(X) \text{ irreducible}$$



# Step 3: Reduction to Smooth Case

Alterations?? (Introduces Ramification)

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \widetilde{X} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

Neron Smoothenings (Sebag-Loeser, Nicaise-(Chambert-Loir)):

$$\exists h : Y \to X$$

- $Y$ smooth, $\widehat{Y}$ irreducible.
- $Y(W_{p,\infty}(\mathbf{F}_p^{alg})) \to X(W_{p,\infty}(\mathbf{F}_p^{alg}))$ surjective

$$\exists Z \subset J_{p,\infty}(X)$$

- $Z$ is closed irreducible subset.
- $Z$ contains an open.
- $Z$ has all of the closed points.

**Claim:**

$X/W_{p,\infty}(\mathbf{F}_p^{alg})$

$\widehat{X}$ irreducible $+\varepsilon$ $\Longrightarrow$ $J_{p,\infty}(X)$ weakly irreducible

$$\exists h : Y \to X$$

- $Y$ smooth, $\widehat{Y}$ irreducible.
- $Y(W_{p,\infty}(\mathbf{F}_p^{alg})) \to X(W_{p,\infty}(\mathbf{F}_p^{alg}))$ surjective

# THANK YOU

$h^{-1}(D)$

$K =$

$\widetilde{X}$

$\text{Spec } K[[T]]$

$\text{Spec } L$

$L = \text{Frac}(K)$

$h^{-1}(\text{Sing}(X))$

$h$

$\mathbf{C}((T))^{alg}$

$Y = \text{Sing}(X)$

$K =$

$\sim$

$h : X \to X$

$$x^p = zy^p$$

$$y^2 = x^2(x+p)$$

$$y^2 = x^2(x-1)$$

$$\mathcal{R} = W_{p,\infty} \qquad x \in J_{p,\infty}(X)$$

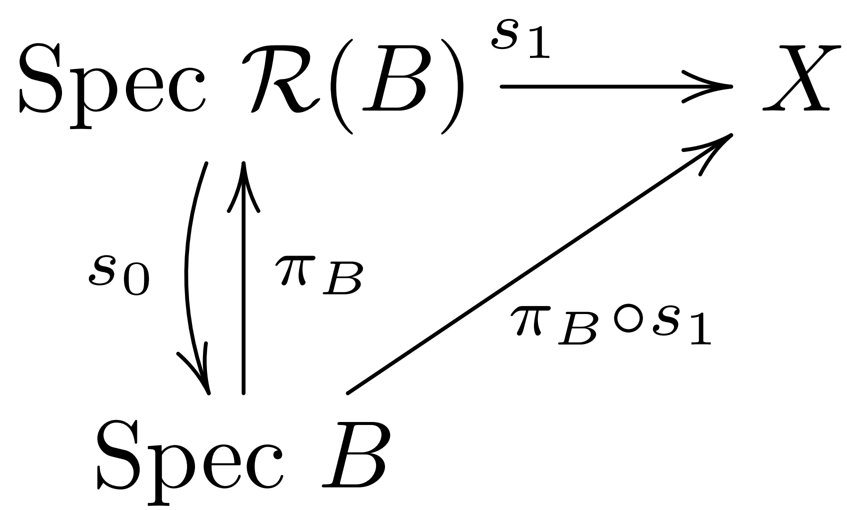
$$X/R$$

$$\kappa(x)$$

$$W_{p,\infty}(k)$$

$$\widetilde{X} \to X \qquad \mathrm{char}(K) \neq p$$

$$\widetilde{X}(R) \to X(R)$$


Spec $\mathcal{R}(B)$
$s_1$
$X$
$s_0$
$\pi_B$
$\pi_B \circ s_1$
Spec $B$

$\lim_n J_n(X)$

$\mathrm{Spec}(\mathcal{R}(C))$

$\mathrm{Spec}(\mathcal{R}(B))$

$Q \odot A$

$Q \odot C$

$J_Q(X)$

$B$

$\Lambda_{p,1}$

$J_Q(X)$

$\mathcal{R} = \mathsf{CRing}(Q, -)$

$C$

$J_1(X)$

$$x_1^a + \cdots + x_n^a$$

$$n/a$$

$$\mathrm{lct}(X, D)$$

# Step 3: Reduction to Smooth Case (arithmetic)

$$\exp : D_\infty \to D_\infty \circ D_\infty$$

$$\exp_A : A[[t]] \to A[[T,S]]$$

$$t \mapsto T + S$$

$$f(t) \mapsto \sum_{n \geq 0} \frac{f^{(n)}(T)}{n!} S^n$$

$$\exp : W_{p,\infty} \to W_{p,\infty} \circ W_{p,\infty}$$