Some Arithmetic Deformation Theory

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Lifts of the Frobenius

Prop

If R is a ring of characteristic p then the map $F : x \mapsto x^p$ is a ring endomorphism.

Lift of Frobenius

A lift of the frobenius on R is a map $\sigma : R \to R$ such that

$$\sigma(x) \equiv x^p \mod p.$$

Absolute Frobenius

k perfect of characteristic p. X a smooth scheme defined over k,

Absolute Frobenius

Morphism of schemes

$$F : X \to X$$

1. Identity on topological space
2. $f \mapsto f^p$ on sheaf.

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2. $f \mapsto f^p$ on sheaf. We are changing the sheaf:

$$\mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$$

What is the Deligne-Illusie Class?

- Obstruction to lift of frobenius $\mod p^2$.
- Used in a paper by Deligne and Illusie in 1987 to give an algebraic proof of Kodaira Vanishing'

$$H^i(X, L \otimes \omega) = 0 \text{ for } i > 0$$

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(for example in Hartshorne X is taken to be a nonsingular projective variety over $\mathbb{C}$)

Deligne and Illusie Class

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The Deligne-Illusie Obstruction

$$\mathrm{DI}_X \in H^1(\underbrace{X_p}_{\text{reduction mod}_p}, \underbrace{F^* T X_p)}_{\text{Frobenius Tangent Bundle}}$$

$X_p = X \otimes_R R/pR$, R/pR perfect

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Recall:

- $\mathrm{char}(R) = p \implies \exists x \mapsto x^p$; the frobenius always makes sense in characteristic p.
- $\theta \in \Gamma(U, F^*TX_p)$ means $\theta : \mathcal{O}(U) \to \mathcal{O}(U)$ with

$$\begin{aligned} \theta(fg) &= \theta(f)g^p + f^p\theta(g) \\ \theta(f+g) &= \theta(f) + \theta(g) \end{aligned}$$

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Lift: $\sigma : \mathcal{O}_X \otimes_R R/p^2 R \to \mathcal{O}_X \otimes_R R/p^2 R$ such that $\sigma(f) \equiv f^p \mod p$.

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Recipient Class? $Mod_{\widehat{\mathbb{Z}_p^{\mathrm{ur}}}}(\mathcal{O},\mathcal{O})$ (Explain later)

Alternative Viewpoint on Lifts of the Frobenius

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$$\exists \sigma : R \to R \iff \exists \delta : R \to R$$

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$$R \text{ has } \sigma \quad \Longrightarrow \quad R \text{ has } \delta, \quad \text{When } R \text{ is p-torsion free} \tag{2}$$

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Given a δ

$$\sigma(x) = x^p + p\delta(x)$$

What is a p-derivation?

WARNING: p-derivations are nonlinear.

$$\begin{aligned} \delta(xy) &= \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y) \\ \delta(x+y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p} \end{aligned}$$

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Example:

$$\delta(\frac{1}{x}) = \frac{1}{x^p}\frac{1}{1 + p\frac{\delta(x)}{x^p}}$$

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As before: unreduced are they a cocycle in the sheaf defined by $U \mapsto Nat(\mathcal{O}|_U, \mathcal{O}|_U)$ "Sheaf Hom"?

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Easier Examples:

- Affine X. Always
- $g(\mathbb{P}^1) = 0$. yes
- E. sometimes
- A. sometimes

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PARTIAL ANSWER:

Necessary condition: $\mathrm{DI}_X = 0$.

HOW IS DI_X A DEFORMATION CLASS?

A Similar Construction using Derivations

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The differences $\delta_i - \delta_j$ give an R-linear derivation. These give a cohomology class

$$[\delta_i - \delta_j] \in H^1(X, TX).$$

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Kodaira-Spencer Map:

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3. X *descends to* $R^\delta = \{r \in R : \delta(r) = 0\}$ *meaning*

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where X_0 *is some schemes defined over* R^δ*.*

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When X is a variety, $R = \mathbb{C}(x)$ defining equations have coefficients in $\mathbb{C}$.

Some Nonsense:

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Vague (and popular) Analogy is employed:

$$\mathrm{KS} = 0 \leftrightarrow \mathrm{DI} = 0$$

Interpretation: When trying to modify theorems in the function field setting, view a lift of the frobenius which keeps the topology fixed as "descent to $\mathbb{F}_1$".

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- We are now going to define $\mathrm{KS}^{\mathsf{ext}}$.

PREPARATIONS

Let $\mathfrak{X}$ over S be a smooth and flat. We view it as a family of varieties

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Important Examples

$$\begin{array}{ccccc} \mathcal{E} & & \mathcal{C}_g & & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1 & & M_g^n & & N_{1,g}^{(n)} \end{array}$$

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$$\mathrm{KS}^{\text{ext}}(\delta_P) := [\delta_i - \delta_j] \in H^1(\mathfrak{X}_p, T\mathfrak{X}_p).$$

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Deformation + Derivation on Base:

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Let $\kappa_S($ generic pt of $S) = K$.

- Construct $\eta \in H^1(\mathfrak{X}, T\mathfrak{X})$ by lifting the δ on affine open sets as we have been doing previously.

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- Specializing η at the closed points gives the external construction

$$\eta \otimes \kappa(\text{closed point } P) = \mathrm{KS}^{\text{ext}}$$

Remark: There is a third way to get the map using the relative tangent sequence. Specializing that map gives different versions of this map.

More than Just an Analogy

Theorem (DI can detect deformation theoretic information)

If A is an abelian variety then

$$F^*\mathrm{KS}^{ext} = \mathrm{DI}$$

PROBLEM: Find external constructions in the p-derivation setting. And relate them to known classes.

Infinitesimal Deformations

Infinitesimal Deformations

"The Kodaira Spencer map for any deformation factors through infinitesimal deformations."

$$\begin{array}{ccc} X^{\epsilon} & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k[\epsilon]) & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

Lemma

For all deformations families of X and choice tangent vector at a point who fiber is X, there exists an infinitesimal deformation that gives rise to the same cohomology class.

Infinitesimal Deformation

Define a functor

$$\mathrm{Def}_X : \{\ \text{Local Artin Rings}\ \} \to \text{Sets}$$

Infinitesimal Deformation

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$$\mathrm{Def}_X : \{\text{ Local Artin Rings }\} \to \text{Sets}$$

where

$$\mathrm{Def}_X(A) = \frac{\{\mathfrak{X} \to \mathrm{Spec}(A), \mathfrak{X} \otimes}{\sim}$$

Prop

$$\mathrm{Def}_X(K[\epsilon]) \leftrightarrow H^1(X, TX)$$

Factoring properties

Differentiation

$$
\begin{array}{ccccc}
R & \longrightarrow & D_1(R) & = & R[\epsilon] \\
 & \searrow & \swarrow & & \\
 & R & & &
\end{array}
$$

Wittferentiation

$$
\begin{array}{ccc}
R & \longrightarrow & W_1(R) \\
 & \searrow \quad \swarrow & \\
 & R &
\end{array}
$$

These allow lifting of derivations and p-derivations for smooth maps.

Wittfinitesimal Deformations

We can construct a map similar to the one for wittfinitesimals in the case when $\mathrm{char}(k) = p$

$$\mathrm{Def}(W_p(k)) \to H^1(X, F^*TX).$$

PROBLEM:

Understand the wittfinitesimal deformations. Understand wittfinitesimal versions of the torelli map.

The Big Class

Big Class

$$B_X \in H^1(X^{\hat{p}}, \underline{\mathrm{Aut}}(\mathbb{A}^1)^{\hat{p}})$$

(that is terrible to TEX)

Let X be a smooth scheme over $W_{p^\infty}(\overline{\mathbb{F}}_p) = \underbrace{\widehat{\mathbb{Z}_p^{\text{ur}}}}_{\text{has unique lift of frob}}$.

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Define cohomology class in the sheaf $\underline{\mathrm{Aut}}(\mathbb{A}^1)^{\hat{p}})$ via the local trivializations of the jet space.

The Big Class

[Jet Rings and Globalizing]

What the sheaf looks like (hat's omitted):

$$\Gamma(U, \underline{\mathrm{Aut}}(\mathbb{A}^1)^{\hat{p}}) = \left\{ \varphi : \begin{array}{ccc} U \times \mathbb{A}^1 & \longrightarrow & U \times \mathbb{A}^1 \\ & \searrow \quad \swarrow & \\ & U & \end{array} \right\}$$

The Big Class

The cocycle is induced by transition maps between trivializations
Let $X_i \subset X$ be trivializing sets
If X is smooth over $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ of relative dimension d then

$$J^n(X_i) \to^{\sim} X_i \times \mathbb{A}^{dn}$$

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*Characteristic Zero

The Big Class

What does the B actually look like?

The Big Class

What does the B actually look like?
Let $X = C$ a curve and $n = 1$.

$$\mathcal{O}(\hat{C}_i \hat{\times} \hat{A}^1) = \mathcal{O}(C_i)[\dot{x}]^{\hat{p}}$$

consisting of restricted powerseries $\sum_{j=0}^{\infty} f_j \dot{x}^j$ satisfing $|f_j|_p \to 0$ as $p \to \infty$.
***Transition maps happen by plugging in restricted power series.

Lemma

Let R be a ring of characteristic p. Then

$$\mathrm{Aut}_R(\mathbb{R}[x]) \cong \mathrm{AL}_1(R).$$

The automorphisms look like

$$f(x) \mapsto f(ax+b)$$

where $a \in R^\times$ and $b \in R$.

The Big Class Controls DI

When we reduce the cocycle in $\underline{\mathrm{Aut}}(\hat{A}^1) \otimes_{\widehat{\mathbb{Z}_p^{\mathrm{ur}}}} \overline{\mathbb{F}}_p \cong \underline{\mathrm{AL}_1}$.

$$\mathrm{AL}_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$$

$$\beta_{ij} = \underbrace{c_{ij}}_{\text{gives } [F^*TX] \in H^1(C, \mathcal{O}^\times)} \dot{x} + \underbrace{d_{ij}}_{\text{gives} \mathrm{DI}}$$

PROBLEM

We want to show $B(X)$ is nontrivial.