Some Arithmetic Deformation Theory

Taylor Dupuy

September 15, 2010

Taylor Dupuy Some Arithmetic Deformation Theory

Prop

If R is a ring of characteristic p then the map $F: x \mapsto x^p$ is a ring endomorphism.

Lift of Frobenius

A lift of the frobenius on R is a map $\sigma:R\to R$ such that

$$\sigma(x) \equiv x^p \mod p.$$

Absolute Frobenius

k perfect of characteristic $p.\ X$ a smooth scheme defined over k,

Absolute Frobenius

Morphism of schemes

$$F: X \to X$$

- Identity on topological space
- **2** $f \mapsto f^p$ on sheaf.

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Morphism of schemes

$$F: X \to X$$

Identity on topological space

2 $f \mapsto f^p$ on sheaf. We are changing the sheaf:

 $\mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$

What is the Deligne-Illusie Class?

- Obstruction to lift of frobenius $\mod p^2$.
- Used in a paper by Deligne and Illusie in 1987 to give an algebraic proof of Kodaira Vanishing'

$$H^i(X, L \otimes \omega) = 0$$
 for $i > 0$

What is the Deligne-Illusie Class?

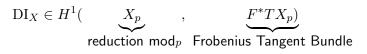
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$$H^i(X, L \otimes \omega) = 0$$
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(for example in Hartshorne X is taken to be a nonsingular projective variety over \mathbb{C})

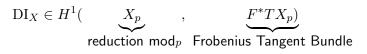
Deligne and Illusie Class

The Deligne-Illusie Obstruction



 $X_p = X \otimes_R R/pR$, R/pR perfect

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The Deligne-Illusie Obstruction

$$\mathrm{DI}_X \in H^1(\underbrace{X_p}_{\mathsf{reduction}}, \underbrace{F^*TX_p}_{\mathsf{Frobenius Tangent}}$$
 Bundle

 $X_p = X \otimes_R R/pR$, R/pR perfect Recall:

- $char(R) = p \implies \exists x \mapsto x^p$; the frobenius always makes sense in characteristic p.
- $\theta \in \Gamma(U, F^*TX_p)$ means $\theta : \mathcal{O}(U) \to \mathcal{O}(U)$ with

$$\begin{aligned} \theta(fg) &= \theta(f)g^p + f^p\theta(g) \\ \theta(f+g) &= \theta(f) + \theta(g) \end{aligned}$$

Obstruction

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Theorem

$DI_X = 0 \iff \mathcal{O}_X$ admits a lift of the frobenius $\mod p^2$

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Lift: $\sigma : \mathcal{O}_X \otimes_R R/p^2 R \to \mathcal{O}_X \otimes_R R/p^2 R$ such that $\sigma(f) \equiv f^p \mod p$.

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R admits a lift of the frobenius, and R/pR a perfect field.

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$$x \mapsto \left(\frac{\sigma_i(x) - \sigma_j(x)}{p}\right) \mod p$$
 derivation of F .

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Recipient Class? $Mod_{\widehat{\mathbb{Z}^{ur}}}(\mathcal{O}, \mathcal{O})$ (Explain later)

Prop

 $R \ \text{a ring}$

$\exists \sigma: R \to R \iff \exists \delta: R \to R$

 δ a p-derivation

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$$R \text{ has } \sigma \implies R \text{ has } \delta,$$

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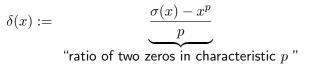
$$R \text{ has } \sigma \iff R \text{ has } \delta \tag{1}$$

$$R \text{ has } \sigma \implies R \text{ has } \delta, \text{ When } R \text{ is p-torsion free} \tag{2}$$

What is a *p*-derivation?

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Given a σ :



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What is a *p*-derivation?

Given a σ :

$$\delta(x) := \underbrace{\frac{\sigma(x) - x^p}{p}}_{\text{``ratio of two zeros in characteristic } p ~`}$$

Given a δ

$$\sigma(x) = x^p + p\delta(x)$$

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What is a *p*-derivation?

WARNING: *p*-derivations are nonlinear.

$$\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$$

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$

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Example:

$$\delta(\frac{1}{x}) = \frac{1}{x^p} \frac{1}{1 + p\frac{\delta(x)}{x^p}}$$

 DI revisited:

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$$\frac{\sigma_i(x) - \sigma_j(x)}{p}$$

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$$\frac{\sigma_i(x) - \sigma_j(x)}{p} = \frac{(x^p + p\delta_i(x)) - (x^p + p\delta_j(x))}{p}$$

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• Differences of *p*-derivations are derivations of the frobenius when reduced mod *p*.

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- They are $\widehat{\mathbb{Z}_p^{ur}}$ linear maps.

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As before: unreduced are they a cocycle in the sheaf defined by $U \mapsto Nat(\mathcal{O}|_U, \mathcal{O}|_U)$ "Sheaf Hom"?

Reynaud's Theorem

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Theorem

If C is a smooth projective curve of genus $g \ge 2$ then C does not admit a lift of the frobenius for all primes p.

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(Say defined over \mathbb{Z} then get C over $\widehat{\mathbb{Z}_p^{ur}}$ by base extension).

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Easier Examples:

- Affine X. Always
- $g(\mathbb{P}^1) = 0$. yes
- E. sometimes
- A. sometimes

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X/R smooth scheme

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When does X admit a lift of the absolute frobenius frobenius agreeing with σ on R?

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PARTIAL ANSWER:

Necessary condition: $DI_X = 0$.

HOW IS DI_X A DEFORMATION CLASS?

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FUNCTION FIELD SETTING

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The differences $\delta_i-\delta_j$ give an R-linear derivation. These give a cohomology class

 $[\delta_i - \delta_j] \in H^1(X, TX).$

Kodaira-Spencer Map:

 $\operatorname{KS}_{X/R}$: $\operatorname{Der}(R, R) \to H^1(X, TX).$

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A Cocycle Construction Using Lifts of Derivations

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 J¹(X) ≅ TX

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Kodaira-Spencer Map:

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Theorem

- The following are equivalent
- $I KS(\delta) = 0$
- $2 J^1(X) \cong TX$
- **3** X descends to $R^{\delta} = \{r \in R : \delta(r) = 0\}$ meaning

$$X \cong X_0 \otimes_{R^\delta} R$$

where X_0 is some schemes defined over R^{δ} .

A Cocycle Construction Using Lifts of Derivations

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- $\textbf{ S} \ X \ \textit{descends to} \ R^{\delta} = \{r \in R : \delta(r) = 0\} \ \textit{meaning}$

$$X \cong X_0 \otimes_{R^\delta} R$$

where X_0 is some schemes defined over R^{δ} .

When X is a variety, $R=\mathbb{C}(x)$ defining equations have coefficients in $\mathbb{C}.$

Some Nonsense:

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Vague (and popular) Analogy is employed:

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KS = 0 \leftrightarrow DI = 0
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Interpretation: When trying to modify theorems in the function field setting, view a lift of the frobenius which keeps the topology fixed as "descent to \mathbb{F}_1 ".

WHERE IS THE DEFORMATION THEORY?

Taylor Dupuy Some Arithmetic Deformation Theory

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WHERE IS THE DEFORMATION THEORY?

- $\bullet~\mathrm{KS}^{\text{ext}}$ denote the class before.
- We are now going to define KS^{ext}.

PREPARATIONS

Let $\mathfrak X$ over S be a smooth and flat. We view it as a family of varieties

 $\mathfrak{X}_P := \mathfrak{X} \otimes_S \kappa_S(P)$

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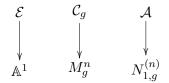
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Important Examples



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- **②** Fatten to some open affine neighborhood U of S which contains P. Call the extended derivation δ .
- **③** Cover \mathfrak{X} by affine \mathfrak{X}_i ,

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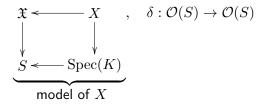
$$\mathrm{KS}^{\mathsf{ext}}(\delta_P) := [\delta_i - \delta_j] \in H^1(\mathfrak{X}_p, T\mathfrak{X}_p).$$

Link Between Internal and External

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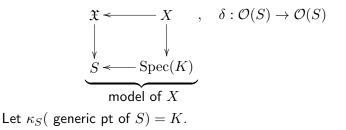
 \boldsymbol{X} be defined over a function field \boldsymbol{K}

X be defined over a function field KDeformation + Derivation on Base:



Link Between Internal and External

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 $\eta \otimes \kappa$ (generic) = KS^{ext}

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\eta \otimes \kappa(generic) = KS<sup>ext</sup>
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• Specializing η at the closed points gives the external construction

 $\eta \otimes \kappa(\text{closed point } P) = \text{KS}^{\text{ext}}$

Remark: There is a third way to get the map using the relative tangent sequence. Specializing that map gives different versions of this map.

Theorem (DI can detect deformation theoretic information) If A is an abelian variety then

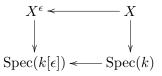
 $F^* KS^{ext} = DI$

PROBLEM: Find external constructions in the p-derivation setting. And relate them to known classes.

Infinitesimal Deformations

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"The Kodaira Spencer map for any deformation factors through infinitesimal deformations."



Lemma

For all deformations families of X and choice tangent vector at a point who fiber is X, there exists an infinitesimal deformation that gives rise to the same cohomology class.

Define a functor

$\operatorname{Def}_X : \{ \text{ Local Artin Rings } \} \to \mathsf{Sets}$

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$$Def_X : \{ Local Artin Rings \} \rightarrow Sets$$

where

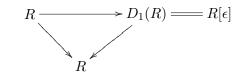
$$\operatorname{Def}_X(A) = \frac{\{\mathfrak{X} \to \operatorname{Spec}(A), \mathfrak{X} \otimes$$

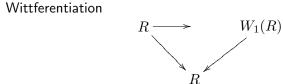
Prop

 $\operatorname{Def}_X(K[\epsilon]) \leftrightarrow H^1(X,TX)$

Factoring properties

Differentiation





These allow lifting of derivations and p-derivations for smooth maps.

Wittfinitesimal Deformations

We can construct a map similar to the one for wittfinitesimals in the case when ${\rm char}(k)=p$

$$\operatorname{Def}(W_p(k)) \to H^1(X, F^*TX).$$

PROBLEM:

Understand the wittfinitesimal deformations. Understand wittfinitesimal versions of the torelli map.

Big Class

$$B_X \in H^1(X^{\hat{p}}, \underline{\operatorname{Aut}}(\mathbb{A}^1)^{\hat{p}})$$

(that is terrible to T_EX)

Let X be a smooth scheme over $W_{p^{\infty}}(\overline{\mathbb{F}}_p) = \underbrace{\mathbb{Z}_p^{ur}}_{p}$ has unique lift of frob

Big Class

$$B_X \in H^1(X^{\hat{p}}, \underline{\operatorname{Aut}}(\mathbb{A}^1)^{\hat{p}})$$

(that is terrible to T_EX)

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Let X be a smooth scheme over $W_{p^{\infty}}(\overline{\mathbb{F}}_p) = \underbrace{\mathbb{Z}_p^{ur}}_{has unique lift of frob}$. Define cohomology class in the sheaf $\underline{\operatorname{Aut}}(\mathbb{A}^1)^{\hat{p}}$) via the local trivializations of the jet space.

[Jet Rings and Globalizing]

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The Big Nasty Class

What the sheaf looks like (hat's omitted):

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$$\Gamma(U,\underline{\operatorname{Aut}}(\mathbb{A}^1)^{\hat{p}}) = \left\{ \varphi: \ U \times \mathbb{A}^1 \xrightarrow{} U \times \mathbb{A}^1 \right\}$$

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The cocycle is induced by transition maps between trivializations Let $X_i \subset X$ be trivializing sets If X is smooth over $\widehat{\mathbb{Z}_p^{ur}}$ of relative dimension d then

$$J^n(X_i) \to \sim X_i \times \mathbb{A}^{dn}$$

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*Characteristic Zero

What does the ${\cal B}$ actually look like?

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What does the B actually look like? Let X = C a curve and n = 1.

$$\mathcal{O}(\hat{C}_i \times \hat{A}^1) = \mathcal{O}(C_i)[\dot{x}]^{\hat{p}}$$

consisting of restricted powerseries $\sum_{j=0}^{\infty} f_j \dot{x}^j$ satisfing $|f_j|_p \to 0$ as $p \to \infty$.

***Transition maps happen by plugging in restricted power series.

Lemma

Let R be a ring of characteristic p. Then

 $\operatorname{Aut}_R(\mathbb{R}[x]) \cong \operatorname{AL}_1(R).$

The automorphisms look like

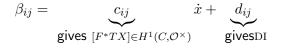
$$f(x) \mapsto f(ax+b)$$

where $a \in R^{\times}$ and $b \in R$.

The Big Class Controls DI

When we reduce the cocycle in $\underline{\operatorname{Aut}}(\hat{A}^1) \otimes_{\widehat{\mathbb{Z}_p^{\operatorname{ur}}}} \overline{\mathbb{F}}_p \cong \underline{\operatorname{AL}}_1.$

$$AL_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$$



PROBLEM

We want to show B(X) is nontrivial.

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