# WEIGHT TWO EICHLER-SHIMURA CRASH COURSE

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Abstract. These notes are an extremely quick introduction to Eichler-Shimura.

## 1. What is the Eichler-Shimura Map?

**Theorem 1.1** (Eichler-Shimura Theorem)**.** *For every type of "special" modular form $f$ we can give a map from a modular curve to an elliptic curve which depends on $f$.*

The word "special" in the above theorem means that $f$ is a so-called Eigencusp-form. This just means that $f$ is a cusp form and it is simultanesouly an eigenvector for all of the Hecke operators. Given such an $f$ the we construct an elliptic curve $E_f$ such that there exists a map

$$X_0(N) \to E_f.$$

This map is called the **Eichler-Shimura Map** or a **modular parametrization**.

In particular curve $E_f$ will be a quotient of $J_0(N) := \mathrm{Jac}(X_0(N))$ by a particular abelian subvariety $H_f$ that we will construct:

$$E_f = J_0(N)/H_f. \tag{1.1}$$

and the map will be the composition

$$X_0(N) \to J_0(N) \to J_0(N)/H_f = E_f. \tag{1.2}$$

There are three stages

- The construction.
- Showing $E_f$ is nonzero.
- Showing showing the dimension of $E_f$ is less than or equal to one.

*All three stages use actions of Hecke operators in the proof.* That's the main idea.

## 2. Construction of the Quotient $J_0(N)$

Let $\mathbb{T}$ be the collection of Hecke operators. For every $f \in S_2(\Gamma_0(N))$ an eigenform we can define a character $\chi_f : \mathbb{T} \to \mathbb{Q}$ via

$$\chi_f(T) := Tf/f.$$

Let $I_f$ be the ideal in the endomorphism algebra:

$$I_f = \ker \chi_f \tag{2.1}$$

and define the subvariety of the Jacobian

$$H_f = \{ \sum_{T \in I_f} T(P) : P \in J_0(N) \} \tag{2.2}$$

*Date*:

**Proposition 2.1.** *$H_f$ is an abelian subvariety defined over $\mathbb{Q}$.*

*Proof.* Since $T$ acts linearly, the set $H_f$ is definitely closed under addition. □

## 3. The Quotient is Nontrivial

**Theorem 3.1.** *$E_f$ is nonzero.*

**Proposition 3.2.** $\ker(\chi_f)$ *is both a maximal an minimal ideal.*

*Proof.* Since the targe of $\chi_f : \mathbb{T} \to \mathbb{Q}$ is an integral domain this implies that $\chi_f$ is prime. □

**Proposition 3.3.** *If* $\ker \chi = \operatorname{ann}(\beta)$ *show that* $\beta H_f = 0$.

**Proposition 3.4.** *By supposing that $H_f = J$ is nonzero you should be able to derive the contradiction that $\beta(J) = 0$ which implies t*

$$\ker \chi_f = \operatorname{ann}(B).$$

*This implies that $\beta = 0$.*

**Lemma 3.5.** *Let $A$ be artinian. For all $M \lhd A$ maximal there exists a $\beta \in A$ such that*

$$A = \operatorname{ann}(\beta)$$

**Lemma 3.6** (Equivariance Lemma)**.** *Show that the action of the Hecke Operators on $J_0(N)$ is equivariant with the action on $E_f$ by multiplication by $a_n$.*

*Proof.* This statement is equivalent to showing that $T_n - [a_n]$ as an endomorphism gives you zero after quotienting. This in turn is equivalent to show that the image of $T_n - [a_n]$ is in the kernel of the next map.This amounts to showing $(T_n - [a_n])(J) \subset H_f$. □

The main idea of what follows is:

$$\text{Multiplicity One Theorem} \implies \text{Quotient of Jacobian Has Dimension One}$$

## 4. The Quotient has Dimension 1 (is an Elliptic Curve)

**Proposition 4.1.** $\dim(E_f) \leq 1$

*Proof.* We will show that the dimension of the cotangent space of $E_f$ is equal to 1. Let $\pi$ be a huge composition of maps

$$\pi : \mathcal{H} \to X_0(N) \to J_0(N) \to E_f.$$

The pullback of every one form on $E_f$ will be a multiple of the modular form associated to $f$. Since the dimension of an abelian variety is the same as the dimension of its space of global one forms we are done (note that an abelian variety just looks like $A = \mathbb{C}^g/L$).

Let $\omega$ be a global one form on $E_f$. Pull it back via $\pi$. We will now that

$$\pi^*\omega = g(z)dz = \alpha f(z)$$

for some nonzero $\alpha$. Using the Equivariance Lemma (Lemma 3.6) we get

$$T_n(\pi^*\omega) = \pi^*([a_n] \cdot \omega) = a_n \pi^* \omega$$

The right hand side of the equality gives $a_n g(z)dz$ and the left hand side gives $(T_n g)(z)dz$ which tells us that $g$ is a Hecke Eigenform. We know that $g$ must be a multiple of our original modular form $f$ by it's coefficients and the multiplicity one theorem. ☐

**Proposition 4.2.** *Show that* $H^0(E,\Omega) \hookrightarrow H_0(X_0(N), Omega) = S_2$ *in an inclusion which is Hecke Equivariant.*

## 5. The Multiplicity One Theorem

**Proposition 5.1** (Multiplicity One Theorem)**.** *If* $g(q) = a_0 + a_1 q + a_2 q + \cdots$ *an element* $S_2(\Gamma_0(N))$ *is an new eigenform for* $\mathbb{T}$ *then for all* $n$ *we have*

$$(T_n g)(q) = a_n g(q). \tag{5.1}$$

**Proposition 5.2.** *If* $f(q)$ *is an eigenform then all its coeff are in* $\mathbb{Z}$.

## 6. Hecke Operators

The vector space of modular forms $M_k(\Gamma_0(N))$ is equipt with an action of a commuting algebra $\mathbb{T} = \{T_n : n \geq 1\}$.

(1) How do Hecke Operators act on Modular forms?

$$(T_p f)(z) = \frac{1}{p+1}\left(f(pz) + \sum_{j=0}^{p-1} f(\frac{z+j}{p})\right) \tag{6.1}$$

$$T_m T_n = \sum_{d|(m,n)} T_{nm/d^2} \tag{6.2}$$

(2) How do Hecke Operators act on $X_0(N)$? (They don't) See next Question,

(3) How do Hecke Operators action on Divisors of $X_0(N)$? If we view $f$ as a function on the points of $X_0(N)$ then we can extend $f$ to a function on the divisors of $X_0(N)$ by linearity. Defining an action on the divisors of $X_0(N)$ is then enough to define an action on the functions of $f$ itself. Analytically an elliptic curve is just $\mathbb{C}/L$ for some lattice. So $X_0(N)$ is really just parametrizing lattices modulo equivalence. We could then ask the Hecke operators to be an action on the formal group of Lattices:

$$(T_p f)(L) = \frac{1}{p+1} \sum_{[L:L']=p} f(L') \tag{6.3}$$

(4) How do Hecke Operatrors act on $\mathrm{Div}^0(X_0(N)) = J_0(N)$?

**Proposition 6.1.** *The Hecke Algebra is Artinian.*

## 7. Prerequisites

**Proposition 7.1.** *If* $X_0(N)$ *is defined over* $\mathbb{Q}$ *then* $J_0(N)$ *is defined over* $\mathbb{Q}$.

More generally if $X$ is a scheme then the group $\mathrm{Pic}^0(X)$ actually has the structure of a scheme and satisfies a universal property. In the case when $X$ is a curve $\mathrm{Pic}^0(X)(\mathbb{C}) \cong \mathrm{Jac}(X)(\mathbb{C})$

If $H \subset A$ is an abelian subscheme then $A/H$ makes sense in the category of schemes and is defined over mathbb $\mathbb{Q}$. If $G$ is any group action on a scheme $X$ then $G$ acts on the graded ring defining $X$. We take the quotient of the graded ring by the group action (the invariants) to get the categorical quotient.

**Corollary 7.2.** *$E_f := J_0(N)/H_f$ is defined over $\mathbb{Q}$*

7.0.1. *Artinian Rings.* An **Artinian Ring** is one which any descending chain of ideals terminates.

Intuitively these are rings with lots of torsion like $k[\epsilon]$ or $W_p(\mathbb{F}) \cong \mathbb{Z}/p^2\mathbb{Z}$. A non-artinian ring would be something like $\mathbb{Z}_p$ where if $\mathfrak{m}$ is the local ring $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ has an infinite descending chain

**Proposition 7.3.** (1) *Every Artinian local ring is Noetherian.*
(2) *Every Artinian ring is Noetherian.*
(3) *Show that $A$ is Artinian if and only if it is Noetherian and Dimension zero.*
(4) *Any maximal ideal is the Annihilator of some $\beta$*

*Proof.* If $A$ is local then then an infinite ascending chain would take the form $\mathfrak{I}_1 \supset I_2 \supset I_2 \subset \cdots$ where $I_j = \mathfrak{m}^{n_j}$ for some $n_j$ a decreasing sequence of natural numbers. There is no such thing as a descreasing sequence of natural numbers.

Suppose that $A$ has an infinite ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$, by Zorns Lemma there exists a maximal ideal $M$ containing this sequence. The ring $A$ includes into its localization $A_M$ at $M$. The ideals in $A_M$ are in one-to-one correspondence with ideals in $A$ which are contained in $M$. This would give an infinite ascending chain in an artinian local ring which is a contradiction.

We need to show that if $A$ is artinian then the maximal length chain of prime ideals is one. If $A$ is local this is trivial. Since the inverse image of a prime ideal is prime, supposing there was a chain of size greater than two would imply that there would be a chain of size greater than two in the localization of $A$ at the maximal ideal at the end of the chain of prime ideals. This is a contradiction.

We will first show that $M$ is annihilated by some $\beta$ if $A$ is local. □

If $A$ has dimension zero then it is a PID.

For our two artinian examples the annihilator of the rings $\langle \epsilon \rangle$ and $\langle p \rangle$ are the elements $\epsilon$ and $p$ resectively.

## References