IMUS Lecture Notes on Harmonic Analysis, Metric Spaces and PDES, Sevilla 2011

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Chapter 1

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1.1 Lecture One: Sobolev Spaces

1.1.1 Differentiability and Smoothness in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be open, let $x_0 \in \Omega$ define and let $f : \Omega \to \mathbb{R}$

- Define what it means for f to be **differentiable** at x_0.
- Define what it means for f to be **smooth** at x_0.

We will denote n times differentiable functions by $\mathbb{C}^n(\Omega)$ and smooth functions by $\mathbb{C}^\infty(\Omega)$.

1.1.2 Sobolev spaces in $\mathbb{R}^n$ by completing the smooth functions

Let $p \in (1, \infty)$ and $f \in \mathbb{C}^\infty(X)$, define the **first Sobolev norm** by

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

- Show that $\| \cdot \|_{W_{1,p}}$ is a norm.

The set of smooth functions $\mathbb{C}^\infty$ with the first Sobolev, $\| - \|_{W_{1,p}(\Omega)}$ is a normed vector space.

- Show that the normed vector space $(\mathbb{C}^\infty(\Omega), W^{1,p}$ is not a Banach space.

The completion of the smooth functions on Ω with respect to the norm $\|-\|_{W^{1,p}(\Omega)}$ is called the **first** p **Sobolev space** and is denoted by $W^{1,p}(\Omega)$. It consists of functions in $L^p(\Omega)$ with a gradient in $L^p(\Omega, \mathbb{R}^n)$.

1.1.3 Integration by Parts and Weak Derivatives

If $f \in \mathbb{C}^\infty(\Omega)$ and $\varphi \in \mathbb{C}_c^\infty(\Omega, \mathbb{R}^n)$ then

$$\int_\Omega \langle \varphi(x), \nabla f(x) \rangle dx = -\int_\Omega f(x) \nabla \cdot \varphi(x) dx \tag{1.1.1}$$

by applying integration by parts and the fact that φ has compact support. For $f : \Omega \to \mathbb{R}$ measurable there exists an operator $df : \mathbb{C}_c^\infty(\Omega, \mathbb{R}^n) \to \mathbb{R}$ given by

$$df(\varphi) = -\int_\Omega f(x) \nabla \cdot \varphi(x) dx.$$

The idea is that when f is differentiable we have

$$df = \langle -, \nabla f \rangle,$$

by equation 7.1.1, so this operator is a replacement for derivatives for functions that don't have them.

- What are the functions f such that there exists some $V : \Omega \to \mathbb{R}^n$ such that for all $\varphi :\in \mathbb{C}_c^\infty(\Omega)$ we have

$$df(\varphi) = \int_\Omega \langle V(x), \varphi(x) \rangle?$$

Functions which have these are called **weakly differentiable**. The following exercise shows that they don't always exist.

- We claim that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1, & x > 0 \\ 0. & x \leq 0 \end{cases}$$

  does not have such a V.

When a function does have a weak derivative we should observe we can change the definition of V on a set of measure zero and get another weak derivative. The weak derivitive is unique up to a set of measure zero.

1.1.4 Sobolev spaces by existence of weak derivatives

If $f \in L^p$ has a weak derivative ∇f then on the subset of functions which have a weak derivative we can define the first Sobolev norm as

$$\|f|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

The Sobolev Space could then be defined as the set of weak differentiable functions in L^p with bounded first Sobolev norm.

1.1.5 Equivalence of two ways of defining Sobolev spaces

We have given two ways of defining the Sobolev Spaces. The first is as the completion of $\mathbb{C}^\infty(\Omega)$ with respect to $\|-\|_{W^{1,p}}$ and the second as the subspace of $L^p(\Omega)$ of weakly differentiable functions with bounded Sobolev norm.

Theorem 1.1.1 (Serrin)**.** *Both approaches to defining the Sobolev space*

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with compact support. Define $\eta(x) = \psi(|x|)$. For $f \in W^{1,p}(\Omega)$ we can define for every $\varepsilon > 0$

$$(f * \eta_\varepsilon)(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(y)\eta(\frac{x-y}{\varepsilon})dy.$$

By approximations of the identity this converges to f in L^p sense as $\varepsilon \to 0$. Next we have

$$\begin{aligned} \partial_j(f * \eta_\varepsilon) &= f * \partial_j \eta_\varepsilon \\ &= \partial_j f * \eta_\varepsilon \end{aligned}$$

which approaches $\partial_j f$ as $\varepsilon \to 0$ for every j. This gives a sequence of approximations of $f \in W^{1,p}(\Omega)$ by smooth functions.

1.1.6 absolute continuity

A function $f : \Omega \to \mathbb{R}$ is **absolutely continuous** along $\gamma : [0,1] \to \Omega$ provided the following holds:

For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k] \subset [0,1]$ we have

$$\sum_i |b_i - a_i| < \delta \implies \int_{\sum[a_i,b_i]} f \circ \gamma < \varepsilon.$$

1.1.7 convention: parametrizing in terms of arclength

Let $\gamma \subset \Omega$ be a curve of finite length and let $f \in \mathbb{C}^\infty(\Omega)$. We can view γ as $\gamma : [0,1] \to \Omega$. We have the inequality

$$|f(y_\gamma) - f(x_\gamma)| \leq \int_0^1 |\nabla f(\gamma(t))||\gamma'(t)|dt,$$

where $y(\gamma)$ is the end point of the curve and x_γ is the end point of the curve. We will parametrize curves in term of arclength so the the norm of the derivative will be equal to one reducing the above expression to

$$|f(y_\gamma) - f(x_\gamma)| \leq \int_0^1 |\nabla f(\gamma(t))|dt.$$

1.1.8 criteria for absolute continuity

Proposition 1.1.2. *Suppose that $f : \Omega \to \mathbb{R}$ and γ is a rectifiable curve in Ω. Also suppose that there exists some $g : \Omega \to [0, \infty]$ measurable such that*

$$\int_\gamma g ds < \infty$$

and such that for all subcurves $\beta \subset \gamma$ we have

$$|f(y_\beta) - f(x_\gamma)| \leq \int_\beta g ds.$$

Then, $f \circ \gamma$ is absolutely continuous and $|(f \circ \gamma)'| \leq g \circ \gamma$ pointwise.

- Using the definition of absolute continuity prove the above proposition.

1.1.9 the Poincaré inequality

This number is about the **Poincaré inequality**. For a set A we use the following notation for integral average

$$\fint_A f = \frac{1}{|A|} \int_A f.$$

Proposition 1.1.3 (Poincaré Inequality)**.** *For $f \in \mathbb{C}^\infty(\Omega)$ and $x_0 \in \Omega$ let $r > 0$ be such that $\overline{B(x_0, r))} \subset \Omega$*

$$\fint_{B(x_0,r)} |f(y) - f(x_0)| \leq c_n \int_{B(x_0,r)} \frac{|\nabla f(z)|}{|z - x_0|^{n-1}}.$$

Proof. The left hand side of the above we convert to polar coordinated

$$\frac{1}{a_n r^n}\int_B |f(y)-f(x)|dy = \frac{1}{a_n r^n}\int_0^r \rho^{n-1}\int_{S^{n-1}} |f(\rho,\theta)-f(x_0)|d\theta d\rho$$

where the volume of the n dimensional ball or radius r in $a_n r^n$ we let $y=(\rho,\theta)$ where $\theta\in S^{n-1}$ and $\rho\in[0,r]$. Since we have

$$|f(\rho,\theta)-f(x_0)| = |\int_0^\rho \nabla f(s,\theta)ds| \leq \int_0^\rho |\nabla f(s,\theta)|ds$$

we have

$$\begin{aligned}\frac{1}{a_n r^n}\int_0^r \rho^{n-1}\int_{S^{n-1}} |f(\rho,\theta)-f(x_0)|d\theta d\rho &\leq \frac{1}{a_n r^n}\int_0^r\int_0^\rho\int_{S^{n-1}} \frac{|\nabla f(s,\theta)|}{s^{n-1}} s^{n-1} d\theta ds d\rho \\ &= \frac{1}{a_n r^n}\int_0^r\int_{B(x_0,\rho)} \frac{|\nabla f(z)|}{|z-x_0|} dz d\rho \\ &= \frac{1}{n a_n}\int_0^r \frac{|\nabla f(z)|}{|z-x_0|^{n-1}} dz.\end{aligned}$$

as an application of Fubini's Theorem. □

Remark 1.1.4. The expression

$$\frac{|\nabla f(z)|}{|z-x_0|^{n-1}}$$

is called the **Reisz Potential** of $|\nabla f|$

1.2 Lecture 2

1.2.1

Last time we showed the Poincare Inequality. This Inequality still holds if we enlarge the Ball we are taking the integral over.

Proposition 1.2.1. *For $x\in B$ and $f\in L^1_{loc}(\Omega)$ we have*

$$⨍_B |f(x)-f(y)| \leq C\int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz \tag{1.2.1}$$

.

Consider the operator I_B which takes in functions $V:\mathbb{R}^n\to R$ and is defined by

$$I_B(V)(x) := \int_{4B} \frac{|V(z)|}{|z-x|^{n-1}} dz.$$

Proposition 1.2.2. *The operator I_B is a bounded operator $L^p(4B) \to L^p(4B)$.*

The strategy is to control this operator by the **Hardy-Littlewood maximal function**

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B f(y)dy$$

which is known to be a bounded function. The proof will allso use the Lebesgue differentiation theorem which states that

$$\frac{1}{|B|} f(y)dy \to f(x) \text{ as } |B| \to 0$$

where the balls are contain x and the limit exists almost everywhere.

Proof. Let $B_i = B(x, 2^{-i}r)$ for $i \geq 0$ then

$$\begin{aligned} I_B(V)(x) &= \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{g(z)}{|z-x|^{n-1}} dz \\ &\leq \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{g(z)}{(2^{-i}r)^{n-1}} dz \\ &\leq \sum_{i=0}^{\infty} \frac{1}{(2^{-i}r)^{n-1}} \int_{B_i} g(z)dz \\ &\leq \sum_{i=0}^{\infty} \frac{a_n 2/r}{|B_i|} \int_{B_i} g(z)dz \\ &\leq \sum (Mg)(x) \end{aligned}$$

then use the boundedness of the maximal function. □

$$|f(x) - \fint_{B(x,r)} f(y)dy \leq \fint_{B(x,r)} |f(x) - f(y)|dy$$

$$|f - f_B| \leq C \int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz$$

Let $g \in W^{1,p}$ and take $g_n \in \mathbb{C}^\infty(\mathbb{R}^n)$ such that $g_n \to g$ as $n \to \infty$ in the $W^{1,p}$ norm.

Proposition 1.2.3. *For $f \in W^{1,p}(\mathbb{R}^n)$ and B a ball containing x,*

$$|f(x) - f_B| \leq C \int_{4B} \int \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz.$$

Proposition 1.2.4. *Let $f \in W^{1,p}(\mathbb{R}^n)$. For almost every x and y in $\mathbb{R}^n$ we have*

$$|f(x) - f(y)| \leq Cd(x,y)\left[M|\nabla f|(x) + M|\nabla f|(y)\right]$$

Proof. Let $B = B(x, 2d)$ where $d = d(x,y)$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_B| + |f(y) - f_B| \\ &= \leq C\left(\int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}}dz + \int_{4B} \frac{|\nabla f(z)|}{|z-y|^{n-1}}dz\right) \\ &= C(M|\nabla f|(x) + M|\nabla f|(y)) \end{aligned}$$

□

Remark 1.2.5. If $\Omega = \mathbb{R}^n$ then we need $\frac{1}{8}\operatorname{dist}(x, \partial\Omega) = d(x,y)$

1.2.2

We have the following characterization for $f \in W^{1,p}(\mathbb{R}^n)$

Proposition 1.2.6. *Let $f \in L^p(\mathbb{R}^n)$ measurable. If there exists a $g \in L^p(\mathbb{R}^n)$ non-negative function such that*

$$|f(y) - f(x)| \leq d(x,y)\left[g(x) + g(y)\right]$$

then $f \in W^{1,p}(\mathbb{R}^n)$.

1.2.3

We will now use the result in the previous section to define a new space $M^{1,p}(X)$ where (X, μ, d) is a certain measure space with a metric. Here

- μ is Borel regular.
- All balls have positive measure.
- Bounded sets have finite measure.

Remark 1.2.7. Non σ-finite measures have stupid pieces of the measure space.

Definition 1.2.8. $M^{3p}(X)$ is called the **Hajtasz-Sobolev Space**. It consists of functions $f \in L^p(X)$ such that there exists some $g \geq 0$ which satisifies

$$|f(x) - f(y)| \leq d(x,y)\left[g(x) + g(y)\right].$$

1.2.4

$f : X \to \mathbb{R}$ is in $M^{1,p}$ if and only if f is in L^p and there exists some positive measure L^p such that

$$|f(x) - f(y)| \leq d(x,y)(g(x) + g(y)).$$

One can check easily that this defined a vectorspace. To do this suppose that For f_1 and f_2 in $M^{p,1}$ with corresponding g_1 and and g_2 then $f_1 + f_2$ has $g_1 + g_2$ as its L^p bounder. We have to show that g_1 and g_2 We can also make it a normed vector space with

$$\|f\| := \|f\|_{L^p} + \inf_{g \in S(f)} \|g\|_{L^p}$$

where $S(f)$ consists all of the possible positive, measurable $g \in L^p$ satisfying the inequality.

1.2.5

We claim that the vector space $M^{1,p}$ is complete. Suppose that f_k is a convergent sequence in $M^{1,p}$. We need to show that $f_k \to f$ such that there exists a g for the f with the appropriate bound. It is that f exists as a function in L^p since the $M^{p,1}$ norm dominates the L^p norm of the sequence of f_k's. What is not clear is that there exist some g which gives the

$$|f(x) - f(y)| \leq d(x,y)(g(x) + g(y)) \text{ a.e.}$$

This is what we will prove.

Since $\|f_{k+1} - f_k\|_{M^{1,p}}$ can find some $g_{k+1,k}$ with $\|g_{k+1,k}\|_{L^p} \to 0$ as $k \to \infty$. Note that for every n we can write f_{n+1} as a sum of functions

$$f_{n+1} = \sum_{k=1}^{n} (f_{k+1} - f_k) + f_1.$$

By the remark in section 7.2.4 the function $g_{n+1} := \sum_{k=1}^{n} g_{k+1,k} + g_1$ works as a sufficient bounder for the f_{n+1}'s. We can assume that g_{n+1} converges to some g in L^p by taking a subsequence if necessary to make the convergence more rapid.

For every n we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(y) - f(y)| + |f_n(x) - f_n(y)| \\
&\leq \varepsilon_n(x) + \varepsilon_n(y) + d(x,y)(g_n(x) + g_n(y)).
\end{aligned}$$

Which implies

$$\begin{aligned}|f(x)-f(y)| &\leq \inf_n \left(\varepsilon_n(x)+\varepsilon_n(y)+d(x,y)(g_n(x)+g_n(y))\right)\\ &\leq \inf_n \left(\varepsilon_n(x)+\varepsilon_n(y)\right)+d(x,y)\sup_n(g_n(x)+g_n(y))\\ &\leq d(x,y)\left(\sup_n g_n(x)-\sup_n g_n(y)\right)\end{aligned}$$

Since $g_n(x)$ is monotonically increasing with n we must have that $\sup_n g_n(x) = \sum_{n=1}^{\infty} g_{n+1,n}(x) + g_1(x)$. This limit exists in L^p which gives the g we were looking for.

1.3 Lecture 3

1.3.1

Let $f \in L^1(X)$ and $g : X \to [0,\infty]$ be a possible unbounded curve. Then for almost every rectifiable γ in X we have

1. $\int_\gamma g ds < \infty$
2. for all $\beta \subset \gamma$

$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta g ds$$

implies that f is absolutely continous on the curve γ with $|f \circ \gamma| \leq g \circ \gamma$ almost everywhere in γ.

1.3.2

Another way to generalize the Sobolev spaces in the metric space setting. For $f \in \mathbb{C}^\infty(X)$ and $g = |\nabla f|$ works to get the above conditions.

For $f \in W^{1,p}(X)$ we can find smooth f_k such that $\|f_k - f\|_{W^{1,p}(X)} \to 0$ as $k \to \infty$. For smooth functions we have

$$|f_k(x_\beta) - f_k(y_\beta)| \leq \int_\beta |\nabla f_k| ds$$

almost everywhere and f_k converges pointwise almost everywhere. Note that convergence could be bad at the endpoints of the curves β. We need to throw a small collection of curves to make the above statement make sense. This boils down to *computing capacities.*

1.3.3

μ is an **exterior measure** provided

- $T_1 \subset T_2$ implies that $\mu(T_1) \leq \mu(T_2)$
- $\mu(\emptyset) = 0$
- For a countable family $T_1, T_2, \dots$ we have
$$\mu(\bigcup_{i\geq 0} T_i) \leq \sum_{i\geq 0} \mu(T_i)$$

1.3.4

Let T be a family of curves in X. The family of admissible functions is

$$A(T) = \{g : X \to [0.\infty] | \forall\gamma \in T, \int_\gamma g ds \geq 1\}$$

Note that these functions detect the family T in the sense that $\int_\gamma g ds \geq 1$. For $p \in (0, 1)$, we define the p-**modulus** of the family T to be

$$\mu_p(T) = \int_{g\in A(T)} \int_X g(x)^p dx.$$

Lemma 1.3.1. *μ_p is an exterior measure on the collection of curves in X.*

In the above statment we are considering the collection of curves which are arclength parametrized and are of finite length.

- Prove lemma 7.3.1.

1.3.5

Consider a cylindrical family K of line segments in $\mathbb{R}^n$ where $A \subset \{x \in \mathbb{R}^n : x_n = 0\}$ and above every point in A there is a straight line starting at $x_n = 0$ going to $x_n = h$. The family is $C = K \times [0, h]$.

The set $A(T)$ is nonempty. The function

$$g(x_1, \dots, x_n) := \frac{1}{h}\chi_K(x_1, \dots, x_n)$$

gives us

$$\begin{aligned}
\mu_p(T) &\leq \int_K g^p dx \\
&= \frac{1}{h^p}\int_K dx \\
&= \frac{1}{h^p}{}_{[0,h]\times A} dydt \\
&= \frac{1}{h^p}\cdot |A| \cdot h
\end{aligned}$$

Let $\rho \in A(T)$. This means that $\int_{L_y} \rho ds \geq 1$ where L_y denotes the fiber above $y \in A$ in K. By Hölder's inequality this is less than or equal to

$$(\int_{L_y} \rho^p ds)^{1/p}(\int_{L_y} 1^{p'})^{1/p'}$$

which tells us that

$$1 \leq \left(\int_{L_y} \rho^p ds\right) h^{p-1}.$$

Computing further we have

$$\begin{aligned}
\int_K \rho^p dx &= \int_{A\times[0,h]} \rho^p dyds \\
&= \int_A \left(\int_0^h \rho^p ds\right) dy \\
&= (\rho^p ds)\,|A| \\
&\geq |A| h^{1-p}
\end{aligned}$$

1.3.6

Let $E \subset X$ and define the family

$$T_E^+ := \{\gamma : |\gamma^{-1}(E)| > 0\}.$$

Lemma 1.3.2. *If $E \subset X$ has measure zero them $\mu_p(T_E^+) = 0$.*

- Show that a function in admissible class is zero.
- If S has measure zero then there exists covers of S which shrink to zero.

Let $E_0 \subset X$ such that $E \subset E_0$ and define

$$g(x) = \infty \chi_{E_0}(x).$$

if $\gamma \in T_E^+$ then

$$\int_\gamma g ds = \infty$$

which means that $g \in A(T_E^+)$ but we have that

$$\int_X g^p dx = 0$$

because g is supported on a set of measure zero.

1.3.7

Lemma 1.3.3 (Fuglede's Lemma). *Let g_k be a sequence of non-negative borel measurable functions on X, $g : X \to (0, \infty]$ borel measurable with*

$$\|g - g_k\|_{L^p(X)} \leq 2^{-(k+1)p}.$$

The set of curves for which $\int_\gamma g_k$ does not approach $\int_\gamma g$ has μ_p measure zero.

We will actually be proving a stronger statement. The family

$$T = \{\gamma : \limsup_{k\to\infty} \int_\gamma |g_k - g| ds > 0\}$$

has $\mu_p(T) = 0$.

1.3.8

Lemma 1.3.4 (Koskela-MacManis). *Let T be a family of curves in X. $\mu_p(T) = 0$ if and only iff there exists a borel measurable $g \in L^p(X)$ such that for all curves γ we have*

$$\int_\gamma g ds = \infty$$

This is a generalization of the above.

1.3.9

Theorem 1.3.5. *$f \in W^{1,p}(X)$ if and only if*

1. *$f \in L^1(X)$*
2. *There exists some $g \in L^p(X)$ a non-negative borel measurable function such that the family of curves T where one of the conditions for absolute continuity fails has measure zero.*

Observe that this theorem fixes exactly what was wrong in the beginning of this lecture.

Proof. For the forward implication observe that $f \in W^{1,p}(X)$ implies that we can find f_k smooth where $f_k \to f$ as functions in $W^{1,p}(X)$ and $\|f_k - f\|_{W^{1,p}} \leq 2^{-(k+2)p}$. Let

$$E = \{x : \lim_k f_k(x) \neq f(x)\}.$$

This set has measure zero which implies that $\mu_p(T_E^+)$ has measure zero.

By Fugleday's lemma we have

$$\mu_p\{\gamma : \lim_{k\to\infty} \int_\gamma ||\nabla f_k| - |\nabla f||ds > 0\} = 0.$$

Observe that $|\nabla f| \in L^p(X)$ does not have to be Borel measurable bu can be modified on a set of measure zero that that it becomes measurable.

Finally, observe that the set of curves on which $\int_\gamma |\nabla f|ds = \infty$ has measure zero.

If γ is not a member of one of these three bad sets of measure zero we have

$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta |\nabla f_k|ds$$

for every subcurve β such that

$$\int_\beta |\nabla f_k|ds \to \int_\beta |\nabla f|ds.$$

The complement of any set of measure zero is dense so we can fine x_β and y_β not in E that give

$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta |\nabla f|ds.$$

We need to show that $E \cap \gamma = \emptyset$. Suppose that $x_0 \notin E$ but $x_0 \in \gamma$. We can choose x_0 as close to y as we want.

$$\lim_{k \to \infty} f_k(y) = f(y)$$

$$f_k(y) - f_k(x_0) = \int_{\beta_y} \nabla f_k ds$$

and since

$$\int_{\beta} |\nabla f_j| ds < \varepsilon$$

we get that

$$|f_k(y) - f_k(x_0)| < \epsilon$$

Use some triple triangle inequality to get what we want. □

The converse depends heavily on the coordinates

- The line segments parallel to the coordinate axes
- get partial derivatives

Remark 1.3.6. The family of integrals $\int_\gamma |\nabla f_j| ds$ are equicontinuous.

1.4 Lecture 4

1.4.1

In the previous section we gave a characterization of Sobolev functions. For X a nice enough metric space we had $f \in W^{1,p}(X)$ if and only if there existed some Borel measurable $g \in L^p(X)$ with a family T such that $\mu_p(T) = 0$ and for all $\gamma \notin T$

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds$$

And recall that by nice enough we means that (X, d, μ) was a triple consisting of a set a metric and a measure where the measure was Borel regular, bounded sets had finite measure and open sets had positive measure.

1.4.2

We define the collection $\widetilde{N}^{1,p}(X)$ to be the set of $f \in L^p(X)$ such that there exists a $g \in L^p(X)$ borel measurable and nonnegative and for almost every γ we have

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds.$$

- Verify that the definition of μ_p did not use any Euclidean structure.

1.4.3

Consider the functions

$$f(x,y) = 0 \text{ and } g(x,y) = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \end{cases}$$

Observe that since these two functions differ only on a set of measure zero we have $g \sim f$ as Sobolev functions.

1.4.4

Let $p \leq 2$ and let $T = \{$ curves going through x_0 $\}$

- Show that $\mu_p(T) = 0$.

  g as in definitino of $\widetilde{N}^{1,p}(X)$ is called a **weak upper gradient**.

  $$\|f\|_{\widetilde{N}^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)}.$$

- Show that $N^{1,p}(X)$ is a vector space.
- Show that $\|-\|_{N^{1,p}}$ is a seminorm (some nonzero elements may have $\|f\|_{N^{1,p}(X)} = 0$.
- Quotient out by the kernel of this semi-norm to get a norm

The **Newton-Sobolev** space is defined by

$$N^{1,p}(X) := \widetilde{N}^{1,p}(X)/\sim$$

where $f \sim g$ if and only if $\|f - g\|_{N^{1,p}} = 0$. The word Newton is because the Fundamental Theorem of Calculus applies.

1.4.5

Theorem 1.4.1. *If X is a subset on $\mathbb{R}^n$ with the induced metric and measure then*

$$N^{1,p}(X) \cong W^{1,p}(X)$$

as Banach Spaces.

Remark 1.4.2.

1. The vector spaces above are not equal as sets since they consist of different collections of equivalence classes.
2. Suppose that f_1 and f_2 have upper gradients g_1 and g_2. It is not the case that $g_1 - g_2$ is a upper gradients.

Proposition 1.4.3. *$N^{1,p}(X)$ is a Banach space.*

Let $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions in $N^{1,p}(X)$ such that

1. $f_k \to f$ in $L^p(X)$
2. f_k has p-weak upper gradients $g_k \in L^p(X)$ with $g_k \to g$ in $L^p(X)$.

Then there exists some $f_0 : X \to [-\infty, \infty]$ with $f_0 = f$ almost everywhere such that $f_0 \in N^{1,p}(X)$ and a Borel representative g_0 of g is a weak upper gradients of f_0.

By passing to a subsequence if necessary we have

$$\begin{aligned} \|f - f_k\|_{L^p(X)} &< 2^{-kp}, \\ \|g - g_k\|_{L^p(X)} &< 2^{-kp}. \end{aligned}$$

The set

$$A = \{x \in X : \lim_{k\to\infty} f_k(x) \neq f(x) \text{ or } \lim_{k\to\infty} g_k(x) \neq g(x)\}$$

has $\mu(A) = 0$. The implies that $mu_p(T_A^+) = 0$ since the set A has measure zero.

Also, the family

$$T = \{\gamma : \lim_{k\to\infty} \int_\gamma |g_k - g| ds$$

has $\mu_p(T) = 0$ by Fugleday's lemma.

By the K MacManus lemma we have

$$\mu_p\{\gamma : \int_\gamma g_k ds = \infty \text{ for some } k \text{ or } \int_\gamma g ds = \infty\} = 0.$$

Also for each $k \geq 1$ the family

$$T_k = \{\gamma : (f_k.g_k) \text{ do not satisfy } |f_k(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g_k ds\}$$

has $\mu_p(T_k) = 0$.

Now the union of these families

$$S = T_A \cup T \cup \bigcup_{k \geq 1} T_k$$

has $\mu_p(S) = 0$.

For $\gamma \notin S$, we have

$$|f_k(x_\gamma) - f_k(y_\gamma)| \leq \int_\gamma g ds$$

with

$$\int_\gamma g_k ds \to \int_\gamma g_k ds$$

and

$$|f_k(x_\gamma) - f_k(y_k)| \to |f(x_\gamma) - f(y_\gamma)|$$

by finding a dense set of points where this works and the triangle inequality.

1.4.6

$M^{1,p}(X)$ is the set of $f \in L^p(X)$ such that there exists some $g \geq 0$ in $L^p(X)$ with the property that for almost every $x, y \in C$ we have

$$|f(x) + f(y)| \leq d(x, y)\left[g(x) + g(Y)\right].$$

$N^{1,p}(X)$ is the set of $f \in L^p(X)$ such that there exists some $g \geq 0$ where

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds.$$

We have proved that

$$\begin{aligned} M^{1,p}(\mathbb{R}^n) &= W^{1,p}(\mathbb{R}^n) \text{ for } p¿1 \\ N^{1,p}(\Omega) &= W^{1,p}(\Omega) \end{aligned}$$

Proposition 1.4.4. *In general we do not have $M^{1,p}(X) = N^{1,p}(X)$.*

counter-example. Suppose that D is the unit disc in $\mathbb{R}^2$ and consider $X = D \setminus [0,1]$. We have

$$M^{1,p}(D \setminus [0,1]) = M^{1,p}(D)$$

since the bounding g in the definition does not see the deleted line segment. On the other hand

$$W^{1,p}(D \setminus [0,1]) \neq W^{1,p}(D)$$

since $\arg(z) \in W^{1,p}(D \setminus [0,1])$ but not in $W^{1,p}(D)$. □

Remark 1.4.5. The example $D \setminus [0,1]$ is called a **Sobolev extension domain**.

1.4.7

Proposition 1.4.6. $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$

Proof. Given $f \in M^{1,p}(X)$ we can modify it on a set of measure zero to get

$$|f(x) - f(y)| \leq d(x,y)|g(x) + g(y)|,$$

with $g \in L^p(X)$. By the Kaskela-MacManis theorem

$$\mu_p\{\gamma : \int_\gamma g ds = \infty\} = 0.$$

since the inequality fails on a set of measure zero the set of curves which see E as a large set has measure zero,

$$\mu_p(T_E^+) = 0.$$

Let γ be not one of the bad curves and partition it into small pieces γ_j each of length $l(\gamma)/N$ having some $z_j \in \gamma_j$ such that

$$g(z_j) \leq \fint_\gamma g ds.$$

$$\begin{aligned}
|f(z_1) - f(z_n)| &\leq \sum_i |f(z_i) - f(z_{i+1})| \\
&\leq \sum_i 2\frac{l(\gamma)}{N}[g(z_i) + g(z_{i+1}) \\
&\leq \sum_i 2\frac{l(\gamma)}{N}\left(\frac{1}{l(\gamma_i)}\int_{\gamma_i} g + \frac{1}{l(\gamma_{i+1})}\int_{\gamma_{i+1}} gds\right) \\
&\leq 2\sum_i 2\int_{\gamma_i} gds \\
&\leq 4\int_\gamma gds
\end{aligned}$$

If f is continuous then we have

$$|f(z_1) - f(z_n)| \to |f(x_\gamma) - f(y_\gamma)|.$$

This show s that

$$M^{1,p}(X) \cap \mathbb{C}(X) \hookrightarrow N^{1,p}(X)$$

is bounded. If we can show that $M^{1,p}(X) \cap \mathbb{C}(C)$ is dense in $M^{1,p}(X)$ then we are done.

We claim that the Lipschitz Functions are dense in $M^{1,p}(X)$.

□

- Show that the Lipschitz functions are dense
- $\mu\{g > \lambda\} \leq \frac{1}{\lambda^p}\|g\|_{L^p(X)}$
- $\mu\{g \leq \lambda \ni x, y$

1.4.8

Proposition 1.4.7. *Let $f \in W^{1,p}(\Omega)$ and $4B \subset \Omega$. Then for all $x \in B$ we have*

$$⨍_B |f(y) - f(x)|dy \leq C\int_{4B} \frac{|\nabla f(z)|}{|z - x|^{n-1}}dz.$$

Remark 1.4.8. The integral on the righthand side above is not a singular integral.

- Show that

$$\int_B \frac{1}{|z-x|^{n-1}} \approx r_B$$

$$\begin{aligned}
\fint_B \fint_B |f(y)-f(x)| dy dx &\leq \fint_B C \int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz dx \\
&= C \int_{4B} |\nabla f(z)| \fint_B \frac{1}{|z-x|^{n-1}} dx dz \\
&= C r_B \fint_{4B} |\nabla f(z)| dz
\end{aligned}$$

This give the 1-**Poincaré inequality**:

Proposition 1.4.9. *For $f \in W^{1,p}(\Omega)$, $4B \subset \Omega$ and $x \in B$ we have*

$$\fint_B \fint_B |f(x)-f(y)| dy dx \leq C r_B \fint_{4B} |\nabla f(z)| dz \tag{1.4.1}$$

Remark 1.4.10. Using Hölder's inequality on the right hand side we get the p-Poincaré inequality which is weaker.

1.5 Lecture 5

In this section we will compare for four different Sobolev spaces in the Metric Setting:

$$M^{1,p}(X), N^{1,p}(X), P^{1,p}(X), KS^{1,p}(X)$$

1.5.1

The p-**Poincaré Inequality** is satisfied on a space (X, d, μ) if there exists some $\lambda \geq 1$ and some positive constant C such that for all f

$$⨍_B |f - f_B| d\mu \leq C r_B (⨍_{\lambda B} |\nabla f|^p d\mu)^{1/p}.$$

Let $\Omega \subset \mathbb{R}^2$ be rectangle with a deleted center. Recall that balls are of the form $B \cap \Omega$ in this space. The λ is needed in this region to engulf the center.

1.5.2

Proposition 1.5.1. *For $X \subset \mathbb{R}^n$,*

$$\textit{Sobolev} \implies \textit{Haiwatch} .$$

The trick is to consider a telescoping series of balls (binoculars) at two different points.

Proposition 1.5.2. *If X supports a p-Poincare Inequality for upper gradient pairs and $p > 1$ we have*

$$⨍ |f - f_B| d\mu \leq C r_B (\int_{\lambda B} |g|^p d\mu)^{1/p}$$

then $M^{1,p}(X) = N^{1,p}(X)$

Suppose that $\mu(x, 2r) \leq C\mu(B(x, r))$

Remark 1.5.3. The property of **asymptotic doubling measures** are enough to show the boundedness of Maximal functions.

$$\lim_{r \to 0} \frac{B(x, r)}{B(x, 2r)} = C(x)$$

exists and is bounded for all x.

Proof. Let $f \in N^{1,p}(X)$. Let E be the set of non-Lesbegue points (where the Mean Values on balls don't converge to the value at the point). By the Lebesgue differentiation theorem $\mu(E) = 0$.

Let $x, y \in X \setminus E$, let $d = d(x, y)$ and define

$$B_i = \begin{cases} B(x, 2^{i-1}d), & i \geq 0 \\ B(y, 2^i d), & i < 0 \end{cases}.$$

Notice that

$$\lim_{i \to \infty} f_{B_i} = f(x) \text{ and } \lim_{i \to -\infty} f_{B_i} = f(y).$$

By a triangle inequality we have

$$|f(x) - f(y)| \leq | \sum_{i=-\infty}^{\infty} |f_{B_i} - f_{B_{i+1}}| = *$$

and since $B_i, B_{i+1} \subset 2B_{i+1}$ we have

$$* \leq \sum |f_{B_i} - f_{2B_{i+1}}| + |f_{2B_{i+1}} - f_{B_{i+1}}|$$

and

$$|f_{B_i} - f_{B_{i+1}}| \leq \int_{B_i} |f - f_{B_{i+1}}| d\mu$$

gives us

$$* \leq 2C \sum_{i=-\infty}^{\infty} \int |f - f_{2B_{i+1}}| d\mu$$

□

Remark 1.5.4. Keith-Zhang have proven for X a doubling space with a p Poincare inequality that there exists some $q \in (1, p)$ where X has a q Poincare inequality. They require the metrix space to be complete.

$$\begin{aligned} |f(x) - f(y)| &\leq C \sum_{i=-\infty}^{\infty} r_{2B_{i+1}} (\fint_{2\lambda B_{i+1}} q^q d\mu)^{1/q} \\ &\leq C \left(\sum_{i \geq 0} (2^{1-i} d M(g^q)(x)^{1/q} + \sum_{i<0} 2^{i+1} M(g^q)(x)^{1/q} \right) \\ &\leq Cd \left(M(g^q)(x)^{1/q} + M(g)(x)^{1/q} \right) \end{aligned}$$

The Hajtase quotient is $M(g^q)^{1/q} \in L^p(X)$

1.5.3

A topological space X is **quasi-convex** if for every $x, y \in X$ there exist some path γ with endpoints x and y such that $l(\gamma) < \infty$ and

$$l(\gamma) \leq Cd(x, y)$$

Remark 1.5.5. Observe that by the triangle inequality $l(\gamma)Cd(x, y)$.

The domain enclosed by the Cardiod with the subspace metric and measure in $\mathbb{R}^2$ is not quasi-convex since there are points arbitrarily close to each other whose minimal path has a positive length.

1.5.4

Proposition 1.5.6. *If X supports a p-Poincaré inequality for (f, g) pairs then X is quasi-convex.*

1.5.5

The $P^{1,p}$-space is the collection of $f \in L^p(X)$ for which there exists a non-negative $g \in L^p(X)$ with the property that for all ball $B \subset X$ we have

$$\fint_B |f - f_B| d\mu \leq r_B \left(\int_{\lambda B} g^p d\mu\right)^{1/p}$$

We call this space the $(1, p)$ **Poincaré space**.

Remark 1.5.7. The space $M^{1,p}(X)$ requires

$$|f(x) - f(y)| \leq d(x, y) \left[g(x) + g(y)\right].$$

It does not have the property that if a function is contains along an open set that "its derivative" g must be zero. We say that it fails to have **strong locality** .

The space $N^{1,p}(X)$ requires

$$|f(x) - f(y)| \leq \int_\gamma g ds.$$

this space *does* have the strong locality property.

The space $P^{1,p}(X)$ is somewhere in between. It has **locality**.

1.5.6

Proposition 1.5.8. *If X supports a p-Poincaré inequality then $P^{1,p}(X) = N^{1,p}(X)$*

Proof. We will prove that $P^{1,p}(X) \subset N^{1,p}(X)$ the other direction is left as an exercise. Let $f \in P^{1,p}(X)$. After modifying on a set of measure zero we will get something in the Newton-Sobolev space. We will be using that $N^{1,p}(X)$ is close to a Banach space in the sense that there exists some $f_k \to f$ which $f_k \in P^{1,p}(X)$.

Fix some $\varepsilon > 0$ we can find an ε-**net** which consists of a sequence of points $\{x_i\}_{i\in I}$ such that for all $x \in X$ there exists some $i \in I$ such that $d(x, x_i) < 2\varepsilon$ and the balls $B_j = B(x_j, 2\varepsilon)$ have small overlap:

$$\sum_i \chi_{10B_i} \leq C.$$

Form a Lipschitz partition of unity subordinate to the cover. There are functions φ_i which are $\frac{C}{\epsilon}$-Lipschitz where $\varphi_i : X \to [0,1]$ have $\mathrm{supp}(\varphi_i) \subset 2B_i$ and

$$\sum_i \varphi_i = 1.$$

We define

$$f_\varepsilon(x) = \sum_i f_{B_i}\varphi_i.$$

For each epslion the function is locally Lipschitz and the function

$$\mathrm{Lip}h(x) := \limsup_{y\to x, y\neq x} \frac{|h(x) - h(y)|}{d(x,y)}$$

is an uppergradient for h.

1. $|f_\varepsilon - f| \to 0$ as $\varepsilon \to 0$.
2. $\{\mathrm{Lip}f_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^p(X)$.
3. Since $p > 1$, L^p is reflexive.
4. Look at a convex combination of the f_ε to

□

Remark 1.5.9. The spaces $M^{1,p}$ and $N^{1,p}$ are used in potential theory. The space $P^{1,p}$ is not used so much.

1.5.7

The **Korevaar-Schoen Space** $KS^{1,p}(X)$ consists of $f \in L^p(X)$ such that

$$\int_X \left(\fint_{B(x,\varepsilon)} \left| \frac{f(x)-f(y)}{\varepsilon} \right|^p d\mu(y) \right) d\mu(x) < \infty$$

The inside can be viewed as an integral average of the derivative or the integral average of energy.

Remark 1.5.10. The above space is used (if one looks at the papers closely) in Diffusion processes on Fractals. See papers by Kigami, Kumagai and Strichartz.

Observe that

$$\int_{B(x,\varepsilon)} \left| \frac{f(x)-f(y)}{\varepsilon} \right|^p d\mu(y) \le \frac{|f(x)-f_B|^p}{\varepsilon^p} + \frac{1}{\varepsilon^p} \int_B |f(y)-f_{B(x,\varepsilon)}|^p d\mu(y)$$

by the triangle inequality.

- Show that all the spaces $N^{1,p}(X), M^{1,p}(X), P^{1,p}(X), KS^{1,p}(X)$ are the same if X supports a Poincare inequality for upper gradient pairs.

Chapter 2

Rafa

Introduction

These notes are intended as reading material for the series of lectures that will be delivered by the author as part of the *Unit 2: Analysis on metric spaces* of the DocCourse IMUS 2011: *Harmonic Analysis, Metric Spaces and Applications to P.D.E.s*, to be held in Seville, Spain, May 15 till July 15, 2011.

These notes provide a first step into some of the main techniques to develop analysis on metric spaces and δ-hyperbolicity. Our lectures are intended for beginning graduate students with a basic knowledge of real and complex analysis, measure theory and functional analysis. Some of the topics developed in these notes have been specifically chosen as basics for other courses included in the Unit 2 of this summer school. Our course will be given in 7 sessions with the following contents:

- *General measure theory and main covering theorems.* Brief overview of most fundamental elements from measure theory. We will describe and prove some of the main covering theorems.

- *Differentiation of Radon measures and Lebesgue differentiation.* We will apply the covering theorems to obtain results on differentiation of Radon measures as well as Lebesgue differentiation results. We will see these results for $\mathbb{R}^n$ with the n-dimensional Lebesgue measure as well as for more general measure spaces. We will also consider the maximal Hardy-Littlewood operator, which plays a very important role in Harmonic Analysis, to obtain these results.

- *Hausdorff measures and elementary property of functions.* We will revise some of the basic results and definitions on Hausdorff measures and how they interplay with certain functions.

- *Lipschitz functions and Rademacher theorem.* Here we will focus on Lipschitz functions and the very famous and important Rademacher theorem on differentiation of Lipschitz functions.
- *Linear maps, Jacobian and the area formula.* The aim of this session will be to reach to the area formula.
- *Geodesic spaces and δ-hyperbolicity.* We focus at this point on geodesic spaces and the notion of geodesic space of uniform nonpositive curvature (so-called CAT(0) spaces) and δ-hyperbolicity given by Gromov.
- *Stability of geodesics.* In our last section we continue with the study of geodesic δ-hyperbolic spaces to finish with the result on stability of geodesic in these spaces.

The material of this course can be found in many text books. Those used for these notes were [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]. There are many results which have been stated as exercises and may be found in some of these books. Working these exercises will help for a deeper and better understanding of the contents of this course.

2.1 General Measure Theory. A review.

2.1.1 Basic facts on measures

Let X denote a set and 2^X the collection of its subsets.

Definition 2.1.1. *A mapping $\mu\colon 2^X \to [0,\infty]$ is called a measure on X if*

1. $\mu(\emptyset) = 0$, *and*
2. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subseteq \cup_{k=1}^{\infty} A_k$.

The above definition corresponds in many texts to the notion of *outer measure.* Notice also that if μ is a measure on X and $A \subseteq B \subseteq X$ then

$$\mu(A) \leq \mu(B).$$

Given a measure μ on X there will be a special class of subsets of X with good properties with respect to this measure, these are the μ-measurable sets.

Definition 2.1.2. *A set $A \subseteq X$ is μ-measurable if for each $B \subseteq X$,*

$$\mu(B) = \mu(B \cap A) + \mu(B - A).$$

Next we give some immediate properties.

Exercise 2.1.3. *Let μ be a measure on X and $A \subseteq X$, then*

1. *If $\mu(A) = 0$ then A is μ-measurable.*
2. *If A is μ-measurable and $B \subseteq X$, then*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

3. *If A is μ-measurable then $X \setminus A$ is μ-measurable.*

Definition 2.1.4. *Let X be a nonempty set, then $\mathcal{M} \subseteq 2^X$ is a σ-algebra on X if*

1. *$\emptyset \in \mathcal{M}$ and $X \in \mathcal{M}$,*
2. *$\mathcal{M}$ is closed under countable unions, and*
3. *$\mathcal{M}$ is closed under taking complements in X.*

After Proposition 2.1.3 the class of μ-measurable sets of X is a σ-algebra.

Definition 2.1.5. *A subset $A \subseteq X$ is σ-finite with respect to μ if $A = \cup_{k=1}^{\infty} B_k$, where B_k is μ-measurable and $\mu(B_k) < \infty$ for each k.*

Definition 2.1.6. *The Borel σ-algebra of a topological space X is the smallest σ-algebra of X containing the open subsets of X.*

Next we give some properties of sequences of measurable sets.

Exercise 2.1.7. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ-measurable sets.*

1. *The sets $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$ are μ-measurable.*
2. *If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint then*

$$\mu\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

3. *If the sequence is increasing with respect to the set inclusion, then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

4. *If the sequence is decreasing with respect to the set inclusion and $\mu(A_1) < \infty$, then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right).$$

The notions given next define different classes of measures which will be useful given to their particular structural properties.

Definition 2.1.8. 1. *A measure μ on X is regular if for each set $A \subseteq X$ there exists a μ-measurable set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.*

2. *A measure μ on a topological space is called Borel if every Borel set is μ-measurable.*

3. *A measure μ on a topological space X is Borel regular if μ is Borel and for each $A \subseteq X$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.*

Theorem 2.1.9. *Let μ be a Borel measure on a metric space X. Let B be a Borel set. Then,*

1. *If $\mu(B) < \infty$ and $\varepsilon > 0$ then B contains a closed set C such that $\mu(B \setminus C) < \varepsilon$.*

2. *If B is contained in the union of countably many open sets V_i with $\mu(V_i) < \infty$ and given $\varepsilon > 0$, then B is contained in an open set W for which $\mu(W \setminus B) < \varepsilon$.*

Exercise 2.1.10. *Show that if μ is a Borel regular measure on a metric space and A is a μ-measurable set with finite measure then there exist Borel sets B and D with*

$$D \subseteq A \subseteq B \text{ and } \mu(B \setminus D) = 0.$$

Theorem 2.1.11. *If μ is a Borel regular measure on a metric space then Theorem 2.1.9 holds for any μ-measurable set.*

Under regularity of the measure we have results as the next one.

Theorem 2.1.12. *Let μ be a regular measure on X. If $A_1 \subseteq A_2 \subseteq \cdots A_k \subseteq A_{k+1} \cdots$, then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right),$$

where the considered sets are not supposed to be measurable.

A lot of work in measure theory is developed on a more restrictive class of measures than that of Borel measures. We define it next.

Definition 2.1.13. *Suppose that μ is a measure on a proper topological space X. We say that μ is a Radon measure if the following conditions hold:*

1. *Every compact set in X has finite measure.*
2. *Every open set is μ-measurable and if $V \subseteq X$ is open then*

$$\mu(V) = \sup\{\mu(K) \colon K \text{ is compact and } K \subseteq V\}.$$

3. *For every $A \subseteq X$,*

$$\mu(A) = \inf\{\mu(V) \colon V \text{ is open and } A \subseteq V\}.$$

Exercise 2.1.14. *Radon measures on $\mathbb{R}^n$ are defined in [**?**] as measures μ which are is Borel regular and such that $\mu(K) < \infty$ for each compact set $K \subseteq \mathbb{R}^n$. Then this definition is shown to imply Definition 2.1.13. Find which properties from $\mathbb{R}^n$ are used to obtain this result and state it in a more general context than $\mathbb{R}^n$ if possible.*

The alternative definition of Radon measure given by the above exercise is useful, for instance, to prove the next result.

Theorem 2.1.15. *Let μ be a Borel regular measure on $\mathbb{R}^n$. Suppose that $A \subseteq X$ is μ-measurable and $\mu(A) < \infty$. Then μ restricted to A (that is, $\mu_{|A}(B) = \mu(A \cap B)$) is a Radon measure.*

Determining if a given measure is a Borel measure is not an easy task in general. However, if the space X is a metric space then there is a tool to determine whether a certain measure is Borel or not. Let X be a metric space and $A, B \subseteq X$ then we denote

$$\operatorname{dist}(A, B) = \inf\{d(a, b) \colon a \in A, b \in B\}.$$

Theorem 2.1.16 (Caratheodory criterion)**.** *Suppose μ is a measure on a metric space X. All open subsets of X are μ-measurable if and only if*

$$\mu(A) + \mu(B) \leq \mu(A \cup B)$$

holds whenever $A, B \subseteq X$ with $0 < \operatorname{dist}(A, B)$.

2.1.2 Lebesgue measure on $\mathbb{R}$

The Lebesgue measure on $\mathbb{R}$ can be understood as an extension of the notion of length of an interval and it is one of the most natural and useful non-trivial measures we may think of.

Definition 2.1.17. *For $A \subseteq \mathbb{R}$, the (one-dimensional) Lebesgue measure of A is denoted by $\mathcal{L}^1(A)$. Denoting $\mathcal{I}$ as a family of bounded open intervals, $\mathcal{L}^1(A)$ is defined by*

$$\inf\left\{\sum_{I\in\mathcal{I}} \operatorname{length}(I) \colon A \subseteq \bigcup_{I\in\mathcal{I}} I\right\}.$$

It is easy to apply Caratheodory's criterion to show that $\mathcal{L}^1$ is a Borel measure.

Theorem 2.1.18. *The Lebesgue measure of the closed bounded interval $[a,b]$ is equal to $b-a$. Moreover, it can be shown that the Lebesgue measure is the only translation-invariant measure on $\mathbb{R}$ giving measure 1 to the unit interval.*

The next example is standard in the theory and shows that not every set is $\mathcal{L}^1$-measurable.

Example 2.1.19. *Let $\mathbb{Q}$ denote the rational numbers. Notice that for each $a \in \mathbb{R}$ the set X_a defined by*

$$X_a = \{a + q \colon q \in \mathbb{Q}\}$$

intersects the unit interval $[0,1]$. If $a_1 - a_2$ is a rational number then $X_{a_1} = X_{a_2}$ and also the converse is true. By axiom of choice there exists a set C such that

$$C \cap [0,1] \cap X_a$$

has exactly one element for every $a \in \mathbb{R}$. Therefore the sets

$$C - q = \{c - q \colon c \in C\},$$

for $q \in [0,1] \cap \mathbb{Q}$, must be pairwise disjoint. Since $\mathcal{L}^1$ is translation-invariant, all sets $C - q$ must measure the same and if one is measurable then all of them are.

Now, if $t \in [0,1]$ then there exists $c \in [0,1] \cap X_t$, that is, $c = t + q$ with $q \in \mathbb{Q}$. Equivalently, we can write $q = c - t$, so we see that $-1 \leq q \leq 1$ and $t \in C - q$. Thus we have

$$[0,1] \subseteq \bigcup_{q\in[-1,1]\cap\mathbb{Q}} (C - q) \subseteq [-1,2]$$

where the sets in the union are all pairwise disjoint.

If C is $\mathcal{L}^1$-measurable, then the above left-hand contention would tell us that $\mathcal{L}^1(C) > 0$, while the right-hand term would tell us that $\mathcal{L}^1(C) = 0$. Therefore C cannot be $\mathcal{L}^1$-measurable.

It can also be shown that if μ is a Borel regular measure on a complete, separable metric space such that there are sets with positive and finite measure and with the property that no singleton has positive measure, then there must exist a set that is not μ-measurable (see [**?**, 2.2.4]). Finally, let us remark that the construction of non-measurable sets requires the use of the axiom of choice.

2.2 Integration

2.2.1 Measurable functions

We start with the definition of measurable function. Let X be a set and Y a topological space. Assume μ is a measure on X.

Definition 2.2.1. *A function $f: X \to Y$ is called μ-measurable if for each open $U \subseteq Y$, $f^{-1}(U)$ is μ-measurable.*

A function f is σ-finite with respect to μ if f is μ-measurable and $\{x: f(x) \neq 0\}$ is σ-finite with respect to μ.

Proposition 2.2.2 (Properties of measurable functions)**.**

1. *If $f, g : X \to \mathbb{R}$ are μ-measurable, then so are $f+g$, fg, $|f|$, $min(f,g)$, and $amx(f,g)$. The function f/g is measurable provided g is not 0 on X.*

2. *If the functions $f_k: X \to [-\infty, \infty]$ are μ-measurable ($k = 1, 2, \cdots$), then $\inf_{k\geq 1} f_k$, $\sup_{k\geq 1} f_k$, $\liminf_{k\to\infty} f_k$, and $\limsup_{k\to\infty} f_k$ are μ-measurable too.*

Theorem 2.2.3. *Let $f: X \to [0, \infty]$ be μ-measurable. Then there exist μ-measurable sets $\{A_k\}_{k=1}^{\infty}$ in X such that*

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}.$$

Next we state two very important results in measure theory which gives regularity properties of measurable functions.

Theorem 2.2.4 (Egorov's theorem)**.** *Let μ be a measure on X and let $f_1, f_2, \cdots$ with values in a separable metric space, μ-measurable functions. If $A \subseteq X$ with finite measure, $\lim_{n\to\infty} f_n(x) = g(x)$ exists for μ-almost every x in A, and given $\varepsilon > 0$, then there exists a μ-measurable set B with $\mu(A \setminus B) < \varepsilon$ such that f_n converges uniformly to g on B.*

Theorem 2.2.5 (Lusin's theorem)**.** *Let X be a metric space and μ a Borel regular measure on X. If f with values on a separable metric space is μ-measurable, $A \subseteq X$ is μ-measurable with finite measure and $\varepsilon > 0$, then there exists a closed set $C \subseteq A$, with $\mu(A \setminus C) < \varepsilon$ such that f is continuous on C.*

2.2.2 Integrals and limit theorems

For a real function f we denote $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Notice that $f = f^+ - f^-$. Let μ be a measure on a set X. The following definitions may vary depending of different textbooks.

Definition 2.2.6. *A function $f\colon X \to [-\infty, \infty]$ is called a simple function if the image of f is countable.*

Definition 2.2.7. *If f is nonnegative, simple and μ-measurable, we define*

$$\int f d\mu = \sum_{0 \leq y \leq \infty} y\mu(f^{-1}\{y\}).$$

Definition 2.2.8. *If f is simple μ-measurable and either $\displaystyle\int f^+ d\mu < \infty$ or $\displaystyle\int f^- d\mu < \infty$, then*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

Definition 2.2.9. *If $f\colon X \to [-\infty, \infty]$ is μ-measurable then its integral with respect to μ is defined as follows:*

1. *If f is nonnegative then*

$$\int f d\mu = \sup\left\{\int h d\mu\colon 0 \leq h \leq f,\ h \text{ simple, } \mu\text{-measurable}\right\}.$$

2. *In case at least one of $\displaystyle\int f^+ d\mu$ and $\displaystyle\int f^- d\mu$ is finite, we set*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

Definition 2.2.10.

1. *To integrate f over a subset A of X, we multiply f by the characteristic function of A, that is,*

$$\int_A f d\mu = \int f \cdot \chi_A d\mu.$$

2. *The definition of $\displaystyle\int f d\mu$ extends to complex-values and $\mathbb{R}^n$-valued functions by separating f into real and imaginary parts or components, respectively.*

3. *If $\displaystyle\int |f| d\mu$ is finite the we say that f is μ-integrable.*

The following basic facts hold for integration of nonnegative functions.

Theorem 2.2.11. *Let μ be a measure on the nonempty set X. Suppose that $f, g\colon X \to [0, \infty]$ are μ-measurable.*

1. *If $A \subseteq X$ is μ-measurable and $f(x) = 0$ holds for μ-almost all $x \in A$, then*

$$\int_A f d\mu = 0.$$

2. *If $A \subseteq X$ is μ-measurable and $\mu(A) = 0$, then*

$$\int_A f d\mu = 0.$$

3. *If $0 \leq c < \infty$, then*

$$\int (c \cdot f) d\mu = d \int f d\mu.$$

4. *If $f \leq g$, then*

$$\int f d\mu \leq \int g d\mu.$$

5. *If $A \subseteq B \subseteq X$ are μ-measurable then*

$$\int_A f d\mu \leq \int_B f d\mu.$$

Lemma 2.2.12. *Let μ be a measure on the nonempty set X. If $f\colon X \to [0, \infty]$ is μ-measurable and $0 \leq h_1 \leq h_2 \leq \cdots \leq f$ is a sequence of simple, μ-integrable functions with $\lim_{n\to\infty} h_n = f$, then*

$$\lim_{n\to\infty} \int h_n d\mu = \int f d\mu.$$

Notice that from Theorem 2.2.3 such a sequence of simple functions always exists. The next theorem follows as a consequence of the previous lemma and has very strong consequences from a structural point of view.

Theorem 2.2.13. *The μ-integrable functions form a vector space and the μ-integral is a linear functional on the space of μ-integrable functions.*

Next we state some of the most fundamental results in integration theory: Fatou's lemma and the monotone and dominated convergence theorems of Lebesgue.

Theorem 2.2.14. *Let μ be a measure on the nonempty set X. Then,*

Fatou's lemma *If $f_1, f_2, \cdots$ are nonnegative μ-measurable functions, then*

$$\liminf_{n\to\infty} \int_X f_n d\mu \geq \int_X \liminf_{n\to\infty} f_n d\mu.$$

Lebesgue's monotone convergence theorem *If $f_1 \leq f_2 \leq \cdots$ are nonnegative μ-measurable functions, then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

Lebesgue's dominated convergence theorem *If $f_1, f_2, \cdots$ are complex-valued μ-measurable functions that converge μ almost everywhere to f and such that there exists a nonnegative μ-measurable function g such that*

$$\sup_n |f_n(x)| \leq g(x) \text{ and } \int_X g d\mu < \infty,$$

then

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0 \text{ and } \lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Exercise 2.2.15. *Let f and $\{f_n\}$ be μ-integrable real functions such that*

$$\lim_{k\to\infty} \int |f_k - f| d\mu = 0.$$

Then there exists a subsequence $\{f_{k_j}\}$ such that

$$f_{k_j} \to f$$

μ almost everywhere.

2.2.3 Lebesgue measure in $\mathbb{R}^n$

We have already defined the Lebesgue measure on the real line. It is also possible to define a similar measure which extends the notion of the volume of an n-dimensional ball. One of the easiest way to construct such measures is following an inductive procedure through products of 1-dimensional Lebesgue measures.

Let X and Y be sets.

Definition 2.2.16 (Product measure)**.** *Let μ be a measure on X and ν a measure on Y. We define the measure $\mu \times \nu \colon 2^{X\times Y} \to [0, \infty]$ by setting for each $S \subseteq X \times Y$:*

$$(\mu \times \nu)(S) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) \right\},$$

where the infimum is taken over all collections of μ-measurable sets $A_i \subseteq X$ and ν-measurable sets $B_i \subseteq Y$ $(i = 1, 2, \cdots)$ such that

$$S \subseteq \cup_{i=1}^{\infty} (A_i \times B_i).$$

The measure $\mu \times \nu$ is called the product measure of μ and ν.

Definition 2.2.17. *The n-dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is given, inductively, by*

$$\mathcal{L}^n = \mathcal{L}^{n-1} \times \mathcal{L} = \mathcal{L}^1 \times \cdots \times \mathcal{L}^1 \ (n \text{ times}).$$

2.3 Covering theorems

Covering theorems are fundamental to obtain results on differentiation of integrals. The main idea is to have certain tools which allow us to take a subcovering from a certain covering which is as disjoint as possible. Usually these coverings are made up of nice sets as, for instance, balls in a metric space. We will typically consider a metric space with a measure μ, so, if $\mathcal{F} = \{B\}$ is a covering of a set A by balls, then

$$\mu(A) \leq \sum_{\mathcal{F}} \mu(B).$$

However what we would like to have is something like

$$\mu(A) \geq C \sum_{\mathcal{F}'} \mu(B)$$

for a certain a subcollection $\mathcal{F}' \subseteq \mathcal{F}$ which still covers A and some positive constant C which is independent from the covering and A. The variety of these results and its different versions is large in the literature. We present next some of them.

Definition 2.3.1.

1. *A collection $\mathcal{F}$ of closed balls in a metric space X is a cover of $A \subseteq X$ if*

$$A \subseteq \bigcup_{B \in \mathcal{F}} B.$$

2. *$\mathcal{F}$ is a fine cover of A if, in addition,*

$$\inf\{\operatorname{diam} B \colon x \in B, B \in \mathcal{F}\} = 0$$

for each $x \in A$.

3. *The cover $\mathcal{F}$ is said to be of uniformly bounded diameter if*

$$\sup\{\operatorname{diam}(B) \colon B \in \mathcal{F}\} < \infty.$$

Exercise 2.3.2. *Show that any fine covering admits a uniformly bounded diameter sub-covering.*

We give first the Wiener covering lemma.

Lemma 2.3.3 (Wiener's lemma)**.** *Let $K \subseteq X$ be a compact set with a covering $\mathcal{F} = \{B_\alpha\}_{\alpha\in A}$, $B_\alpha = B(c_\alpha, r_\alpha)$, by open balls. Then there is a subcollection $B_{\alpha_1}, B_{\alpha_2}, \cdots, B_{\alpha_n}$, consisting of pairwise disjoint balls, such that*

$$\bigcup_{j=1}^{m} B(c_{c_{\alpha_j}}, 3r_{\alpha_j}) \supseteq K.$$

Wiener's lemma is the constructive version of the next result, which in many textbooks is presented as the *basic covering theorem.* Remember that Zorn's lemma claims that *if every chain in a nonempty partially ordered set has an upper bound, then the partially ordered set has a maximal element, that is, an element which is not strictly smaller than any other.*

Theorem 2.3.4 (Basic covering theorem)**.** *Every family $\mathcal{F}$ of closed balls of uniformly bounded diameter in a metric space X contains a disjoint subfamily $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} 5B,$$

where $5B$ stands for the ball of center the same as B and radius five times that of B. Moreover, every ball B from $\mathcal{F}$ meets a ball B' from $\mathcal{G}$ with radius at least half that of B, so, in particular, $B \subseteq 5B'$.

Remark 2.3.5. *In most applications the family $\mathcal{G}$ will be countable.*

Exercise 2.3.6. *Why do we enlarge balls by 3 in Wiener's lemma and by 5 in the basic covering theorem? Is the proof of the basic covering theorem working with a 3 instead than 5? How close can we get to 3?*

Corollary 2.3.7. *Assume that $\mathcal{F}$ is a fine cover of A by closed balls of uniformly bounded diameter. Then there exists a family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, \cdots, B_m\} \subseteq \mathcal{F}$, we have*

$$A \setminus \bigcup_{k=1}^{m} B_k \subseteq \bigcup_{B\in\mathcal{G}\setminus\{B_1,B_2,\cdots,B_m\}} 5B.$$

Corollary 2.3.8 (Vitali covering theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be open, $\delta > 0$. There exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that* $\operatorname{diam} B \leq \delta$ *for all $B \in \mathcal{G}$ and*

$$\mathcal{L}^n\left(U \setminus \bigcup_{B\in\mathcal{G}} B\right) = 0.$$

The same corollary can be obtained with basically the same proof in more general contexts. Doubling space is a main notion in analysis on metric spaces.

Definition 2.3.9 (Doubling space)**.** *A measure μ in a metric space is called doubling if non-degenerate balls have finite and positive measure and there is a constant $C(\mu) \geq 1$ such that*

$$\mu(2B) \leq C(\mu)\mu(B)$$

for all balls B. The metric measure space (X, μ) is then called doubling.

The next result provides an useful characterization of doubling spaces.

Theorem 2.3.10. *A measure μ with finite and positive values on non-degenerate balls is doubling if and only if there exist constants $C' > 0$ and s such that*

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C' \left(\frac{r}{R}\right)^s$$

for all $x, y \in E$ and $R \geq r > 0$ such that $x \in B(y, R)$.

Theorem 2.3.11 (Vitali covering theorem in doubling spaces)**.** *Let A be a bounded subset in a doubling metric measure space (X, μ) and let $\mathcal{F}$ be a fine collection of closed balls centered at A. Then, there is a countable disjointed subfamily $\mathcal{G}$ of $\mathcal{F}$ such that*

$$\mu\left(A \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$

Exercise 2.3.12. *Show that the same result is true for A not necessarily bounded.*

As we will see, Vitali's covering theorem is very useful to study the Lebesgue differentiability of locally integrable functions. However, if we want to study the differentiability of Radon measures then we find the problem in the above covering results that balls are enlarged by a factor. Enlarging balls should not be a problem if the measure is doubling, but not all Radon measures are doubling. Next we state the fundamental Besicovitch's covering theorem where such enlargements are not required.

Theorem 2.3.13 (Besicovitch covering theorem)**.** *There exists a constant N_n, depending only on n, with the following property: if $\mathcal{F}$ is any collection of nondegenerate closed balls of uniformly bounded diameter in $\mathbb{R}^n$ and if A is the set of centers of balls in $\mathcal{F}$, then there exist $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{N_n} \subseteq \mathcal{F}$ such that each $\mathcal{G}_i$ is a countable collection of disjoint balls in $\mathcal{F}$ and*

$$A \subseteq \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}} B.$$

This theorem is very particular of $\mathbb{R}^n$ and hard to extend to more general context. However an extension to a more metric context can be found in [**?**, p. 7].

Corollary 2.3.14. *Let μ be a Borel measure on $\mathbb{R}^n$, and $\mathcal{F}$ any collection of non-degenerate closed balls. Let A be the set of centers of the balls in $\mathcal{F}$. Assume $\mu(A) < \infty$ and that $\mathcal{F}$ is a fine covering of A. Then for each open set $U \subseteq \mathbb{R}^n$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that*

$$\bigcup_{B \in \mathcal{G}} B \subseteq U$$

and

$$\mu\left((A \cap U) \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$

2.4 Differentiation of Radon measures

We start defining the notions of superior and inferior derivatives of Radon measures in $\mathbb{R}^n$. Let μ and ν be two Radon measures in $\mathbb{R}^n$

Definition 2.4.1. *For each point $x \in \mathbb{R}^n$, define*

$$\overline{D}_\mu \nu(x) = \limsup_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \text{ if } \mu(B(x,r)) > 0 \text{ for all } r > 0,$$

or $+\infty$ otherwise. And,

$$\underline{D}_\mu \nu(x) = \liminf_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \text{ if } \mu(B(x,r)) > 0 \text{ for all } r > 0,$$

or $+\infty$ otherwise.

Definition 2.4.2. *If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$, we say that ν is differentiable with respect to μ at x and write*

$$D_\mu \nu(x) = \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x).$$

In this case, we say that $D_\mu \nu$ is the derivative of ν with respect to μ. We also call $D_\mu \nu$ the density of ν with respect to μ.

The goal of differentiation of Radon measures is to determine when $D_\mu \nu$ exists and when ν can be recovered by integrating $D_\mu \nu$.

Lemma 2.4.3. *For $0 < \alpha < \infty$,*

1. *$A \subseteq \{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) \le \alpha\}$ implies that $\nu(A) \le \alpha\mu(A)$,*

2. *$A \subseteq \{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \ge \alpha\}$ implies that $\nu(A) \ge \alpha\mu(A)$.*

Theorem 2.4.4. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Then $D_\mu \nu$ exists and is finite μ almost everywhere. Moreover, $D_\mu \nu$ is μ-measurable.*

Next we present results on integration of derivatives and Lebesgue decomposition.

Definition 2.4.5. *Let ν and μ be two measures on a topological space X. Then the measure ν is absolutely continuous with respect to μ, denoted by $\nu \ll \mu$, provided that $\mu(A) = 0$ implies that $\nu(A) = 0$ for all $A \subseteq X$.*

The measures ν and μ are mutually singular, denoted by $\nu \perp \mu$, if there exists a Borel subset $B \subseteq X$ such that

$$\mu(X \setminus B) = \nu(B) = 0.$$

Theorem 2.4.6 (Radon-Nikodym theorem)**.** *Let ν, μ be Radon measures on $\mathbb{R}^n$ with $\nu \ll \mu$. Then*

$$\nu(A) = \int_A D_\mu \nu d\mu$$

for all μ-measurable sets $A \subseteq \mathbb{R}^n$.

Theorem 2.4.7 (Lebesgue decomposition theorem)**.** *Let ν, μ be Radon measures on $\mathbb{R}^n$. Then,*

1. *Then $\nu = \nu_{ac} + \nu_s$, where ν_{ac}, ν_s are Radon measures on $\mathbb{R}^n$ with*

$$\nu_{ac} \ll \mu \text{ and } \nu_s \perp \mu.$$

2. *Furthermore,*

$$D_\mu \nu = D_\mu \nu_{ac} \text{ and } D_\mu \nu_s = 0 \ \mu\text{-almost everywhere,}$$

and so,

$$\nu(A) = \int_A D_\mu \nu d\mu + \nu_s(A)$$

for each Borel set $A \subseteq \mathbb{R}^n$.

Of course, there are far less restrictive versions of Radon-Nikodym theorem. The next is one of them.

Theorem 2.4.8 (General Radon-Nikodym theorem)**.** *If μ and ν are σ-finite measures on the same σ-algebra and ν is absolutely continuous with respect to μ, then there exists a finite valued measurable function f on X such that*

$$\nu(E) = \int_E f d\mu$$

for every measurable set E. The function f is unique.

2.5 Lebesgue differentiation theorem

2.5.1 Classical approach

Following [**?**, Sect. 1.7] we first show the Lebesgue differentiation theorem for Radon measures on $\mathbb{R}^n$.

Definition 2.5.1. *The average of a function f over a set E with respect to a measure μ is generically given by*

$$f_E = \fint_E f d\mu = \frac{1}{\mu(E)} \int_E f d\mu,$$

provided $0 < \mu(E) < \infty$ and the integral on the right is defined.

Theorem 2.5.2 (Lebesgue-Besicovith differentiation theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r\to 0} \fint_{B(x,r)} f d\mu = f(x)$$

for μ almost every $x \in \mathbb{R}^n$.

Corollary 2.5.3. *let μ be a Radon measure on $\mathbb{R}^n$, $1 \leq p < \infty$, and $f \in L^p_{\text{loc}}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r\to 0} \fint_{B(x,r)} |f - f(x)|^p d\mu = 0$$

for μ almost every x.

Definition 2.5.4 (Lebesgue point)**.** *A point x for which the above corollary holds is called a Lebesgue point of f with respect to μ.*

The above theorem can be strengthened if μ is the Lebesgue measure.

Corollary 2.5.5. *If $f \in L^p_{loc}(\mathbb{R}^n, \mathcal{L}^n)$ for some $1 \leq p < \infty$, then*

$$\lim_{B \downarrow \{x\}} \fint_B |f - f(x)| dy = 0$$

for $\mathcal{L}^n$ almost everywhere, where the limit is taken over all closed balls B containing x as diam $B \to 0$.

Corollary 2.5.6. *Let E be a $\mathcal{L}^n$-measurable. Then almost every $x \in E$ is a point of density* 1 *for E, that is,*

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 1,$$

and almost every $x \in \mathbb{R}^n \setminus E$ is a point of density 0 *for E, that is,*

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 0$$

for almost every x in $\mathbb{R}^n \setminus E$.

The Lebesgue differentiation theorem stands in more general contexts ([**?**, p. 4]).

Theorem 2.5.7. *Let $f \in L^1_{\text{loc}}(X, \mu)$ where (X, μ) is a doubling Borel regular metric measure space, then*

$$\lim_{r \to 0} \fint_{B(x,r)} f d\mu = f(x)$$

for almost every $x \in X$.

2.5.2 Maximal operator approach

Another very interesting approach to the Lebesgue differentiation theorem is through the Hardy-Littlewood maximal operator.

Definition 2.5.8 (Hardy-Littlewood maximal operator)**.** *Given two positive measures μ and ν on a metric space X, the maximal operator of the measure ν with respect to μ is defined by*

$$M_\nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

for all x in X.

The maximal operator is usually defined for locally integrable functions. See the next definition for the Lebesgue measure on $\mathbb{R}^n$.

Definition 2.5.9. *If f is a locally integrable function on $\mathbb{R}^n$, then*

$$M_f(x) = \sup_{r>0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |f| d\mathcal{L}^n.$$

Some properties of the maximal operator for locally integrable functions are given next.

Exercise 2.5.10. *1. M is not linear but it is sub-linear*

$$M[f+g](x) \leq Mf(x) + Mg(x).$$

2. Mf is always nonnegative and it may be always identically equal to infinity.

3. Mf makes sense for any locally integrable function.

Theorem 2.5.11 (Weak (1-1)-estimate)**.** *Let μ be a doubling measure which support coincides with the whole of X. Then there exists $C > 0$ (depending only on the doubling constant of μ) such that*

$$\mu(\{x \in X \colon M_\nu(x) > \lambda\}) \leq \frac{C}{\lambda} \nu(X)$$

for all $\lambda > 0$.

Exercise 2.5.12. *1. Prove this result for the case when $M_\nu \in L^1(X, \mu)$.*

2. If $X = \mathbb{R}^n$, $\mu = \mathcal{L}^n$ determine the measure ν so $M_\nu \in L^1(X, \mu)$.

From the weak (1-1) estimate it is possible to give a quite simple proof of the Lebesgue differentiation theorem for doubling measures.

Theorem 2.5.13. *Let X be a metric space, μ a doubling Borel regular measure on X, and $f \in L^1_{\mathrm{loc}}(X, \mu)$. then we have*

$$\lim_{r \to 0} ⨍_{B(x,r)} |f(y) - f(x)| d\mu(y) = 0$$

for μ-almost every $x \in X$.

Exercise 2.5.14. *Is the above result still working for $f \in L^p_{\mathrm{loc}}(X, \mu)$ and*

$$\lim_{r \to 0} ⨍_{B(x,r)} |f(y) - f(x)|^p d\mu(y) = 0$$

for μ-almost every $x \in X$.

2.6 Hausdorff Measure

Hausdorff measure is the most relevant example of lower dimensional measures on $\mathbb{R}^n$. These measures are usually applied to measure the size of different sets and constitute a fundamental tool in geometric measure theory.

2.6.1 Definition and main properties

Definition 2.6.1. *Let X be a metric space. For $s \geq 0$ let*

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}, \text{ where } \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$

If $\delta \in (0, +\infty]$ and $A \subseteq X$, we define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{j=1}^\infty \alpha(s) \left(\frac{\text{diam } C_j}{2} \right)^s : A \subseteq \cup_{i=1}^\infty C_i, \text{ diam } C_i \leq \delta \right\}.$$

Finally, we define

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

as the s-dimensional Hausdorff measure on X.

Remark 2.6.2.

1. *Notice that the covering sets C_i in the definition of Hausdorff measure can be assumed to be closed.*

2. *For $s = 0$ we obtain the counting measure, so these measures are not Radon in general.*

3. *Observe that for $B(x, r)$ the ball of an n-dimensional Euclidean space*

$$\mathcal{L}^n(B(x, r)) = \alpha(n) r^n$$

where $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure.

The Hausdorff measure is just a particular case of those measures obtained through the Carethéodory construction. Under this construction a more general case is considered and a family of measures is defined as follows:

$$\phi_\delta(A) = \inf \left\{ \sum_{j=1}^\infty \xi(C_j) \colon A \subseteq \cup_{i=1}^\infty C_i, \{C_i\} \subseteq \mathcal{F}, \text{ diam } C_i \leq \delta \right\}.$$

where $\mathcal{F}$ is a certain collection of subsets of X and ξ is a positive function defined on $\mathcal{F}$ called the *gauge* of the measure. Then a limit measure is obtained by taking δ going to 0. If $\mathcal{F}$ is the collection of balls in the definition of the Hausdorff measure then we have the *spherical Hausdorff measure*

Theorem 2.6.3. *$\mathcal{H}^s$ is a Borel regular measure for $0 \leq s < \infty$.*

Remark 2.6.4. *$\mathcal{H}^s_\delta$ is not Borel in general.*

Exercise 2.6.5. *Consider $\mathcal{H}^{1/2}_1$ on the real line and show that not every open set is measurable for it.*

There is another approach to Cartheodory's construction due to Federer which does not require the of a metric structure. More precisely, $\Phi(A)$ can be characterized as the infimum of all numbers t with the property that for each open covering $\mathcal{U}$ of A there exists a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ such that each member of $\mathcal{G}$ is contained in some member of $\mathcal{U}$, $\mathcal{G}$ covers A and

$$\sum_{S\in\ \mathcal{G}} \xi(S) < t.$$

Next we list some of the elementary properties of Hausdorff measures on $\mathbb{R}^n$.

Exercise 2.6.6.

1. *$\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$.*
2. *$\mathcal{H}^s = 0$ on $\mathbb{R}^n$ for all $s > n$.*
3. *$\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0$ and $A \subseteq \mathbb{R}^n$.*
4. *$\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for each isometry $L : \mathbb{R}^n \to \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$.*

The notion of Hausdorff measure is related to the idea of dimensionality of a set.

Lemma 2.6.7. *Let $A \subseteq X$ such that $\mathcal{H}^s_\delta(A) = 0$ for some $0 < \delta < \infty$ then $\mathcal{H}^s(A) = 0$.*

Exercise 2.6.8. *Show the case $s = 0$.*

Lemma 2.6.9. *Let $A \subseteq X$ and $0 \leq s < t < \infty$, then*

1. *If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.*
2. *If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = +\infty$.*

Definition 2.6.10. *The Hausdorff dimension of a set $A \subseteq X$ is defined to be*

$$\mathcal{H}_{\dim}(A) = \inf\{s \colon \mathcal{H}^s(A) = 0\}.$$

Remark 2.6.11. *It is very well-known that the Hausdorff dimension of a set does not have to be an entire number. Fractal sets are typical examples of sets with no entire dimension.*

2.6.2 Isodiametric inequality

Next thing we want to show is the isodiametric inequality for the n-dimensional Legesgue measure $\mathcal{L}^n$. Remember that $\mathcal{L}^n$ can be described as

$$\mathcal{L}^n(A) = \inf\left\{\sum_i \mathcal{L}^n(Q_i) \colon Q_i \text{ are cubes and } A \subseteq \cup Q_i\right\}.$$

We will need the Steiner symmetrization. For $a, b \in \mathbb{R}^n$ with $\|a\| = 1$ we denote

$$L_b^a = \{b + ta \colon t \in \mathbb{R}\},$$

and

$$P_a = \{x \in \mathbb{R}^n \colon x \cdot a = 0\}.$$

Definition 2.6.12. *Choose $a \in \mathbb{R}^n$ with $\|a\| = 1$ and let $A \subseteq \mathbb{R}^n$. We define the Steiner symmetrization of A with respect to the plane P_a to be the set*

$$S_a(A) = \bigcup_{b \in P_a,\ A \cap L_b^a \neq \emptyset} \left\{b + ta \colon |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)\right\}.$$

Lemma 2.6.13 (Properties of Steiner symmetrization)**.** *1.* $\mathrm{diam\ S_a(A)} \leq \mathrm{diam\ A}$.

2. If A is $\mathcal{L}^n$-measurable the so is $S_a(A)$ and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Theorem 2.6.14 (Isodiametric inequality)**.** *For all $A \subseteq \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\mathrm{diam\ A}}{2}\right)^n.$$

Finally we obtain the equality between the n-dimensional Hausdorff measure and $\mathcal{L}^n$.

Theorem 2.6.15. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.*

2.6.3 Hausdorff measures and Lipschitz functions

Definition 2.6.16. *Given two metric spaces X and Y, a mapping $f \colon X \to Y$ is called Lipschitz if there exists a constant C such that*

$$d_Y(f(x), f(y)) \leq C \cdot d_X(x, y)$$

for all $x, y \in X$. The Lipschitz constant of such a mapping is defined as the lowest of those constants C, that is,

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_Y(f(x), f(y))}{d_X(x, y)} \colon x, y \in X,\ x \neq y\right\}.$$

Theorem 2.6.17. *If $f\colon X \to Y$ is Lipschitz, $A \subseteq X$, $0 \le s < \infty$, then*

$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s(A).$$

Corollary 2.6.18. *Suppose $n > k$. Let $P : \mathbb{R}^n \to \mathbb{R}^m$ be the orthogonal projection, $A \subseteq \mathbb{R}^n$, $0 \le s < \infty$. Then*

$$\mathcal{H}^s(P(A)) \le \mathcal{H}^s(A).$$

For $f\colon X \to Y$, $A \subseteq X$, the graph of f over A, $G(f; A)$, is given by

$$G(f; A) = \{(x, f(x))\colon x \in A\} \subseteq X \times Y.$$

Theorem 2.6.19. *Assume $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{L}^n(A) > 0$. Then,*

1. $\mathcal{H}_{\mathrm{dim}}(G(f; A)) \ge n$,
2. *if f is Lipschitz,* $\mathcal{H}_{\mathrm{dim}}(G(f; A)) = n$.

2.7 Extension of Lipschitz mappings

2.7.1 The extension property

A very relevant property of Lipschitz mappings is that they can be extended as Lipschitz mappings depending on the geometries of their domains and images. The typical situation, for X and Y metric spaces, is a Lipschitz mapping $f\colon D \to Y$ where $D \subseteq X$ which we want to extend as a Lipschitz mapping to the whole of X. That is, under which conditions of X and Y we can guarantee that for any such D there exists $\tilde{f}\colon X \to Y$ such that $\tilde{f}$ is Lipschitz and $\tilde{f}(x) = f(x)$ for any $x \in D$. Moreover, we wonder about the relation of $\mathrm{Lip}(\tilde{f})$ and $\mathrm{Lip}(f)$. Maybe the most well-known result of this kind if the McShane theorem.

Theorem 2.7.1 (McShane Theorem)**.** *Assume $A \subseteq \mathbb{R}^n$ and let $f\colon A \to \mathbb{R}^m$ be Lipschitz. There exists a Lipschitz function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

1. *$\tilde{f} = f$ on A.*
2. $\mathrm{Lip}(\tilde{f}) \le \sqrt{m}\mathrm{Lip}(f)$.

McShane finds many and important applications when working with spaces of Lipschitz functions, however, there is still a very large and fruitful literature on the subject of extending Lipschitz mappings. Many authors have studied conditions which guarantee a bound of $\mathrm{Lip}(\tilde{f})$ in terms of $\mathrm{Lip}(f)$ or, even, set conditions under which it can be assured that $\mathrm{Lip}(\tilde{f}) = \mathrm{Lip}(f)$.

Definition 2.7.2. *Let X be a metric space and $D \subseteq X$. Then X is said to have the contraction extension property (E) if every Lipschitz mapping on D to X extends to Lipschitz mapping on the whole of X with same Lipschitz constant.*

This property has been studied by many authors. In particular, it is not hard to see that property (E) is equivalent to M. D. Kirszbraun intersecting property.

Theorem 2.7.3. *A metric space X has property (E) if and only if for any $x_i, y_i \in X$, $(i \in I)$, with $\|y_i - y_j\| \leq \|x_i - x_j\|$ for any $i, j \in I$, with $r_i > 0$ for $i \in I$ and the family of closed balls $\{B(x_i, r_i) \colon i \in I\}$ has a nonempty intersection then the same holds for the family $\{B(y_i, r_i) \colon i \in I\}$.*

Theorem 2.7.4 (Kirszbraun's theorem)**.** *If X is a (real) Hilbert space then X has property (E).*

Exercise 2.7.5. *Show that if $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ then it is the case that any Lipschitz mapping f from a subset of E into F can be extended to E with the same Lipschitz constant.*

Example 2.7.6. *The Kirszbraun theorem does not hold for Lipschitz maps between finite-dimensional Banach spaces. Let E be $\mathbb{R}^2$ with the maximum norm and F the same space with the Euclidean norm. Let $A = \{(-1,1), (1,-1), (1,1)\}$ and let $f \colon A \to F$ be given by*

$$f(-1,1) = (-1,0),\ f(1,-1) = (1,0),\ f(1,1) = (0,\sqrt{3}).$$

Then $\mathrm{Lip}(f) = 1$ but f has no 1-Lipschitz extension to $A \cup \{(0,0)\}$.

Exercise 2.7.7. *Show that there is no such an extension.*

Hilbert spaces are not yet the only Banach spaces with property (E). Another class of such spaces are spaces $C(\Omega)$ of continuous functions with Ω compact, Hausdorff and extremely disconnected as, for instance, $L^\infty(I)$. These spaces are also known in the literature as $\mathcal{P}_1$*-spaces or injective spaces.*

2.7.2 Injective spaces

Injective spaces have a metric counterpart which are called *hyperconvex spaces* introduced by Aronszajn and Panitchpakdi.

Definition 2.7.8. *A metric space X is called hyperconvex (or injective) if*

$$\bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any indexed class of closed balls $\{B(x_\alpha, r_\alpha) : \alpha \in \mathcal{A}\}$ in X, satisfying the condition $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for all α and β in $\mathcal{A}$.

The injectivity condition has strong consequences by itself.

Exercise 2.7.9. *Show that:*

1. *any injective metric space is complete,*
2. *if A is a subset of an injective space X then there exists $x \in X$ such that $A \subseteq B\left(x, \dfrac{\text{diam (A)}}{2}\right)$.*

Remark 2.7.10. *Let X be a hyperconvex space, if $A, B \subseteq X$ are hyperconvex then $A \cap B$ does not need to be hyperconvex.*

Remark 2.7.11. *If A is a hyperconvex subset of a linear space then A need not be convex.*

Injective spaces behave extraordinarily well with respect to the extension of Lipschitz functions.

Theorem 2.7.12. *A metric space M is injective if and only if whenever A is a subspace of another metric space X and $f\colon A \to M$ is Lipschitz, then there is a Lispschitz extension $\tilde{f}$ of f to the whole of X with same Lipschitz constant.*

Corollary 2.7.13. *Hyperconvex spaces are absolute nonexpansive retract. That is, if M is hyperconvex and $M \subseteq X$ then there exists a nonexpansive retraction from X onto M.*

2.7.3 Injective hull

The injectivity condition is a very restrictive one. In facet, the only injective Banach spaces are the spaces of continuous functions on an extremely disconnected compact Hausdorff space, as, for instance, $L^\infty(I)$. It has been proved, however, by Isbell [?] that any metric space admits an injective hull. The injective hull of a metric space M is an injective metric space $\tilde{M}$ which contains an isometric copy of M and which is isometric with a subspace of any hyperconvex metric space which contains an isometric copy of M. Injective hulls are constructed as follows.

Let M be a metric space. For any $x \in M$ define the positive real valued function $f_x\colon M \to [0, \infty)$ by $f_x(y) = d(x, y)$. Then

Exercise 2.7.14.

$$d(x, y) \leq f_a(x) + f_a(y)$$

and

$$f_a(x) \leq d(x, y) + f_a(y)$$

for any $x, y, a \in M$. Moreover, if $f\colon M \to [0,\infty)$ is such that $d(x,y) \le f(x) + f(y)$ for any $x, y \in M$, and for some $a \in M$ is the case that $f(x) \le f_a(x)$ for all $x \in M$, then $f = f_a$.

Definition 2.7.15. *Given M a metric space and $A \subseteq M$, a function $f\colon A \to [0,\infty)$ is extremal if*

$$d(x,y) \le f(x) + f(y)$$

for all $x, y \in A$, and is pointwise minimal, that is, if $g\colon A \to [0,\infty)$ is such that $d(x,y) \le g(x) + g(y)$ and $g(x) \le f(x)$ for all $x \in A$ then $f = g$.

Definition 2.7.16 (Isbell's injective hull)**.** *If $\epsilon(A)$ stands for the set of all extremal functions on A, then $\epsilon(A)$ with the supremum norm is the injective hull of A.*

Exercise 2.7.17. *Show that A can be isometrically embedded into $\epsilon(A)$.*

We give next some properties of extremal functions.

Proposition 2.7.18.

1. *If $f \in \epsilon(A)$ then $f(x) \le d(x,y) + f(y)$ for all x, y in A. Moreover,*
$$f(x) = \sup_{y\in A} |f(y) - f_x(y)| = d(f, e(x)).$$
2. *For any $f \in \epsilon(A)$, $\delta > 0$ and $x \in A$, there exists $y \in A$ such that*
$$f(x) + f(y) < d(x,y) + \delta.$$
3. *If A is compact, then $\epsilon(A)$ is compact.*
4. *If s is an extremal function on the metric space $\epsilon(A)$ then $s \circ e$, where $e : A \to \epsilon(A)$ is such that $e(a) = f_a$, is extremal on A.*

Theorem 2.7.19. *For any metric space A its injective hull $\epsilon(A)$ is hyperconvex and no proper subset of $\epsilon(A)$ which contains A is hyperconvex.*

2.8 Rademacher theorem

Rademacher theorem is a fundamental result which assures that Lipschitz functions are $\mathcal{L}^n$-almost everywhere differentiable. Therefore in geometric measure theory Lipschitz functions play the same role as differentiable functions play in elementary analysis. Versions of this result have recently been obtained in certain metric structures under conditions involving that the space is doubling and p-Poincaré inequality.

Definition 2.8.1. *A function $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$ is absolutely continuous if for every sequence of disjoint intervals (x_k, y_k) with $\sum |x_k - y_k| < \infty$ then $\sum |f(x_k) - f(y_k)| < \infty$.*

It is immediate to see that Lipschitz functions are absolutely continuous. The following is a well-known result from real analysis.

Theorem 2.8.2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous then it is $\mathcal{L}^1$-almost everywhere differentiable.*

Definition 2.8.3. *The function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|x-y|} = 0.$$

Theorem 2.8.4 (Rademacher's theorem)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$-almost everywhere.*

2.9 Linear maps and Jacobians. The area formula.

2.9.1 Linear maps and Jacobians

In order to describe the *area formula* we need to recall some notions of basic linear algebra. The goal is to be able to define the Jacobian of a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

Definition 2.9.1. *We recall the following kinds of linear maps:*

1. *A linear map $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal if $\langle Ox, Oy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$ (rotations and symmetries).*

2. *A linear map $S\colon \mathbb{R}^n \to \mathbb{R}^m$ is symmetric if if $\langle x, Sy\rangle = \langle Sx, y\rangle$ for all $x, y \in \mathbb{R}^n$.*

3. *A linear map $D\colon \mathbb{R}^n \to \mathbb{R}^m$ is diagonal if there exists $d_1, \cdots, d_n \in \mathbb{R}$ such that $Dx = (d_1 x_1, \cdots, c_n d_n)$ for all $x \in \mathbb{R}^n$.*

4. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. The adjoint of A is the linear map $A^*\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $\langle x, A^* y\rangle = \langle Ax, y\rangle$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.*

The existence of the adjoint operator is not trivial, in fact, it requires of the following representation result.

Theorem 2.9.2 (Representation theorem)**.** *If $L\colon \mathbb{R}^n \to \mathbb{R}$ is a linear functional, then there exists a unique $w \in \mathbb{R}^n$ so that $L(x) = \langle x, w\rangle$ for all $x \in \mathbb{R}^n$.*

Exercise 2.9.3. *Show that the adjoint operator is well defined.*

We describe next some properties from linear algebra.

Theorem 2.9.4.

1. *$A^{**} = A$.*
2. *$(A \circ B)^* = B^* \circ A^*$.*
3. *$O^* = O^{-1}$ if $O\colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal.*
4. *$S^* = S$ if $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is symmetric.*
5. *If $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is symmetric, there exists an orthogonal map $O\colon \mathbb{R}^n \to \mathbb{R}^n$ and a diagonal map $D\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*
$$S = O \circ D \circ O^{-1}.$$
6. *$O\colon \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal, then $n \le m$ and $O^* \circ O = I$ on $\mathbb{R}^n$, and $O \circ O^* = I$ on $O(\mathbb{R}^n)$.*

Theorem 2.9.5 (Polar decomposition)**.** *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then,*

1. *If $n \le m$, there exists a symmetric map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ and an orthogonal map $O\colon \mathbb{R}^n \to \mathbb{R}^m$ such that*
$$L = O \circ S.$$
2. *If $n \ge m$, there exists a symmetric map $S\colon \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O\colon \mathbb{R}^m \to \mathbb{R}^n$ such that*
$$L = S \circ O^*.$$

Next we define the Jacobian of a linear map.

Definition 2.9.6. *Assume $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then,*

1. *if $n \le m$, we write $L = O \circ S$ as above and define the Jacobian of L as*
$$\mathrm{J}(L) = |\det S|,$$
2. *if $n \ge m$, we write $L = S \circ O^*$ as above and define the Jacobian of L as well as*
$$\mathrm{J}(L) = |\det S|.$$

The next result states that the Jacobian of a linear map is independent of the representation.

Theorem 2.9.7. *1. If $n \leq m$, then*

$$\mathrm{J}(L)^2 = \det(L^* \circ L),$$

2. if $n \geq m$, then

$$\mathrm{J}(L)^2 = \det(L \circ L^*).$$

We introduce next some notation. If $n \leq m$, then

$$\Lambda(m, n) = \{\lambda\colon \{1, \cdots, n\} \to \{1, \cdots, m\} \text{ s. t. } \lambda \text{ is increasing}\}.$$

Moreover, for $\lambda \in \Lambda(m, n)$, we define $P_\lambda\colon \mathbb{R}^m \to \mathbb{R}^n$ by

$$P_\lambda(x_1, \cdots, x_m) = (x_{\lambda(1)}, \cdots, x_{\lambda(n)}).$$

Notice that P_λ is a nonexpansive (Lipschitz contant 1) projection onto the subspace

$$S_\lambda = \mathrm{span}\{e_{\lambda(1)}, \cdots, e_{\lambda(n)}\} \subseteq \mathbb{R}^n.$$

The Binet-Cauchy formula gives a useful method to compute the Jacobian of a linear map.

Theorem 2.9.8 (Binet-Cauchy formula)**.** *Assume $n \leq m$ and $L\colon \mathbb{R}^n \to \mathbb{R}^m$ in linear. Then,*

$$\mathrm{J}(L)^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

This formula can be regarded as a generalization of the Pythagorean theorem.

Next we define the Jacobian of a Lipschitz function. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. By Rademacher's theorem, f is differentiable $\mathcal{L}^n$ a.e., and therefore $Df(x)$ exists and defines a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathcal{L}^n$ almost every $x \in \mathbb{R}^n$.

Definition 2.9.9. *The Jacobian of f af x is*

$$Jf(x) = \mathrm{J}(Df(x)).$$

2.9.2 The area formula

In this subsection we always assume $n \leq m$.

Lemma 2.9.10. *Suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then*

$$\mathcal{H}^n(L(A)) = \mathrm{J}(L)\mathcal{L}^n(A)$$

for all $A \subseteq \mathbb{R}^n$.

Remark 2.9.11. *Notice in the above statement that although $\mathcal{H}^n = \mathcal{L}^n$, this equality must be understood only on $\mathbb{R}^n$ while $L(A) \subseteq \mathbb{R}^m$.*

Lemma 2.9.12. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Let $A \subseteq \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then*

1. *$f(A)$ is $\mathcal{H}^n$-measurable,*
2. *the mappings that assigns y to $\mathcal{H}^0(A \cap f^{-1}\{y\})$ is $\mathcal{H}^m$-measurable on $\mathbb{R}^m$ and,*
3. $$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n \leq (\mathrm{Lip}(f))^n \mathcal{L}^n(A).$$

Lemma 2.9.13. *Let $t > 0$ and $B = \{x\colon Df(x) \text{ exists}, Jf(x) > 0\}$. Then there is a countable collection $\{E_k\}_{k=1}^{\infty}$ of Borel subsets of $\mathbb{R}^n$ such that*

1. *$B = \cup_{k=1}^{\infty} E_k$,*
2. *$f|_{E_k}$ is one-to-one $(k = 1, 2, \cdots)$,*
3. *for each $k = 1, 2, \cdots$ there exists a symmetric automorphism $T_k : \mathbb{R}^n \to \mathbb{R}^n$ such that*
$$\mathrm{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t,\ \mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t,$$
and
$$t^{-n}|\mathrm{det}T_k| \leq Jf|_{E_k} \leq t^n|\mathrm{det}T_k|.$$

Theorem 2.9.14 (Area formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, with $n \leq m$. Then for each $\mathcal{L}^n$-measurable subset $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf\, dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n(y).$$

Another important theorem is the next change of variable formula.

Theorem 2.9.15 (Change of variables formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, with $n \le m$. Then for each $\mathcal{L}^n$-summable function $g\colon \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x)Jf(x)\,dx = \int_{\mathbb{R}^n} \left[\sum_{x\in f^{-1}\{y\}} g(x)\right] d\mathcal{H}^n(y).$$

Some consequences of the area formula are the next two ones.

Length of a curve **(**$n = 1, m \ge 1$**).** Assume $f\colon \mathbb{R} \to \mathbb{R}^m$ is Lipschitz and one-to-one. For $-\infty < a < b < \infty$, consider the curve $C = f([a, b])$, then

$$\mathcal{H}^1(C) = \int_a^b Jf\,dt.$$

Surface area of a graph **(**$n \ge 1, m = n + 1$**).** (Contrast with Theorem 2.6.19.) Assume $g\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and define $f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $f(x) = (x, g(x))$. Notice that Df is made of the identity matrix plus an extra row with the gradient of g, so, by the Binet-Cauchy, formula we have that

$$(Jf)^2 = 1 + (Jg)^2.$$

Therefore, if $U \subseteq \mathbb{R}^n$ is an open set we can define the graph of g over U by

$$G = G(g; U) = \{(x, g(x))\colon x \in U\} \subseteq \mathbb{R}^{n+1}.$$

Then

$$\mathcal{H}^n(G) = \text{ surface area of G } = \int_U (1 + (Jg)^2)^{1/2} dx.$$

2.10 Geodesic spaces of bounded curvature

In this section we will revise the notions of upper generalized curvature and δ-hyperbolicity in geodesic spaces.

2.10.1 Basic notions and model spaces

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in Y$ (or, more briefly, a *geodesic* from x to y) is a map $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic (or metric) segment* joining x and y. When it is unique this geodesic is denoted $[x, y]$.

The space (X, d) is said to be a *geodesic space* (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ such that $d(x, y) < D$).

Exercise 2.10.1. *Any normed space is geodesic. Usual lines are geodesics joining each two points. Moreover, a normed space is uniquely geodesic if and only if it is strictly convex (that is, if $\|u_1\|$ and $\|u_2\|$ are distinct vectors of normed one, then $\|(1 - t)u_1 + tu_2\| < 1$ for $t \in (0, 1)$).*

Some of the most fundamental examples of geodesic spaces are the model spaces of constant curvature M^n_κ which will be introduced below. Notice that for $\kappa = 0$ we are dealing with $\mathbb{E}^n$ the n-dimensional Euclidean space.

Let $I \subseteq \mathbb{R}$ be an interval. A map $c\colon I \to X$ is said to be a *linearly reparametrized geodesic* or *constant speed geodesic,* if there exists a constant λ such that $d(c(t), c(t')) = \lambda|t - t'|$. A *geodesic ray* is a map $c\colon [0, \infty) \to X$ such that $d(c(t), c(t')) = |t - t'|$. It will be a *geodesic line* if its domain is the whole $\mathbb{R}$.

Definition 2.10.2 (Convexity in geodesic spaces)**.** *Let $Y \subset X$, we denote by $G_1(Y)$ the union of all geodesic segments in X with endpoints in Y. Then Y is said to be convex if $G_1(Y) = Y$ or, equivalently, if every pair of points $x, y \in Y$ can be joined by a geodesic in X and the image of any such geodesic is contained in Y. Y is said to be D-convex if this condition holds for all points $x, y \in Y$ with $d(x, y) < D$. For $n \geq 2$ we inductively define $G_n(Y) = G_1(G_{n-1}(Y))$; then*

$$conv(Y) = \cup_{n=1}^{\infty} G_n(Y)$$

is the convex hull of Y.

Definition 2.10.3 (Triangles)**.** *A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). We will say that the triangle is degenerate if all three vertices belong to a same geodesic.*

2.10.2 Model spaces

Next we introduce the Model Spaces M_k^n, for a more detailed description of them as well as for the proofs of results we state in this section the reader can check [**?**, Chapter I.2]. To begin we need to describe the spaces $\mathbb{E}^n, \mathbb{S}^n$ and $\mathbb{H}^n$.

Let $\mathbb{E}^n$ stand for the metric space obtained by equipping the vector space $\mathbb{R}^n$ with the metric associated to the norm arising from the Euclidean scalar product $(x|y) = \sum_{i=1}^{i=n} x_i y_i$, where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, i.e. $\mathbb{R}^n$ endowed with the usual Euclidean distance. In this space we have the law of cosine:

$$a^2 = b^2 + c^2 - 2bc\cos\gamma,$$

where a, b, c are the sides of a triangle and γ the opposite angle to a.

The n-dimensional sphere $\mathbb{S}^n$ is the set $\{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \,:\, (x|x) = 1\}$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product.

Proposition 2.10.4. *Let $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function that assigns to each pair $(A, B) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique real number $d(A, B) \in [0, \pi]$ such that*

$$\cos d(A, B) = (A|B).$$

Then $(\mathbb{S}^n, d)$ is a metric space.

Geodesics in $\mathbb{S}^n$ coincide with sufficiently small arcs of great circles, i.e. intersections of $\mathbb{S}^n$ with a 2-dimensional vector subspace of $\mathbb{E}^{n+1}$. There is a natural way to parameterize arcs of great circles with respect to arc length which will be useful in this work: given a point $A \in \mathbb{S}^n$, a unit vector $u \in \mathbb{E}^{n+1}$ with $(u|A) = 0$ and a number $a \in [0, \pi]$, the path $c : [0, a] \to \mathbb{S}^n$ given by $c(t) = (\cos t)A + (\sin t)u$ is a geodesic and any geodesic in $\mathbb{S}^n$ can be parameterized this way. The next proposition summarizes some of the properties of the metric space $(\mathbb{S}^n, d)$.

Proposition 2.10.5. *Let $(\mathbb{S}^n, d)$ be as above and $A, B \in \mathbb{S}^n$, then:*

(1) If $d(A, B) < \pi$ then there is just one geodesic segment joining both points.

(2) If $B \neq A$ then the initial vector u of this geodesic is the unit vector, with to the Euclidean norm, in the direction of $B - (A|B)A$.

(3) *Balls of radius smaller than $\pi/2$ are convex sets.*

Definition 2.10.6 (Spherical angle). *The spherical angle between two geodesics from a point of $\mathbb{S}^n$, with initial vectors u and v, is the unique number $\alpha \in [0, \pi]$ such that $\cos\alpha = (u|v)$.*

Given $\triangle(A, B, C)$ a triangle in $\mathbb{S}^n$, the vertex angle at C is defined to be the spherical angle between the sides of $\triangle$ joining C to A and C to B. Then the Spherical Law of Cosines can be described as follows:

Proposition 2.10.7. *Let $\triangle$ be a spherical triangle with vertices A, B, C. Let $a = d(B, C)$, $b = d(C, A)$ and $c = d(A, B)$. Let γ denote the vertex angle at C. Then*

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$$

Now, in order to introduce the Hyperbolic n-Space $\mathbb{H}^n$, let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Then the *real hyperbolic n-space* $\mathbb{H}^n$ is

$$\{u \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} \geq 1\}.$$

Proposition 2.10.8. *Let $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the function that assigns to each pair $(A, B) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique non-negative number $d(A, B)$ such that*

$$\cosh d(A, B) = -\langle A, B\rangle.$$

Then $(\mathbb{H}^n, d)$ is a uniquely geodesic metric space.

Some of the most relevant properties of these spaces are summarized next.

Proposition 2.10.9. *Let $(\mathbb{H}^n, d)$ be as above and $A, B \in \mathbb{H}^n$, then:*

(1) *If u is the unit vector, with respect to the bilinear form, in the direction $B + \langle A|B\rangle A$ then the geodesic segment joining A and B and starting at A is given by $c(t) = (\cosh t)A + (\sinh t)u$.*

(2) *Balls are convex sets.*

(3) (Hyperbolic Law of Cosines) Under the same notation of Proposition 2.10.7,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma,$$

where γ stands for the hyperbolic angle which can be defined in a similar way to the spherical angle.

The Model Spaces M_k^n are defined as follows.

Definition 2.10.10. *Given a real number k, we denote by M_k^n the following metric spaces:*

(1) if $k = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;

(2) if $k > 0$ then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{k}$;

(3) if $k < 0$ then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-k}$.

Proposition 2.10.11. *M_k^n is a geodesic metric space. If $k \leq 0$ then M_k^n is uniquely geodesic and all balls in M_k^n are convex. If $k > 0$ then there is a unique geodesic segment joining $x, y \in M_k^n$ if and only if $d(x, y) < \pi/\sqrt{k}$. If $k > 0$, closed balls in M_k^n of radius smaller than $\pi/2\sqrt{k}$ are convex.*

2.10.3 CAT(κ)-spaces

Let (X, d) be a geodesic metric space. A *comparison triangle* for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_k^2 such that $d_{M_k^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

Proposition 2.10.12. *If $k \leq 0$ then such a comparison triangle always exists (and is unique up to isometries) in M_k^2. If $k > 0$ then such a triangle exists whenever $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$, where $D_k = \pi/\sqrt{k}$.*

A geodesic triangle $\triangle$ in X is said to satisfy the CAT(κ) *inequality* if, given $\bar{\triangle}$ a comparison triangle in M_k^2 for $\triangle$, for all $x, y \in \triangle$

$$d(x, y) \leq d_{M_k^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\triangle}$ are the respective *comparison points* of x, y, i.e., if $x \in [x_i, x_j]$ is such that $d(x, x_i) = \lambda d(x_i, x_j)$ and $d(x, x_j) = (1 - \lambda) d(x_i, x_j)$ then $\bar{x} \in [\bar{x}_i, \bar{x}_j]$ is such that $d(\bar{x}, \bar{x}_i) = \lambda d(\bar{x}_i, \bar{x}_j)$ and $d(\bar{x}, \bar{x}_j) = (1 - \lambda) d(\bar{x}_i, \bar{x}_j)$.

Definition 2.10.13. *If $k \leq 0$, then X is called a $CAT(k)$ space if X is a geodesic space such that all of its geodesic triangles satisfy the $CAT(k)$ inequality.*

If $k > 0$, then X is called a $CAT(k)$ space if X is D_k-geodesic and all geodesic triangles in X of perimeter less than $2D_k$ satisfy the $CAT(k)$ inequality.

Definition 2.10.14. *An $\mathbb{R}$-tree is a metric space T such that:*

(1) it is a uniquely geodesic metric space;

(2) if x, y and $z \in T$ are such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Remark 2.10.15. *Notice that all triangles in an $\mathbb{R}$-tree are degenerate.*

Exercise 2.10.16. *Show that any $\mathbb{R}$-tree is $CAT(\kappa)$ for any real κ.*

Next we define the notion of comparison angle.

Definition 2.10.17. *Let p, q and r be three points in a metric space. We call comparison angle between q and r at p, which will be denoted as $\overline{\angle}_p(q, r)$, to the interior angle of $\bar{\triangle}(p, q, r) \subseteq \mathbb{E}^2$ at $\bar{p}$.*

2.10.4 Notion of angle

Given any two geodesics starting at a same point, it is possible to define their angle at their initial common point. This is given by the notion of (Alexandrov) angle.

Definition 2.10.18. *Let X be a metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\triangle(\overline{c(0)}, \overline{c(t)}, \overline{c'(t')})$ and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$ in $\mathbb{E}^2$. The (Alexandrov) angle or the upper angle between the geodesic paths c and c' is the number $\angle(c, c') \in [0, \pi]$ defined by:*

$$\angle(c, c') = \limsup_{t, t' \to 0^+} \overline{\angle}_{c(0)}(c(t), c'(t')).$$

The angle between the geodesic segments $[p, x]$ and $[p, y]$ will be denoted $\angle_p(x, y)$.

If the above limit exists then it is said that the angle exists in the strict sense.

Remark 2.10.19. *The Alexandrov angle coincides with the spherical angle on $\mathbb{S}^n$ and the hyperbolic angle on $\mathbb{H}^n$.*

Remark 2.10.20. *1. The angle between c and c' depends only on the germs of these paths at 0, that is, if $c''\colon [0,l] \to X$ is any geodesic path for which there exists $\varepsilon > 0$ such that $c''|_{[0,\varepsilon]} = c'_{[0,\varepsilon]}$, then the angle between c and c'' is the same as that between c and c'.*

2. *The angle between the incoming and outgoing germs of a geodesic at any interior point along its image is π.*

3. *In a metric tree, the angle between two geodesic segments which have a common endpoint is either 0 or π.*

4. *Consider $(\mathbb{R}^2, d_\infty)$. For $n > 1$, the map that assigns $(t, [t(1-t)]^n)$ to each t defines a geodesic path on $[0, 1/n]$. These geodesics all issue from a common point and their germs are pairwise disjoint but the angle between any two of them is zero.*

The last statement in the above remark shows that the notion of angle does not define a metric on the set of geodesics starting from a same point. The next proposition tells that instead we have a pseudometric.

Proposition 2.10.21. *Let X be a metric space and let c, c', c'' be three geodesic paths in X issuing from the same point. Then*

$$\angle(c', c'') \leq \angle(c, c') + \angle(c, c'').$$

The other two axioms, symmetry and positivity, trivially hold.

Angles behave in a nice way if the metric space is a CAT(κ) space.

Proposition 2.10.22. *Let X be a CAT(κ) space and let $c\colon [0,a] \to X$ and $c'\colon [0,a']toX$ be two geodesic paths issuing from the same point $c(0) = c'(0)$. Then the κ-comparison angle $\angle^\kappa_{c(0)}(c(t), c'(t))$ is a non-decreasing function of both $t, t' \geq 0$ and the Alexandrov angle $\angle(c, c')$ is equal to*

$$\lim_{t,t'\to 0} \angle^\kappa_{c(0)}(c(t), c'(t')) = \lim_{t\to 0} \angle^\kappa_{c(0)}(c(t), c'(t)).$$

2.10.5 Properties of CAT(κ) and CAT(0) spaces

Theorem 2.10.23. *1. If X is a CAT(κ) space, then it is a CAT(κ') space for every $\kappa' \geq \kappa$.*

2. *If X is a CAT(κ') space for every $\kappa' > \kappa$, then it is a CAT(κ) space.*

Some examples of CAT(0) spaces.

Proposition 2.10.24. *If a normed real vector space V is CAT(κ) for some $\kappa \in \mathbb{R}$, then it is a pre-Hilbert space.*

1. Convex subsets of Hilbert spaces.
2. $\mathbb{R}^2 \setminus \{(x, y) : x > 0, y > 0\}$ endowed with the induced length metric.
3. Product of CAT(0) spaces.

We summarize next some of the properties of CAT(0) spaces which can be found in [**?**, Chapter II] and will be needed in our work.

Proposition 2.10.25. *Let (X, d) be a CAT(0) space, then the following properties hold:*

1. *(X, d) is a uniquely geodesic space.*
2. *If Δ=$\Delta(A, B, C)$ is a triangle in (X, d) and $\bar{\Delta} = \Delta(\bar{A}, \bar{B}, \bar{C})$ is its Euclidean comparison triangle, then for any vertex of Δ, let us say A,*
$$\gamma = \angle_A(B, C) \leq \angle_{\bar{A}}(\bar{B}, \bar{C}).$$
3. *(Law of cosines.) If γ is as above and $a = d(B, C)$, $b = d(A, C)$ and $c = d(A, B)$ then*
$$a^2 \geq b^2 + c^2 - 2bc \cos \gamma.$$
In particular, if $\gamma \geq \pi/2$ then the largest side of Δ is the opposite to γ.
4. **The CN inequality of Bruhat and Tits.** *For all $p, q, r \in X$ and all $m \in X$ with $d(q, m) = d(r, m) = \dfrac{d(q, r)}{2}$, one has:*
$$d^2(p, q) + d^2(p, r) \geq 2d^2(m, p) + (1/2)d^2(q, r).$$

Another important feature from CAT(0) spaces is the behavior of the metric projection. This behavior resembles that of the same projections in Hilbert spaces.

Definition 2.10.26. *Given a metric space X and a nonempty subset K of X, the metric projection (or nearest point map) from X onto K is denoted as P_K and defined by*
$$P_K(x) = \{y \in K : d(x, y) = \operatorname{dist}(x, K)\}$$
where $\operatorname{dist}(x, K) = \inf_{y \in K} d(x, y)$.

The next proposition, which summarizes the properties of the metric projection onto closed and convex subsets of CAT(0) spaces, can be found in [**?**].

Theorem 2.10.27. *Let X be a CAT(0) space, and let C be a convex subset which is complete in the induced metric. Then*

1. *For every $x \in X$, there exists a unique point $\pi(x) \in X$ such that $d(x, \pi(x)) = d(x, C) = \inf_{y \in C} d(x, y)$.*
2. *If x' belongs to the geodesic segment $[x, \pi(x)]$ then $\pi(x') = \pi(x)$.*
3. *Given $x \notin C$ and $y \in C$, if y is different from $\pi(x)$ then $\angle_{\pi(x)}(x, y) \geq \pi/2$.*
4. *The map $\pi(x)$ is nonexpansive, that is, Lipschitz with constant 1.*

2.11 Gromov hyperbolicity

2.11.1 Gromov hyperbolicity

Next we introduce another notion of hyperbolicity given by Gromov. Again, the idea of hyperbolicity will come next to the thinness of triangles. A convenient way to introduce Gromov hyperbolicity is via the idea Gromov product.

Definition 2.11.1. *Let X be a metric space. Fix a base point $0 \in X$. Then, for $x, y \in X$, the Gromov product of x and y with respect to 0 is given by*

$$(x|y)_0 = \frac{1}{2}(d(x,0) + d(y,0) - d(x,y))$$

The Gromov product, thanks to the triangle inequality, is always positive. Moreover, if X is a geodesic space, then $(x|y)_0$ measures how close the triangle $\triangle(x, y, 0)$ is from being a degenerate triangle. The next lemma gives a geometric interpretation of this product.

Lemma 2.11.2. *Let X be a geodesic space and $\triangle(x, y, z)$ a triangle in X. There is a unique collection of points $u \in [y, z]$, $w \in [x, y]$ and $v \in [x, z]$ such that $d(x, v) = d(x, w) = a$, $d(y, u) = d(y, w) = b$ and $d(z, v) = d(z, u) = c$. Moreover,*

$$a = (y|z)_x, \; b = (x|z)_y, \; c = (x|y)_z.$$

The points u, w and v are called equiradial points.

Definition 2.11.3 (Gromov hyperbolicity)**.** *A geodesic metric space is called δ-hyperbolic, $\delta \geq 0$, if for any $\triangle(x, y, z) \subseteq X$ the following holds: If $y' \in [x, y]$, $z' \in [x, z]$ are points with $d(x, y') = d(x, z') \leq (y|z)_x$, then $d(x', z') \leq \delta$. The space is (Gromov) hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.*

An equivalent definition of hyperbolicity is directly through the idea of thinness of triangles.

Definition 2.11.4. *A geodesic space X is said to be hyperbolic if any side of any triangle lies in the δ-neighborhood of the two other sides.*

Remark 2.11.5. *Notice that the above definition is interesting only if the sides of the triangle are large with respect to δ.*

Proposition 2.11.6.

1. *In a δ-hyperbolic space any side of any triangle lies in the δ-neighborhood of the two other sides.*
2. *$\mathbb{R}$-trees are 0-hyperbolic spaces.*

3. *CAT(κ) spaces, with $\kappa < 0$, are δ-hyperbolic with δ only depending on κ.*

4. *Bounded geodesic spaces are trivially hyperbolic for a large enough δ. Therefore, this notion is relevant only if δ is small with respect to the diameter.*

The next lemma is a ver useful extension of the idea of hyperbolicity of a triangle to taken to an n-gon.

Lemma 2.11.7. *Let $x_1, \cdots, x_n$ be an n-gon with $n \leq 2^k + 1$ for some $k \in \mathbb{N}$, then every side is contained in the $k\delta$-neighborhood of the union of the other sides.*

2.11.2 Stability of geodesics

Geodesic is a fundamental notion in all this theory, the idea now is to show that they are stable in hyperbolic spaces. This means that we wish to enlarge the class of geodesic mappings to a new collection of mappings, so-called *quasi-geodesics*, which remain at a uniformly bounded distance to a geodesic.

Definition 2.11.8. *Let X and Y be two metric spaces and $f\colon X \to Y$, then:*

1. *f is bilipschitz if*
$$\frac{1}{a} d(x,y) \leq d(f(x), f(y)) \leq a d(x,y)$$
for some $a \geq 1$.

2. *A subset A of Y is a net if the distances of all points $y \in Y$ to A are uniformly bounded.*

3. *f is quasi-isometric if there are $a \geq 1$ and $b \geq 0$ such that*
$$\frac{1}{a} d(x,y) - b \leq d(f(x), f(y)) \leq a d(x,y) + b,$$
that is, f behaves as a bilipschitz map on large distances.

4. *If a quasi-isometric mapping f is such that $f(X)$ is a net in Y then f is said to be a quasi-isometry and the spaces X and Y are said quasi-isometric.*

5. *A quasi-geodesic in X in a quasi-isometric map defined on a real interval.*

In a general metric space a quasi-geodesic can be very far from a geodesic. Just consider in the Euclidean plane $\gamma(t) = (t(\cos(\ln t)), \sin(\ln t))$, then $|\gamma(t)| = |t|$ and $|\gamma'(t)| = \sqrt{2}$ for all $t > 1$ we have that
$$\frac{1}{\sqrt{2}} d(\gamma(t), \gamma(t')) \leq |t - t'| \leq d(\gamma(t), \gamma(t')).$$

Definition 2.11.9. *An a-path, $a > 0$, in a metric space is a finite or infinite sequence of points $\{x_i\}$ with $d(x_i, x_{i+1}) \leq a$ for each i.*

The next lemma shows that paths avoiding balls in a hyperbolic space must have a somehow exponentially large length with respect to the radius.

Lemma 2.11.10. *Assume that an a-path $f : \{1, \cdots, N\} \to X$ in a geodesic δ-hyperbolic space, $\delta > 0$, lies outside of the ball $B(x, r)$ centered at some point $x \in [f(1), f(N)]$. Then*

$$N \geq c \cdot 2^{r/\delta}$$

for some constant $c > 0$ depending only on a and δ.

Next we state the result on stability of quasi-geodesics.

Theorem 2.11.11. *Let X be a δ-hyperbolic geodesic space and $a \geq 1$, $b \geq 0$. Then there exists $h = h(a, b, \delta) > 0$ such that for every $N \in \mathbb{N}$ the image $\operatorname{im}(f)$ of every (a, b)-quasi-isometric map $f \colon \{1, \cdots, N\} \to X$ lies in the h-neighborhood of any geodesic $c \colon [0, l] \to X$ with $c(0) = f(1)$, $c(l) = f(N)$, and vice versa, c lies in the h-neighborhood of $\operatorname{im}(f)$. As a consequence we have that if a geodesic space X is quasi-isomorphic to a hyperbolic geodesic space Y, then X is also hyperbolic.*

Chapter 3

Zhong

Poincaré inequality, Muckenhoupt Weights and Related Topics Xiao Zhong

3.1 Lecture One. Poincaré inequalities.

Let $1 \leq p < \infty$. The p-Poincaré inequality

$$\int_B |u-u_B|^p dx \leq C(n,p) r^p \int_B |\nabla u|^p dx, \quad \forall u \in \mathcal{C}^1(B),\ \forall B \subset \mathbb{R}^n. \tag{3.1.1}$$

Notation 3.1.1.

- $B = B(x_0, r) \subset \mathbb{R}^n$
- $\lambda B = B(x_0, \lambda),\ \forall \lambda > 0$
- $\nabla u(x) = (\partial_{x_1} u(x), \ldots, \partial_{x_n} u(x)),\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
- $$u_B = \frac{1}{|B|}\int_B u(y)dy = ⨍_B u(y)dy$$

Definition 3.1.1. A measure μ on $\mathbb{R}^n$ is called p-**admissible**, $1 \leq p < \infty$ if

1. μ is doubling i.e. $\exists C_\mu > 0$ s.t. $\mu(2B) \leq C_\mu \mu(B)\ \forall B \subset \mathbb{R}^n$
2. it admits a weak p-Poincaré inequality i.e. $\exists C > 0$ and $\lambda \geq 1$ s.t.

$$⨍_B |u-u_B| dx \leq Cr \left(⨍_{\lambda B} |\nabla u|^p dx \right)^{\frac{1}{p}}, \quad \forall u \in \mathcal{C}^1(B),\ \forall B \subset \mathbb{R}^n \tag{3.1.2}$$

where $u_B = \frac{1}{\mu(B)} \int_B u d\mu$

Remark 3.1.2. p-admissible $\Rightarrow$ q-admissible, $q \geq p$.

Notation 3.1.2. *Weight,* $w \geq 0,\ w \in L^1_{loc},\ d\mu = wdx$

$$\mu(E) = \int_E w(x)dx = w(E)$$

Open problem 3.1.1. *If μ is 1-admissible Radon measure, is it absolutely continuous with respect to Lebesgue measure?*

Recall the **Muckenhoupt A_p weights**

$$\sup_{B\subset\mathbb{R}^n} \fint_B wdx \left(w^{\frac{1}{1-p}}dx\right)^{p-1} \leq C, \qquad p > 1$$

$$\fint_B wdx \leq C \operatorname*{ess\,inf}_B w, \qquad p = 1$$

Theorem 3.1.3 (Muckenhoupt and Wheeden 74'). $1 \leq p < \infty$. $w \in A_p \implies$ *w is p-admissible.*

Proof of 3.1.1. Fix $B = B(x_0, r) \subset \mathbb{R}^n$. We will show the following inequality

$$|u(x) - u_B| \leq c(n)r^n \int_B \frac{|\nabla u(y)|}{|y-x|^{n-1}} \leq c(n)rM(|\nabla u|\chi_B), \quad \forall x \in B \tag{3.1.3}$$

from which (3.1.1) follows. Fix $x \in B$. Then $\forall y \in B$

$$|u(x)-u(y)| = \left|\int_0^1 \frac{d}{dt}u(\underbrace{ty+(1-t)x}_{z})dt\right| = \left|\int_0^1 \nabla u(z)\cdot(y-x)dt\right| \leq \int_0^1 |\nabla u(z)||y-x|dt.$$

Therefore

$$\int_B |u(x)-u(y)|dy \leq \int_B \int_0^1 |\nabla u(z)||y-x|dt\,dy.$$

By introducing the change of variables $z = x + t(y-x)$, $z_0 = x_0 + t(y - x_0)$ we get

$$\int_B |u(x)-u(y)|dy \leq \int_{B(z_0,tr)} \int_0^1 |\nabla u(z)|\frac{|y-x|}{t}\frac{1}{t^n}dt\,dy \leq \int_B |\nabla u(z)||z-x| \int_{\frac{|z-x|}{2r}}^1 \frac{1}{t^{n+1}}dt\,dz$$

and using

$$\int_{\frac{|z-x|}{2r}}^1 \frac{1}{t^{n+1}}dt \leq c(n)\frac{r^n}{|z-x|^n}$$

we obtain

$$\int_B |u(x)-u(y)|dy \le r^n \int_B \frac{|\nabla u(y)|}{|y-x|^{n-1}}dy. \tag{3.1.4}$$

Denote $g(x) = |\nabla u(x)|\chi_B(x)$, then

$$\begin{aligned}\int_B \frac{|\nabla u(y)|}{|y-x|^{n-1}}dy &\le \int_{B(x,2r)} \frac{g(y)}{|y-x|^{n-1}}dy \\ &\le \sum_{i\ge 0}\int_{B(x,2^{-i+1}r)\setminus B(x,2^{-i}r)} \frac{g(y)}{|y-x|^{n-1}}dy \\ &\le \sum_{i\ge 0} c(n)2^{i(n-1)}r^{1-n}\int_{B(x,2^{-i+1}r)} g(y)dy \\ &= \sum_{i\ge 0} c(n)2^{-i}r ⨍_{B(x,2^{-i+1}r)} g(y)dy \le rM(g)(x).\end{aligned} \tag{3.1.5}$$

Combining (3.1.4) and (3.1.5) and recalling the definition of g we obtain (3.1.3). □

Theorem 3.1.4. *$n=1$, $1\le p<\infty$. Then μ is p-admissible $\Longleftrightarrow$ $d\mu = wdx$, $w\in A_p$.*

Proof. Exercise.
Hint: $\forall f \in C_0^\infty(\mathbb{R})$ define

$$u(x) = \int_{-\infty}^{x} f(t)\chi_I(t)dx$$

and use Poincaré inequality for $u(x)$. □

3.2 Lecture Two

Recall the **Weighted Poincaré Inequality** which states that for all balls B and all $u \in \mathcal{C}^1(B)$ we have

$$⨍_B |u-u_B|d\mu \le C(⨍_{\lambda B} |\nabla u|^p d\mu)^{1/p} \tag{3.2.1}$$

where

$$u_B := \frac{1}{\mu(B)}\int_B u(y)d\mu(y).$$

Theorem 3.2.1. *Let $n\ge 2$, $\delta > -n$. Then $w(x) = |x|^\delta$ is 1-admissible.*

Notation 3.2.1.
- *Df is the Jacobian matrix*
- $(Df(x))_{ij} = (\partial_{x_j} f^i(x))$ *for* $f = (f^1, \dots, f^n)$
- $J_f(x) = \det(Df(x))$ *is the Jacobian determinant.*

Theorem 3.2.2. *If $n \geq 2$ and $p \in [1, n)$ then $Jf(x)^{1-p/n}$ is p-admissible whenever f is quasi-conformal*

Definition 3.2.3. For $K \geq 1$ a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-**quasi-conformal** if

- f homeomorphism
- $f \in W^{1,n}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$
- $|Df(x)|^n \leq K J_f(x)$ for almost every $x \in \mathbb{R}^n$.

Remark 3.2.4. Idea is that the image of the ball B under f q.c. is an ellipsoid where the diameter of $f(B)$ is comparable to B.

Proposition 3.2.5. *If f is K quasi-conformal then*

1. $J_f(x) \in A_\infty$
2. $\log J_f(x) \in \mathrm{BMO}$

We will prove Theorem 3.2.1 given Theorem 3.2.2.

Proof of Theorem 3.2.1.
1. $-n < \delta \leq 0$ is OK
2. for $\delta > 0$ let $f(x) = |x|^{\gamma-1}x$, $\gamma > 0$, then f is K-q.c. with $K = K(\gamma, n) \geq 1$ and
$$J_f(x)^{1-1/n} \approx |x|^{(\gamma-1)n(n-1)/n} = |x|^\delta$$
is one-admissible by Theorem 3.2.2.

□

Proof of Theorem 3.2.2. We want to show for all $u \in \mathbb{C}^1(B)$ and some $\gamma > 0$,
$$\fint_B |u-\gamma|^p J_f(x)^{1-p/n} dx \leq cr^p \left(\fint |\nabla u|^p J_f(x)^{1-p/n} dx \right).$$
Then from that and the fact that $J_f(x)^{1-p/n} \in A_\infty$ is doubling the theorem will follow. Let $f : B \to f(B)$ and set $g = f^{-1}$, and we define a new function

$v(y) = u(g(y))$.

Then

$$\nabla v(y) = Dg(y)^T \nabla u(g(y))$$

$$\begin{aligned}
\int_{f(B)} |\nabla v(y)|^p dy &\leq \int_{f(B)} |\nabla u(g(y))|^p |D(g(y))|^p dy \\
&\leq c \int_{f(B)} |\nabla u(g(y))|^p J_g(y)^{p/n-1} J_g(y) dy \\
&= c \int_{f(B)} |\nabla u(g(y))|^p (J_f(g(y))^{1-p/n}) J_g(y) dy \\
&= c \int_B |\nabla u(x)|^p J_f(x)^{1-p/n} dy
\end{aligned}$$

note that

$$|Df(x)|^n \leq K J_f(x) \implies |Dg(x)|^n \leq K^{n-1} J_g(x)$$

and the last line is change of variables $x = g(y)$.

For the left hand side we use Hölder inequality to get an estimate from above by

$$\left(⨍_B |u - \gamma|^{pn/(n-p)} J_f(x) dx \right)^{(n-p)/n} |B|^{p/n} \leq c \left(\int_{f(B)} |v(y) - \gamma|^{p^*} dy \right)^{(n-p)/n} r^p$$

where $p^* = np/(n-p)$. If we can show

$$\left(\int_{f(B)} |v(y) - v_{f(B)}|^{p^*} dy \right)^{1/p^*} \leq c \left(\int_{f(B)} |\nabla v(y)|^p dy \right)^{1/p}$$

then we are done taking $\gamma = v_{f(B)}$ (exercise). □

Exercise 3.2.6. *Let*

$$A = \{(x, y) : xy \geq 0\}$$

and let $\Omega = \overline{A \cap B(0, r)} \subset \mathbb{R}^2$. *Show that the p-Poincaré inequality holds on this domain. That is* $\exists C = C(\rho, \theta) > 0$ *such that* $⨍_\Omega |u - u_\Omega| dx \leq Cr \left(⨍_\Omega |\nabla u|^p dx \right)^{1/p}$ *for all* u *in Lip* Ω *if and only if* $p > 2$.

Hint: Instead of using a straight path between two points use a path which consists of to segments connected at the origin.

Theorem 3.2.7 (David and Semmes 1990). *Let $n \geq 2$. Suppose that $w \in SA_\infty$ then $w^{1-p/n}$ is p-admissible for $p \in [1, n)$*

Definition 3.2.8. We say that $w \in SA_\infty$ if w is doubling and for all $x,\ y \in \mathbb{R}^n$

$$D_w(x,y) = \left(\int B_{x,y} w(z)dz \right)^{1/n} \approx d(x,y)$$

where $B_{x,y}$ is the smallest ball containing x and y and d is a metric which is comparable to the normal distance.

$$B_{x,y} = B((x+y)/2, |x-y|/2).$$

Proposition 3.2.9.
- *If $w \in SA_\infty$ then $w \in A_\infty$*
- *If $w \in A_1$ then $w \in SA_\infty$*
- *$J_f \in SA_\infty$ for f q.c.*

One way is easy and the other is not and you need to use the maximal function property of A_p weights.
Hint: consider

$$d(x,y) = \inf_T \int_0^T w(\gamma(t))^{1/n} dt$$

with $\gamma(0) = x,\ \gamma(T) = y$.

Remark 3.2.10. There is a converse question one can ask: given a SA_∞ weight is it comparable to a quasi-conformal map.

Quasiconformal Jacobian Problem Characterize weights which are comparable to the Jacobian of a qc map.

Proposition 3.2.11 (Laakso et al 2002). *There exist SA_∞ weights which are not comparable to the Jacobian of any f which is q.c.*

Proposition 3.2.12 (Bishop 2007). *There exists $w \in A_1$ with the same property.*

Proposition 3.2.13 (Bank, Heinonen and Saksman 2008). *Let $w(x) = e^{nu(x)}$. Then $w(x) \approx J_f(x)$ if*

$$u(x) = -\int_{\mathbb{R}^n} \log|x-y| d\mu(x)$$

with $\|\mu\| < C_n$ small.

3.3 Lecture Three. Poncaré $\Longrightarrow$ Sobolev-Poncaré.

We start with an exercise

Exercise 3.3.1. *Let μ be a doubling measure in $\mathbb{R}^n$. Then show that for all $A \subset \mathbb{R}^n$ with $x \in A$ and $0 < r < \operatorname{diam}(A)$ we have*

$$\frac{\mu(B(x,r))}{\mu(A)} \geq 2^{-s}\left(\frac{r}{\operatorname{diam}(A)}\right)^s$$

where $s = \log_2 C_\mu$ and C_μ is the doubling constant.

The following theorem is an improvement of the Poincaré inequality from the left hand side:

Theorem 3.3.2 (Hajtasz-Koskela, 1995). *Suppose that μ is a doubling measure in $\mathbb{R}^n$, $B_0 = B(x_0, r_0) \subset \mathbb{R}^n$, and $u : B_0 \to \mathbb{R}^n$ Lipschitz in B_0 such that for every B with $\lambda B \subset B_0$ we have*

$$\fint_B |u - u_B| \leq c r_B \left(\fint_{\lambda B} |\nabla u|^p d\mu\right)^{1/p} \tag{3.3.1}$$

where $p \in (0, s)$, $c > 0$ and $\lambda \geq 1$. Then for every $q < ps/(s-p)$

$$\left(\fint_{B_0} |u - u_{B_0}|^q d\mu\right)^{1/q} \leq C' r_0 \left(\fint_{B_0} |\nabla u|^p d\mu\right)^{1/p} \tag{3.3.2}$$

where $C' = C'(n, p, q, \lambda, c) > 0$

Remark 3.3.3. When number on the RHS is bigger then p we call it a **Sobolev-Poincaré** inequality.

Proof. Strategy is to apply our given inequality infinitely many times for a certain sequence of balls.

(Take $\lambda = 2$ for convenience) Assume $u_{B_0} = 0$. Fix $t > 0$ and consider the level set at t

$$A_t = \{y \in B_0 : |u(y)| > t\}$$

and take a point $x \in A_t$. Consider the family of balls

$$B((1-t)x_0 + tx, 2^{-i} r_0) = B_i$$

with $t = 2^{-i} r_0 / |x - x_0|$, $r_i \sim 2^{-i} r_0$ (some homotopy to a ball centered at x which is shrinking in radius.

By Lebesgue differentiation theorem

$$-u(x) = \sum_{i\geq 0}(u_{B_i} - u_{B_{i+1}})$$

which implies

$$\begin{aligned} t < |-u(x)| &\leq \sum_{i\geq 0} |u_{B_i} - u_{B_{i+1}}| \\ &\leq \sum_{i\geq 0} |u_{B_i} - u_{B_i\cap B_{i+1}}| + |u_{B_i\cap B_{i+1}} - u_{B_{i+1}}|. \end{aligned}$$

Next,

$$\begin{aligned} &\sum_{i\geq 0} |u_{B_i} - u_{B_i\cap B_{i+1}}| \\ &\qquad \leq \sum_{i\geq 0} \frac{1}{\mu(B_i\cap B_{i+1})} \int_{B_i\cap B_{i+1}} |u - u_{B_i}| d\mu \\ &\qquad \leq \sum_{i\geq 0} c \fint_{B_i} |u - u_{B_i}| d\mu \\ &\qquad \leq \sum_{i\geq 0} r_i \left(\fint_{\lambda B_i} |\nabla u|^p d\mu \right) \end{aligned}$$

where the second inequality is obtained using the doubling property of μ and the last one by the application of (3.3.1).

On the other hand, $\forall \varepsilon > 0$ $\exists C(\varepsilon)$ such that we can write

$$1 \geq C(\varepsilon) \sum_{i\geq 0} (r_i/r_0)^{\varepsilon}$$

which gives some $i \in \mathbb{N}$ such that

$$t(r_i/r_0)^{\varepsilon} \leq c r_i \left(\fint_{\lambda B_i} |\nabla u|^p d\mu \right)^{1/p}$$

and therefore

$$\begin{aligned} t &\leq c(r_i/r_0)^{1-\varepsilon} r_0 \left(\fint_{\lambda B_i} |\nabla u|^p d\mu \right)^{1/p} \\ &\leq c(\mu(B_i)/\mu(B_0))^{(1-\varepsilon)/s} r_0 \left(\fint_{\lambda B_i} |\nabla u|^p d\mu \right)^{1/p}. \end{aligned}$$

Rearranging we get

$$t^p \mu(\lambda B_i)^{1-(1-\varepsilon)p/s} \leq \frac{c r_0^p}{\mu(B_0)^{(1-\varepsilon)p/s}} \int_{\lambda B_i} |\nabla u|^p d\mu \tag{3.3.3}$$

so for each x we find an i so that the above is true. Next, we choose a ball B_x centered at x which covers the whole λB_i. Thus we obtain a covering $\{B_x\}$ and by covering lemma we can choose a subcover $\{5B\}$ such that the balls B are disjoint. Therefore

$$t^p \mu(A_t)^{1-(1-\varepsilon)p/s} \leq \frac{c r_0^p}{\mu(B_0)^{(1-\varepsilon)p/s}} \int_{B_0} |\nabla u|^p d\mu$$

Using the Layer Cake formula we obtain the desired result. □

Open ended property of p-admissible weights We now want to improve the right hand side from going from p to $p-\varepsilon$.

Theorem 3.3.4 (Keith-Zhong, 2008)**.** *Suppose w is p-admissible for $p > 1$. Then there exists an $\varepsilon > 0$ such that w is $(p-\varepsilon)$-admissible.*

To prove this theorem we recall Muckenhoupt A_p-weights.

We are going to use cubes with sides parallel to the axes in our maximal function.

$$Mf(x) := \sup_{Q \ni x} \fint_Q |f(y)| dy$$

Theorem 3.3.5. *We have*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x) dx \leq c \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \tag{3.3.4}$$

if and only if

$$w \in A_p$$

recall that we say $w \in A_p$ when

$$\sup_Q \fint_Q w dx \left(\fint_Q w^{-1/(1-p)} dx \right)^{p-1} \leq K < \infty, \qquad p > 1$$

$$\fint_Q w dx \leq K \inf_Q w, \qquad p = 1$$

Theorem 3.3.6 (Keith-Zhong, 2009)**.** *Let $w \in A_p$, $p > 1$. Then $\forall \delta > 1$ $\exists N_0 = N_0(\delta, p, n, K) > 0$ such that $\forall N \geq N_0$, $\forall \lambda > 0$, $\forall f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$w(\Omega_\lambda) \leq \delta N^p w(\Omega_{N\lambda}) + N^{2p} w\left(\{x \in \mathbb{R}^n : |f(x)| > N^{-2}\lambda\}\right) \tag{3.3.5}$$

where $\Omega_\lambda = \{Mf(x) > \lambda\}$.

We are now ready to prove Theorem 3.3.4.

Proof of Theorem 3.3.4. Let $\delta = 1/4$, and let N_0 and ε be such that $N_0^\varepsilon < 2$.

Integrating both sides of (3.3.5) with respect to the measure $\lambda^{p-\varepsilon}$

$$\begin{aligned}\int_0^\infty w(\Omega_\lambda) d\lambda^{p-\varepsilon} \leq \delta N_0^\varepsilon \int_0^\infty & N_0^{p-\varepsilon} w(\Omega_{N\lambda}) d\lambda^{p-\varepsilon} \\ &+ N_0^{2p} \int_0^\infty w\left(\{x \in \mathbb{R}^n : |f(x)| > N^{-2}\lambda\}\right) d\lambda^{p-\varepsilon}\end{aligned}$$

Next, by the Layer Cake formula

$$\int_0^\infty w(\Omega_\lambda) d\lambda^{p-\varepsilon} = \int_{\mathbb{R}^n} Mf(x)^{p-\varepsilon} w(x) dx$$

and

$$\int_0^\infty N_0^{p-\varepsilon} w(\Omega_{N\lambda}) d\lambda^{p-\varepsilon} = \int_{\mathbb{R}^n} Mf(x)^{p-\varepsilon} w(x) dx.$$

Therefore, using $\delta N_0^\varepsilon < 1/2$ we can bring the last term to the right hand side and using

$$N_0^{2p} \int_0^\infty w\left(\{x \in \mathbb{R}^n : |f(x)| > N^{-2}\lambda\}\right) d\lambda^{p-\varepsilon} \approx \int_{\mathbb{R}^n} |f(x)|^{p-\varepsilon} w(x) dx$$

we obtain the desired result. □

3.4 Lecture Four.

We will prove a version of Theorem 3.3.6.

Theorem 3.4.1. *Let $w \in A_p$, $p > 1$. Then $\forall \alpha \in \mathbb{N}$ $\exists k_0 = k_0(\alpha, n, p, K)$ such that $\forall k \geq k_0$, $\forall \lambda > 0$, $\forall f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$w(\Omega_\lambda) \leq 2^{pk-\alpha} w(U_{2^k\lambda}) + 4^{pk} w\left(\{x \in \mathbb{R}^n : |f(x)| > 4^{-k}\lambda\}\right) \tag{3.4.1}$$

Proof. 1. Starting inequality, $\forall Q \subset \mathbb{R}^n$, $g \geq 0$

$$\frac{1}{|Q|}\int_Q g dx \leq C\left(\frac{1}{w(Q)}\int_Q g^p w dx\right)^{1/p}. \tag{3.4.2}$$

To see this use

$$\int (Mf)^p w dx \leq C \int |f|^p w dx$$

with $f = g\chi$ and write $g = g w^{1/p} w^{-1/p}$.

2. Calderon-Zygmund lemma: $\forall t > 0$ $\exists \{Q_j^t\}$ — dyadic cubes such that

   - $$t < \fint_{Q_j^t} f dx \leq 2^n t$$
   - $f(x) \leq t$ if $x \notin \bigcup_j Q_j^t = \{x \in \mathbb{R}^n : M^d f(x) > t\} =: U_t$

   (we assume $f \geq 0$). Here $M^d f$ is the dyadic maximal function.

3. Fix $Q_0 = Q_j^\lambda$.

   Claim:

$$w(Q_0) \leq 2^{pk-\alpha} w(U_{2^k\lambda} \cap Q_0) + 4^{pk} w\left(\{x \in Q_0 : |f(x)| > 4^{-k}\lambda\}\right). \tag{3.4.3}$$

   To prove the claim assume the contrary, i.e.

$$w(U_{2^k\lambda} \cap Q_0) < 2^{-pk+\alpha} w(Q_0)$$

   and

$$w(A_\lambda) < 4^{-pk} w(Q_0)$$

   where $A_\lambda = \{x \in Q_0 : |f(x)| > 4^{-k}\lambda\}$. We will get a contradiction.

Step 1. C-Z decomposition of f at level $2^k\lambda$

$$u(x) = \begin{cases} f(x), & \text{if} \quad x \in Q_0 \backslash Q_j^{2^k\lambda} \\ \fint_{Q_j^{2^k\lambda}} f(y) dy, & \text{if} \quad x \in Q_j^{2^k\lambda} \subset Q_0. \end{cases}$$

Note, that $|u(x)| \leq 2^{k+n}\lambda$, $\forall x \in Q_0$. Integrating u^p with weight w (we denote $Q_j = Q_j^{2^k\lambda}$) we have

$$\begin{aligned} \int_{Q_0 \backslash \bigcup_j Q_j} u^p w dx = \int_{Q_0 \backslash \bigcup_j Q_j} f^p w dx &\leq \int_{Q_0} f^p w dx \\ &\leq (4^{-k}\lambda)^p w(Q_0) + (2^{n+k}\lambda)^p w(A_\lambda) \\ &\leq C 2 \cdot 2^{-kp} \lambda^p w(Q_0) \end{aligned}$$

$$\int_{\bigcup_j Q_j} u^p w dx \leq (2^{n+k}\lambda)^p 2^{-kp+\alpha} w(Q_0) = 2^{np+\alpha}\lambda^p w(Q_0).$$

Step 2. C-Z decomposition of u at level $2^i\lambda$, $[k/2] \leq i \leq k-1$ to obtain $\{\tilde{Q}_j^i\} := \{\tilde{Q}_j^{2^i\lambda}\}$ such that

$$2^i\lambda \leq \fint_{\tilde{Q}_j^i} u(y)dy \leq 2^{i+n}\lambda$$

and let

$$u_i(x) = \begin{cases} u(x), & \text{if} \quad x \in Q_0 \backslash \tilde{Q}_j^i \\ \fint_{\tilde{Q}_j^i} u(y)dy, & \text{if} \quad x \in \tilde{Q}_j^i. \end{cases}$$

Recall, $\bigcup_j \tilde{Q}_j^i = \{x \in Q_0 : M^d u(x) > 2^i\lambda\} := \tilde{U}_{2^i\lambda}$.
From (3.4.2) we have

$$2^i\lambda \leq \fint_{\tilde{Q}_j^i} u(y)dy \leq C\left(\frac{1}{w(\tilde{Q}_j^i)} \int_{\tilde{Q}_j^i} u^p(y)w(y)dy\right)^p$$

Using that and the definition of $\tilde{U}_{2^i\lambda}$ we have

$$w(\tilde{U}_{2^i\lambda}) \leq C2^{-ip}\lambda^{-p} \int_{\tilde{Q}_j^i} u^p w \leq C2^{-ip} w(Q_0). \tag{3.4.4}$$

Step 3. Let

$$h = \frac{1}{k-[k/2]} \sum_{i=[k/2]}^{k-1} u_i,$$

from (3.4.2)

$$\lambda < \fint_{Q_0} h dx \leq C\left(\frac{1}{w(Q_0)} \int_{Q_0} h^p w dx\right)^{1/p}. \tag{3.4.5}$$

Next,

$$u_i \leq u\chi_{Q_0\backslash \tilde{U}_{2^i\lambda}} + 2^{n+i}\chi_{\tilde{U}_{2^i\lambda}} \leq u\chi_{Q_0\backslash \tilde{U}_{2^i\lambda}} + 2^{n+i}\chi_{\tilde{U}_{2^i\lambda}\cup U_{2^k\lambda}}$$

and therefore

$$h \leq u\chi_{Q_0\setminus\tilde{U}_{2^i\lambda}} + \frac{1}{k-[k/2]}\sum_{i=[k/2]}^{k-1} 2^{n+i}\chi_{A_i}$$

where we denote $A_i = \tilde{U}_{2^i\lambda} \cup U_{2^k\lambda}$. Denoting the last term in the above inequality by g and integrating h^p with weight w we obtain

$$\int_{Q_0} h^p w \leq 2^p \int_{Q_0\setminus U_{2^k\lambda}} u^p w + 2^p \int_{Q_0} g^p w.$$

The first term is bounded above by $2^{-kp}\lambda^p w(Q_0)$ and for the second one we get

$$\begin{aligned} 2^p \int_{Q_0} g^p w &\leq \frac{\lambda^p}{(k-[k/2])^p}\int_{Q_0}\sum_{i=[k/2]}^{k-1}\left(\sum_{j=[k/2]}^{i} 2^{n+j}\right)^p \chi_{A_i} w \\ &\leq \frac{\lambda^p}{(k-[k/2])^p}\sum_{i=[k/2]}^{k-1} 2^{ip} w(A_i) \\ &\leq \frac{C\lambda^p w(Q_0)}{(k-[k/2])^{p-1}} \end{aligned}$$

and therefore

$$\int_{Q_0} h^p w \leq C\lambda^p w(Q_0)\left(2^{-kp} + (k-[k/2])^{1-p}\right). \tag{3.4.6}$$

By taking k large enough from (3.4.5), (3.4.6) we get a contradiction.

□

3.5 Lecture Five

Theorem 3.5.1.

μ is p-admissible for $p > 1 \implies \mu$ is $(p-\varepsilon)$ − admissible, $\varepsilon > 0$, quantitatively

Proof. 1. Starting inequality, which states that

$$\frac{1}{l(Q)}\fint_Q |u - u_Q| d\mu \leq c\left(\fint_Q |\nabla u|^p d\mu\right)^{1/p}$$

for all $Q \subset \mathbb{R}^n$ and $u \in Lip(Q)$.

2. Fix a cube $Q_0 \subset \mathbb{R}^n$ and for all u we define the **sharp maximal function**

$$M^{\#}u(x) = \sup_{Q \ni x, Q || Q_0, Q \subset Q_0} \frac{1}{l(Q)} \fint_Q |u - u_Q| d\mu.$$

It's a maximal function of the mean oscillation and one should think of it as the maximal function of the gradient.

Lemma 3.5.2. *For all $\alpha \in \mathbb{N}$ there exists a $k_0 = k_0(\alpha) \in \mathbb{N}$ such that for all $k \geq k_0$ and all $\lambda > 0$ and all u we have*

$$\mu(U_\lambda) \leq 2^{kp-\alpha}\mu(U_{2^k\lambda}) + 4^{kp}\mu(\{x \in Q_0 : |\nabla u|(x) > 4^{-k}\lambda\}) \tag{3.5.1}$$

where $U_\lambda = \{x \in Q_0 : M^{\#}u(x) > \lambda\}$.

Given the above inequality we obtain the statement of the theorem as follows. Integrate (3.5.1) with respect to the measure $d\lambda^{p-\varepsilon}$ from 0 to ∞ to obtain

$$\int_{Q_0} (M^{\#}u)^{p-\varepsilon} d\mu \leq c \int_{Q_0} |\nabla u|^{p-\varepsilon} d\mu$$

Then using the definition of the sharp maximal function $M^{\#}$ we obtain the desired result.

To finish the proof we now need to prove Lemma 3.5.2.

Proof of Lemma 3.5.2. **Claim:** Fix Q_0 such that $\frac{1}{l(Q_0)} \fint_{Q_0} |u - u_{Q_0}| d\mu > \lambda$. Then

$$\mu(Q_0) \leq 2^{kp-\alpha}\mu(U_{2^k\lambda}) + 4^{kp}\mu(A_\lambda) \tag{3.5.2}$$

where $A_\lambda = \{x \in Q_0 : |\nabla u|(x) > 4^{-k}\lambda\}$.

To prove the claim assume the contrary, we will get a contradiction.

Step 1. Truncate $M^{\#}u$ at level $2^k\lambda$.

- For all x and y in Q_0

  $$|u(x) - u(y)| \leq C|x - y|(M^{\#}u(x) - M^{\#}u(y))$$

  To show this find a sequence of cubes one containing the other with $l(Q_i) \approx 2^{-1}$.

Then

$$\begin{aligned}|u(x) - u_{Q_1}| &\leq \sum_{i\geq 1} |u_{Q_i} - u_{Q_{i+1}}| \\ &\leq \sum_{i\geq 1} \fint_{Q_i} |u - u_{Q_i}| d\mu \cdot \frac{1}{l(Q_i)} \cdot l(Q_i) \\ &\leq M^{\#} u l(Q_i)\end{aligned}$$

and therefore

$$|u(x) - u_{Q_1}| \leq c|x-y| M^{\#} u(x).$$

- This implies that $u|_{Q_0 \setminus U_{2^k\lambda}}$ is $c2^k\lambda$-Lip by boundedness of the sharp maximal function.
- Using a theorem of McShane we extend this function to a function g which is Lipschitz in Q_0 with the same Lipschitz constant.
  (McShane 1934: $u|_A$ is c-Lip. Then $v(x) = \inf_{y\in A}(u(y) + c|y-x|)$ is c-Lip.)
- For the function g we have

$$\begin{aligned}\frac{1}{l(Q_0)} \fint_{Q_0} |g - g_{Q_0}| d\mu &\geq \lambda / c \\ \left(\fint_{Q_0} |\nabla g|^p d\mu \right)^{1/p} &\leq c\lambda\end{aligned}$$

using the definition of g and the countrary to (3.5.2).

Step 2. Truncate $M^{\#} g$ at level $2^i\lambda$ where $\lfloor k/2 \rfloor \leq i \leq k-1$ to obtain g_i. We apply what we did before to get

$$\mu(\{x \in Q_0 : M^{\#} g(x) > 2^i \lambda\} \leq c 2^{-ip} w(Q_0).$$

Step 3. Let

$$h = \frac{1}{k - \lfloor k/2 \rfloor} \sum_{i=\lfloor k/2 \rfloor}^{k-1} g_i$$

Then we have

$$\lambda \leq c \frac{1}{l(Q_0)} \fint_{Q_0} |h - h_Q| \leq C \left(\fint_{Q_0} |\nabla h|^p d\mu \right)^{1/p} \leq C \left(\frac{1}{(k - [k/2])^{p/2}} + 2^{-kp} \right) \lambda$$

and by taking k large enough we get a contradiction.

□

□

3.5.1 Open Problems

Here are three open problems

Open problem 3.5.1. *Characterization of p-admissible weights*

The problem to characterize p-admissible weights. Similar to the A_p weights. Note that in dimension 1 they are exactly the same as A_p weights. The characterization should be similar to the A_p weights

$$\fint_I g \leq \left(\frac{1}{w(I)} \int_I g^p w \right)^{1/p}$$

to show this condition is A_p is easy but the other way you need to use the Maximal function.

If f is a qc then we have the Jacobian to some power would be something, this statement would give a necessary condition.

Open problem 3.5.2. *Does a metric with non-negative Ricci curvature admit a Poincaré inequality?*

Proposition 3.5.3 (Borell-Brascamp-Lieb)**.** *Let $t \in [0,1]$ let $f, g, h \geq 0$ and $\int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} f = 1$. Assume that for all x and y*

$$\frac{1}{h(tx + (1-t)y)^{1/n}} \leq \frac{t}{f(x)^{1/n}} + \frac{1-t}{g(y)^{1/n}}$$

then $\int_{\mathbb{R}^n} h \geq 1$.

Open problem 3.5.3. *Consider the equation*

$$\nabla \cdot (w(x) \nabla u(x)) = 0.$$

What is a condition on $w(x)$ such that $u(x)$ is Lipschitz?

It is known that if $w(x) \in A_2$ then $u(x)$ is Hölder continuous.

Chapter 4

Mateu

4.1 Lecture 1

There notes are about properties and applications of singular integrals.

Properties
- Improvements of Cotlar's Inequality. Which amounts to "control of Tf by T^*f".
- Differences between odd and even kernels.

Applications
- Capacities and uniform rectifiability
- Quasi-conformal mappings
- Euler's Equations
- Sobolev Spaces

4.1.1

We will be working with kernels

$$K(x) = \frac{\Omega(x)}{|x|^n}$$

where Ω satisfies $\Omega(\lambda x) = \Omega(x)$ with zero average over the sphere

$$\int_{|x|=1} \Omega(x) dS(x)$$

where dS is the surface measure on the sphere.

We define operators

$$T^{\varepsilon} f(x) = \int_{|x-y|>\varepsilon} K(x-y) f(y) dy$$

so that

$$Tf(x) = p.v. \int K(x-y) f(y) dy := \lim_{\varepsilon \to 0} T^{\varepsilon} f(x).$$

We will also be interested in

$$T^* f(x) = \sup_{\varepsilon > 0} |T^{\varepsilon} f(x)|.$$

We will often want to control T by T^*. When the limit exists we have

$$Tf(x) \leq T^* f(x)$$

pointwise. We wil be after inequalities of the form

$$\|T^* f(x)\|_p \leq C_p \|Tf\|_p. \tag{4.1.1}$$

4.1.2

We will let M denote the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B f(y) dy. \tag{4.1.2}$$

Theorem 4.1.1 (Cotlar's Inequality)**.** *For all $1 < p < \infty$ and all $f \in L^p$ we have*

$$T^* f(x) \leq C \left(M(Tf)(x) + Mf(x) \right) \tag{4.1.3}$$

pointwise almost everywhere.

We aim to make improvements on this theorem by showing this by proving

$$T^* f(x) \leq CM(Tf)(x)$$

or

$$T^* f(x) \leq CM^2(Tf)(x)$$

4.1.3

In what follows we will be working in $\mathbb{R}^n$. For $d < n$ we can take $K(x) = x/|x|^{d+1}$ and some measure $\mu \geq 0$ and a modified truncated Riesz-Transform

$$R_\mu^\varepsilon := \int_{|x-y|>\varepsilon} f(y)\frac{x_i - y_i}{|x-y|^{d+1}} d\mu(y)$$

and ask the following questions

1. Is R_{mu} bounded in $L^2(\mu)$? When does $\lim_{\varepsilon\to 0} R_\mu^\varepsilon$ exists?
2. What abou the case when μ is a Uniformly Rectifiable Measure?

A measure μ is **uniformly rectifiable** provided there exist $\mathbb{C}^1$ manifolds E_i all of dimension d such that

- $\mathrm{supp}(\mu) = \bigcup_i E_i$
- $\mu|_{E_i} = cH^d$ where H^d denotes the Hausdorff Measure.

David-Semmes showed that (2) implies (1).

4.1.4 Hilbert, Cauchy and Beurling Transforms

The **Hilbert Transform** is defined by

$$Hf(x) = P.V. \int_{\mathbb{R}} f(x-y)\frac{dy}{y}.$$

The **Beurling Transform** is defined by

$$Bf(z) = \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dA(w) = P.V. \left(\frac{1}{z^2}\right) * f \tag{4.1.4}$$

It can be thought of like an even kernel version of the Hilbert Transform. Its definition relies on the fact that the complex numbers are a field. We will also be considering the **Cauchy Transform** which is

$$Cf(z) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{z-w} dA(w) = f * \frac{1}{\pi z}. \tag{4.1.5}$$

Proposition 4.1.2. *1. H is bounded in L^p for $p \in (0, \infty)$*

2. *Its Fourier multiplier is* $m = \left(\frac{1}{x}\right)^{\wedge}(\xi) = -isgn(\xi)$ *(Note that this implies it is an isometry in* L^2*)*

3. *It has* $H : L^\infty \to \mathrm{BMO}$.

Remark 4.1.3. The first follows from general theory of Calderon-Zygmund operators and proves $L^p(\mathbb{R})$ boundedness of the Carleson Operator.

An example of the third property is the following: Let $[a,b] = I$ then

$$H(\chi_I)(x) = \log|\frac{x-a}{x-b}| \in \mathrm{BMO}.$$

Proposition 4.1.4. *For all* $f \in \mathcal{S}(\mathbb{C})$ *we have*

$$\partial f = B(\bar{\partial} f) \tag{4.1.6}$$

in distribution sense.

Proof. We have[1]

$$\partial(\frac{1}{\pi z}) = \frac{-1}{\pi z^2}.$$

We also have

$$\bar{\partial}(\frac{1}{\pi z}) = \delta$$

since $\frac{1}{\pi z}$ is the elementary solution of the $\bar{\partial}$-problem. By definition property of the elementary solution we have

$$f = \bar{\partial}(f * \frac{1}{\pi z}).$$

This gives us

$$\partial f = \bar{\partial} f * \frac{-1}{\pi z^2}$$

which give the theorem. □

Corollary 4.1.5. *If* $f \in L^2(\mathbb{C})$ *and* $\bar{\partial} f \in L^2(\mathbb{C})$ *then* $f \in W^{1,2}$.

---

[1]Recall the notation

$$\begin{aligned} \bar{\partial} &= \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \partial &= \frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}. \end{aligned}$$

Proof. We just need to show that $\bar{\partial} f \in L^2(\mathbb{C})$. Well

$$\|\bar{\partial} f\|_{L^2(\mathbb{C})} = \|B(\partial f)\|_{L^2} \leq \|B\|_2 \|\bar{\partial} f\|_2 < \infty$$

since the Beurling transform is bouded in L^2. □

Remark 4.1.6. This actually works to show L^p boundedness.

Corollary 4.1.7. *For all $f \in \mathcal{S}$ we have*

$$\begin{aligned} \partial C f &= B f, \\ \bar{\partial} C f &= f \end{aligned}$$

‘

Proof. The first equation follows from the fact that $\partial(f * \frac{1}{\pi z}) = f * \partial \frac{1}{\pi z}$ and the second follows from the first and the fact that $f = B(\bar{\partial} f)$. □

Let $D \subset \mathbb{C}$ be the unit disc. We will compute $B(\chi_D)$. We will use the formula $\partial \varphi = B(\bar{\partial} \varphi)$ on the function

$$\varphi = \begin{cases} \bar{z}, & z \in D \\ 1/z, & z \in D^c \end{cases}.$$

Note that since $1/z = \bar{z}/|z|^2$ and $|z| = 1$ on ∂D the function is continuous. Now

$$\partial \varphi = \begin{cases} 0, & z \in D \\ -1/z^2, & z \in D^c \end{cases}.$$

and

$$\bar{\partial} \varphi = \begin{cases} 1, & z \in D \\ 0, & z \in D^c \end{cases} = \chi_D.$$

This gives

$$B(\chi_D) = B(\bar{\partial} \varphi) = \partial \varphi = \frac{-1}{z^2} \chi_{D^c} \in L^\infty.$$

This is an instance of a general phenomena

Theorem 4.1.8. *If D is a bounded domain with $\partial D \in \mathbb{C}^{1+\varepsilon}$ then $B(\chi_D) \in L^\infty$*

But it does not hold for kernels with higher powers in the denominator.

Proposition 4.1.9 (Garnett). *E bdd with $|E| > 0$ and $R(\chi_E) \notin L^\infty(\mathbb{C})$ where*

$$Rf(x) := \int \frac{f(x-y)y}{y^{n+1}} dA(y).$$

Proposition 4.1.10. *The multiplier of B is $\xi/\bar{\xi}$. That is*

$$\widehat{Bf}(\xi) = (\xi/\bar{\xi})\widehat{f}(\xi).$$

Proof.

$$\begin{aligned}
(1/\pi z^2)^\wedge &= (\partial \frac{1}{\pi z})^\wedge \\
&= 2\pi\zeta(\frac{1}{\pi z})^\wedge \\
&= 2\pi\zeta(\frac{2\pi i\bar{\zeta}}{2\pi i\bar{\zeta}}(\frac{1}{\pi z^2})^\wedge) \\
&= \frac{\zeta}{\bar{\zeta}}(\bar{\partial}\frac{1}{\pi z})^\wedge \\
&= \frac{\zeta}{\bar{\zeta}}(\delta)^\wedge
\end{aligned}$$

□

Corollary 4.1.11. *B is an isometry on $L^2(\mathbb{C})$: for all $f \in L^2(\mathbb{C})$*

$$\|Bf\|_2 = \|f\|_2.$$

This implies that for all

Conjecture 4.1.12 (T. Iwaniecz[2]). *For all $p > 2$*

$$\|B\|_p = p - 1.$$

4.1.5 Higher Order Reisz-Transforms

We will now define the Higher-Riesz Transforms which will generalize both the Cauchy and Beurling transforms. Let P_d be a harmonic polynomial in n-variables of degree d. A **higher order Riesz Transform** will mean an operator of the form

$$R_P f(x) := P.V. \int_{\mathbb{R}^n} f(x-y)\frac{P_d(y)}{|y|^{n+d}} dy.$$

[2] Twin brother of H. Iwaniecz the number theorist.

Remark 4.1.13. Note that both B and C are examples of such higher order Riesz Transforms.

By proposition 4.3.1 we have

$$\widehat{R_P f} = \gamma_d \frac{P_d(\xi)}{|\xi|^d} \hat{f}.$$

Since $P_d(\xi)$ and $|\xi|^d$ are homogeneous of the same degree that value of $P_d(\xi)/|\xi|^d$ only depends it's value on S^{n-1}. Since S^{n-1} is compact it achieves its maximum on the sphere which implies that $|P_d(\xi)/|\xi|^d \leq B$ (the value at the origin is comparable to the min and the max on the sphere) so by Plancharel we have boundedness in L^2.

We have more to say about it's L^2 norm

Proposition 4.1.14. *For every $f \in L^2$ we have*

$$\|R_P f\|_2 \leq C\|R_p f\|_2.$$

4.1.6 Type of Kernels

We are going to be working with smooth kernels $K(x) = \Omega(x)/|x|^n$ where $\Omega \in C^\infty(\mathbb{R}^n$ as satisfies

1. $\Omega(\lambda x) = \Omega(x)$
2. $\int_{S^{n-1}} \Omega(x) dS(x) = 0$

Note condition 1 says that Ω is determined by its restriction to S^{n-1}. By density of the harmonic polynomials on S^{n-1} we can write $\Omega = \sum P_j$ where P_j are harmonic polynomials.

Proposition 4.1.15 (Verdera, O, Mateu/+ Perez in odd case)**.** *Let T be an operator with even kernel $\Omega = \sum_j P_{2j}$. The following are equivalent*

1. *$T^* f(x) \leq CMTf(x)$ pointwise a.e.*
2. *$\|T^* f\|_2 \leq C\|Tf\|_2$*
3. *There exists some U invertible CZ-operator and R a higher order Riesz Transform with polynomial P such that*

   $$T = R \circ U,$$

   furthermore $P|P_{2j}$ for all j.

If we have an odd kernel $\Omega = \sum_j P_{2j+1}$ then we replace 1 by

1. $T^*f(x) \leq CM^2Tf(x)$.

Remark 4.1.16. One can try these estimated for $p \neq 2$. Going from 2 to p larger than 2 is easy. Going from 2 to 1 is hard.

There is also a weighted version of this due to A, Bosch, O,M the results are as follows for $1 < p < \infty$

- $\|Tf\|_p \leq C\|Tf\|_p$
- For any $w \in A_p$ we have

$$\|T^*f\|_{p,w} \leq C\|Cf\|_{p,w}.$$

- for $f \in H^1$ and $Tf \in L^1$ we have (in the even case)

$$\|T^*f\|_{1,\infty} \leq C\|Tf\|_1 \tag{4.1.7}$$

4.2 Lecture 2

4.2.1 Belatrami Equation

The **Belatrami Equation** is

$$\bar{\partial} f = \mu \partial f$$

where $\mu \in L^\infty(\mathbb{C})_{cmpt}$ with $\|\mu\|_\infty = K < 1.$ is the so called **Belatrami-Coefficient**.

Proposition 4.2.1. *A solution of the Belatrami Equation is $W^{1,2}_{loc}$,*

Definition 4.2.2. A function is K-**quasi-regular** (**quasi-analytic**) if and only if it is a solution of a Beltrami equation with $\|m\|_\infty \leq K$.

Remark 4.2.3. Note that when $K = 0$ the Belatrami equation becomes $\bar{\partial} f = 0$. This means 0-quasiregularity is equivalent to regularity (analyticity).

Definition 4.2.4. Let $A \subset \mathbb{C}$. If $f : A \to f(A)$ is K-quasiregular and a homeomorphism then we say it is K-quasi conformal.

Quasi-conformal maps on have similar rigidity to conformal maps (for people looking over these notes, what is the most general version of this?)

Proposition 4.2.5. *If f fixed three points the the solution is unique.*

Theorem 4.2.6 (Stoilow Factorization Theorem). *For every quasi-regular f there exists some g analytic and ϕ quasi-conformal such that*

$$f = g \circ \phi$$

Proposition 4.2.7. *A solution of the Belatrami equations of the form $f(z) = z + Ch(z)$ has $h = \sum_{n=0}^{\infty}(\mu B)^n \mu$. This implies that $h \in L^2$, $Ch \in W^{1,2}_{loc}$ and $f = z + Ch$ admits a solution h.*

Proof. We consider the ansatz $f = z + Ch$ where C is the Cauchy transform. We will solve for h. We have that $\bar{\partial} f = h + \bar{\partial} Ch = h$. We also have $\partial f = 1 + \partial Ch = 1 + Bh$. Since $\bar{\partial} f = \mu \partial f$ we have that $h - \mu Bh = \mu$ which gives

$$h = (1 - \mu B)^{-1} \mu = \sum_{n=0}^{\infty} (\mu B)^n \mu.$$

The series on the right is a Neumann series and the powers are compositions. It converges as an operator on L^2

$$\begin{aligned} \|\sum_{n \geq 0} (\mu B)^n \mu\| &= \sum_n \|(\mu B)^n \mu\|_2 && (4.2.1) \\ &= \sum_n \|\mu\|_\infty^n \|\mu\|_2 && \end{aligned}$$

(4.2.2)

which follows from the fact that $\|\mu\|_\infty < 1$ and $\|B\mu\|_2 = \|\mu\|_2$. ☐

Remark 4.2.8. If $p \neq 2$ then the crude in equation **??** becomes

$$\begin{aligned} \sum_{n \geq 0} \|(\mu B)^n \mu\|_p &= \|\mu\|_p \sum_{n \geq 0} (\|\mu\|_\infty \|B\|_p)^n \\ &= \frac{\|\mu\|_p}{1 - \|\mu\|_\infty \|B\|_p} \end{aligned}$$

which exists bounded provided $K\|B\|_p < 1$. If we assume the Iwaniecz conjecture that $\|B\|_p = p - 1$ then

$$(p-1)K < 1.$$

Iwaniecz-Astala obtained some index $p(K)$ such that $f \in W^{1,p(K)}$ for a K-quasiconformal solution of the Belatrami equation.

4.2.2

We will now look at solutions of the Belatrami equation when μ has some regularity[3]

Proposition 4.2.9 (Schander Estimates)**.** *If f is a solution of*

$$\begin{cases} \bar{\partial} f = \mu \partial f \\ \mu \in \mathrm{Lip}(\alpha) \end{cases}$$

where $\alpha \in (0,1)$, then $f \in \mathrm{Lip}_{loc}(1+\alpha)$.

Remark 4.2.10. The general philosophy is that if f is a solutions of Linear PDE with coefficients that are $\mathrm{Lip}(\alpha)$ we expect that f to be $\mathrm{Lip}(1+\alpha)$.

We have an improvement of the above

Theorem 4.2.11 (M, Orobitzm Verdarou)**.** *If f is a solution of*

$$\begin{cases} \bar{\partial} f = \mu \partial f, \\ \mu = g \chi_\Omega, \end{cases}$$

where $g \in \mathrm{Lip}(\alpha, \Omega)$ and $\partial\Omega \in \mathbb{C}^{1+\varepsilon}$ then $f \in \mathrm{Lip}_{loc}(1+\alpha')$

We can see that μ is $\mathrm{Lip}(\alpha)$ inside Ω and 0 on Ω^c.

Proposition 4.2.12. *Let f be as above. The quasi-conformal map associated to the equation is Bi-Lipschitz.*

4.2.3

Proposition 4.2.13. *Suppose that $\Omega \subset \mathbb{R}^2$ is bounded and $\partial\Omega \in \mathbb{C}^{1+\varepsilon}$ then the for all $f \in \mathrm{Lip}(\alpha, \Omega)$ we have*

$$\|Bf\|_{\mathrm{Lip}(\alpha,\Omega)} \leq C \|f\|_{\mathrm{Lip}(\alpha,\Omega)}$$

To prove this we use compactness of certain operators.

Proposition 4.2.14. *If $\mu \in \mathrm{VMO}$ then*

$$[\mu, B]f = \int \frac{\mu(x) - \mu(y)}{(x-y)^2} f(y) dA(y)$$

is compact in L^p but not compact in $\mathrm{Lip}(\alpha)$.

[3] $f \in \mathrm{Lip}(\alpha)$ if and only if there exists some C such that for all x and y we have $|f(x) - f(y)|/|x-y|^\alpha \leq C$. The best of the all possible C's for some given α defined $\|f\|_{\mathrm{Lip}(\alpha)}$.

4.2.4 General Facts About Singular Integrals

Proposition 4.2.15. *Suppose that T is a singular integral operator. Then*

1. $\|T^*f\|_2 \leq C\|Tf\|$
2. $T^*f(x) \leq MTf(x)$ *pointwise a.e.*

Lemma 4.2.16 (Cotlar's Lemma)**.** *Let $f \in L^2$ and T a singular integral operator*

$$T^*f(x) \leq C(MTf(x) + Mf(x))$$

$$\|T^*f\|_2$$

4.3 Appendix

Proposition 4.3.1. *Every polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ can be written uniquely as $\sum_{k=1}^{d} r^{2k} h_k(x)$ where $r^2 = x_1^2 + \cdots x_n^2$ and h_k is a harmonic polynomial.*

Proposition 4.3.2. *If P is a harmonic polynomial of degree d then*

$$\left(\frac{P(x)}{|x|^{n+d}}\right)^\wedge = \gamma_d \frac{P(\xi)}{|\xi|^d}.$$

Chapter 5

Jaramillo

5.1 Lecture One: The basic geometry of length in a metric space

Definition 5.1.1. Let (X, d) be a metric space. *A path* in X is a continuous mapping $\gamma\colon [a,b] \to X$. *The length* of γ is defined

$$\ell(\gamma) = \sup_p \left\{ \sum_{j=1}^{N} d(\gamma(t_j), \gamma(t_{j-1}))\colon p = \{a = t_0 < t_1 < \ldots < t_N = b\} \right\}.$$

Path γ is *rectifiable* if $\ell(\gamma) < +\infty$. *The length function* associated to γ is a mapping

$$s\colon [a,b] \to [0, \ell(\gamma)], \quad s(t) = \ell(\gamma|_{[a,t]}) \text{ for all } a \leq t \leq b.$$

Proposition 5.1.2 (Properties of length)**.** *We have the following properties which are easy to verify:*

1. $\ell(\gamma) \geq d(\gamma(a), \gamma(b))$*;*
2. $\ell(\gamma|_{[a,c]}) + \ell(\gamma|_{[c,b]}) = \ell(\gamma|_{[a,b]}) \quad \forall\, a < c < b$ *(additivity);*
3. *If γ is rectifiable, then the length function associated to γ is continuous (continuity property);*
4. *ℓ is lower semi-continuous, i.e. if $\{\gamma_n\}$ is a sequence of rectifiable paths $\gamma_n\colon [a,b] \to X$ and $\gamma_n \to \gamma$ pointwise on $[a,b]$, then $\ell(\gamma) \leq \liminf_{n\to\infty} \ell(\gamma_n)$.*

Proof. (4) Suppose that γ is a rectifiable path. Given $\varepsilon > 0$, there exists a partition $p = \{a = t_0 < t_1 < \ldots < t_N = b\}$ such that

$$\ell(\gamma) < \sum_{j=1}^{N} d(\gamma(t_j), \gamma(t_{j-1})) + \varepsilon/3,$$

and $n_0 \in \mathbb{N}$ such that $d(\gamma(t_j), \gamma_n(t_j)) < \varepsilon/(3N)$ for all $j = 1, 2, \ldots, N$ and $n \geq n_0$. Hence,

$$\begin{aligned}\ell(\gamma) &< \sum_{j=1}^{N} d(\gamma(t_j), \gamma(t_{j-1})) + \varepsilon/3 \\ &\leq \sum_{j=1}^{N} [d(\gamma(t_j), \gamma_n(t_j)) + d(\gamma_n(t_j), \gamma_n(t_{j-1})) + d(\gamma_n(t_{j-1}), \gamma_n(t_{j-1}))] + \varepsilon/3 \\ &\leq \sum_{j=1}^{N} d(\gamma_n(t_j), \gamma_n(t_{j-1})) + \varepsilon \leq \ell(\gamma_n) + \varepsilon.\end{aligned}$$

□

Remark 5.1.3. Function ℓ is lower semi-continuous, but in general not continuous. Indeed, let $X = \mathbb{R}^2$ with the Euclidean metric and consider paths $\gamma, \gamma_n \colon [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t,t)$, γ_n (see picture). Then $\gamma_n \to \gamma$ uniformly on $[0,1]$, but $\ell(\gamma_n) = 2$ for all n and $\ell(\gamma) = \sqrt{2}$.

Definition 5.1.4 (Main definition)**.** (X,d) is *rectifiably connected* if for all $x, y \in X$ there exists a rectifiable path γ from x to y; (X,d) is *a length space* if for all $x, y \in X$, $d(x,y) = \inf\{\ell(\gamma) \colon \gamma \text{ is a path from } x \text{ to } y\}$; (X,d) is *a geodesic space* if for all $x, y \in X$ there exists a path γ from x to y such that $\ell(\gamma) = d(x,y)$.

Remarks 5.1.1.

1. *If (X,d) is a length space, then (X,d) is rectifiably connected;*

2. *(X,d) is a length space if and only if for all $x, y \in X$ and for all $\varepsilon > 0$, there exists a path γ from x to y such that $\ell(\gamma) \leq d(x,y) + \varepsilon$;*

3. *If (X,d) is a geodesic space, then (X,d) is a length space but the converse is not in general true. (Consider e.g. $X = \mathbb{R}^2 \setminus \{p\}, p \in \mathbb{R}^2$, with the Euclidean metric. Then (X,d) is a length space but it is not a geodesic space.)*

Definition 5.1.5. Suppose (X,d) is rectifiably connected. Then we can define *the intrinsic metric (inner metric)* d_i on X:

$$d_i(x,y) := \inf\{\ell(\gamma) \colon \gamma \text{ a path from } x \text{ to } y\}$$

It is easy to see that d_i is a metric on X and that $(d_i)_i = d_i$ for all metrics d on X. Further, if (X, d) is rectifiably connected then (X, d) is a length space if and only if $d = d_i$. Note that we always have $d \leq d_i$. In particular, (X, d_i) is a length space.

Examples 5.1.2. *1. Let*

$$X = \bigcup_{n=1}^{\infty} [(0,1), (\frac{1}{n}, 0)] \cup [(0,1), (0,0)],$$

an union of segments in $\mathbb{R}^2$ with the Euclidean metric (see picture). Then

$$d_i((\frac{1}{n}, 0), (\frac{1}{m}, 0)) \geq 2 \text{ if } n \neq m$$

showing that (X, d) is not a length space; $\{(1/n, 0)\}$ is not convergent in (X, d_i).

2. Let $X = \mathbb{R}^2$ and $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + \sqrt{|y_1 - y_2|}$. Then d is equivalent to the Euclidean distance. However,

$$d_i((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2|, & \text{if } y_1 = y_2; \\ +\infty & \text{if } y_1 \neq y_2. \end{cases}$$

Indeed, suppose $y_1 < y_2$ and let $\gamma\colon [a, b] \to \mathbb{R}^2$, $\gamma = (\gamma_1, \gamma_2)$, be a path from (x_1, y_1) to (x_2, y_2). Consider $\gamma_2\colon [a, b] \to \mathbb{R}^2$. For every $n \in \mathbb{N}$ there exists a partition $p = \{a = t_0 < t_1 < \ldots < t_n = b\}$ such that

$$\gamma_2(t_j) = y_j + \frac{j}{n}(y_2 - y_1);$$

$$\ell(\gamma) \geq \textstyle\sum_{j=1}^{n} \sqrt{\frac{1}{n}(y_2 - y_1)} = |y_2 - y_1| \frac{n}{\sqrt{n}} \to \infty \text{ as } n \to \infty.$$

Definition 5.1.6. (X, d) is (C-)*quasi-convex* if $\exists\, C > 0$ such that for all $x, y \in X$ there exists a path γ from x to y such that $\ell(\gamma) \leq C d(x, y)$.

Remark 5.1.7. Suppose (X, d) is rectifiably connected. Then (X, d) is quasi-convex if and only if d is Lipschitz-equivalent to a length space, i.e. there exists a metric d' such that (X, d') is a length space and $\exists\, h\colon (X, d) \to (X, d')$ which is bi-Lipschitz. Implication $\Rightarrow$ is easy; let us show $\Leftarrow$: Suppose (X, d) is rectifiably connected. Then (X, d_i) is a length space and there holds $d \leq d_i \leq cd$.

Example 5.1.3. *Consider*

$$X = \mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty} R_n; \quad R_n = (2n+1, 2n+2) \times (0, n+1)$$

with the Euclidean metric (see picture). Then (X, d) is not quasi-convex but is locally 2-quasi-convex: For all $p \in X$, $B(p,1)$ is 2-quasi-convex, i.e. if $x, y \in B(p,1)$ then there exists a path γ from x to y such that $\ell(\gamma) \leq 2d(x,y)$.

Remark 5.1.8. If $\varphi\colon [c,d] \to [a,b]$ is continuous, non-decreasing and surjective (onto) and $\gamma\colon [a,b] \to X$ is a path, then $\ell(\gamma \circ \varphi) = \ell(\gamma)$.

Proposition 5.1.9 (Arc-length parametrization)**.** *Let $\gamma\colon [a,b] \to (X,d)$ be a rectifiable path. Then there exists a unique path $\hat{\gamma}\colon [0, \ell(\gamma)] \to (X,d)$ such that $\hat{\gamma} \circ s = \gamma$, where $s(t) = \ell(\gamma|_{[a,t]})$ for all $t \in [a,b]$ is the length function associated to γ. As a consequence,*

$$\ell(\hat{\gamma}|_{[0,\tau]}) = \tau \text{ for all } 0 \leq \tau \leq \ell(\gamma) \quad \text{and} \quad \ell(\hat{\gamma}) = \ell(\gamma).$$

In fact, $\ell(\hat{\gamma}|_{[\tau_1,\tau_2]}) = \tau_2 - \tau_1$ for all $0 \leq \tau_1 < \tau_2 \leq \ell(\gamma)$. The mapping $\hat{\gamma}$ is called the arc-length parameterization of γ.

Proof. For every $\tau \in [0, \ell(\gamma)]$ choose $t \in [a,b]$ such that $s(t) = \tau$. (If $\tau < \tau'$ with $s(\tau) = s(\tau')$ then $\ell(\gamma|_{[\tau,\tau']}) = 0$ implying that $\gamma|_{[\tau,\tau']}$ is constant and hence, $\gamma(t) = \gamma(t')$.) We can define $\hat{\gamma}(\tau) = \gamma(t)$. Then prove that $\hat{\gamma}$ is 1-Lipschitz (and therefore continuous) and that $\ell(\hat{\gamma}|_{[0,\tau]}) = \tau$. $\square$

5.2 Lecture Two: Defining line integral

Definition 5.2.1. Let $\gamma\colon [a,b] \to X$ be a rectifiable path and suppose $u\colon \mathrm{Im}\gamma \to [0, +\infty]$ is a Borel function. We define

$$\int_\gamma u := \int_0^{\ell(\gamma)} u(\hat{\gamma}(t))dt.$$

5.2.1 Lipschitz speed

Let $\gamma\colon [a,b] \to X$ be a path. We define its *Lipschitz speed* at the point $t \in (a,b)$ as the limit (if it exists)

$$|\dot{\gamma}|(t) := \lim_{s \to 0} \frac{d(\gamma(t+s), \gamma(t))}{|s|} \quad \textit{(“metric derivative”)}$$

Remark 5.2.2. If $X = (V, \|\cdot\|)$ is a normed space and $\gamma\colon [a,b] \to V$ is differentiable, then $|\dot{\gamma}|(t) = \|\gamma'(t)\|$.

Theorem 5.2.3. *Let $\gamma\colon [a,b] \to (X,d)$ be a Lipschitz path. Then the Lipschitz speed exists for a.e. $t \in (a,b)$ and*

$$\ell(\gamma) = \int_a^b |\dot{\gamma}|(t)dt.$$

References:

Burago, Burago & Ivanov: "A course in metric geometry". A.M.S
Survey by Hajtasz: "Sobolev spaces on metric spaces". Contemporary Math. vol. 338 (2003)

5.2.2 Midpoints in length spaces

Definition 5.2.4. Let (X,d) be a metric space and suppose $x, y \in X$. We say that $z \in X$ is *a midpoint of x and y* if

$$d(x,z) = \frac{1}{2}d(x,y) = d(y,z).$$

Remark 5.2.5. Given two points, a midpoint may exist or not, and it need not to be unique. (See picture.)

Example 5.2.1. *Let $(X,d) = (\mathbb{R}^2, \|\|_1)$. (picture)*

Definition 5.2.6. Let (X,d) be a metric space and suppose $x, y \in X$. We say that for $\varepsilon > 0$, $z \in X$ is *ε-midpoint of x and y* if

$$|2d(x,z) - d(x,y)| < \varepsilon \quad \text{and} \quad |2d(y,z) - d(x,y)| < \varepsilon.$$

Theorem 5.2.7. *Suppose (X,d) is a complete metric space. Then*

1. *(X,d) is geodesic space if and only if for all $x, y \in X$ there exists $z \in X$, a midpoint of x and y;*

2. *(X,d) is length space if and only if for all $x, y \in X$, for all $\varepsilon > 0$ there exists $z \in X$, ε-midpoint of x and y.*

Proof. (1) $\Rightarrow$: (Here we do not need compactness.) Given x, y there exists a path $\gamma\colon [a,b] \to X$ s.t. $\ell(\gamma) = d(x,y)$. By continuity, we find $t \in [a,b]$ such that $\ell(\gamma|_{[a,t]}) = \frac{1}{2}d(x,y)$. By additivity, $\ell(\gamma|_{[t,b]}) = \frac{1}{2}d(x,y)$. Thus, $z = \gamma(t)$ is a midpoint of x and y.

$\Leftarrow$: We need to show that there exists a path from x to y. We define $\gamma\colon [0,1] \to X$ as follows. Let $\gamma(0) = x$ and $\gamma(1) = y$; $\gamma(1/2) = z$, midpoint of x and y; $\gamma(1/4) = z_1$, midpoint of x and z and $\gamma(3/4) = z_2$, midpoint of z and y. In this way we define γ on the disjoint numbers $k/2^n$ for all $k \in [0,n]$, $n \in \mathbb{N}$. By construction, $d(\gamma(t), \gamma(t')) = |t - t'| d(x,y)$ for such points, and γ is $d(x,y)$-Lipschitz. Thus, by compactness, we may extend γ to $[0,1]$ with the same Lipschitz constant. We obtain a path $\gamma\colon [0,1] \to X$ from x to y with $\ell(\gamma) = d(x,y)$.

The proof for (2) is similar. $\square$

Corollary 5.2.8 (Characterization of a length space)**.** *Suppose (X,d) is a complete metric space. Then the following conditions are equivalent.*

1. *(X,d) is a length space.*
2. *For all $x, y \in X$ and $\varepsilon > 0$, there exists an ε-chain from x to y, i.e. we can find a set of points $\{x = z_0, z_1, \ldots, z_n = y\}$ with the properties that $d(z_j, z_{j-1}) < \varepsilon$ for all $j = 1, \ldots, n$ and*

$$\sum_{j=1}^{n} d(z_j, z_{j-1}) < d(x,y) + \varepsilon.$$

5.2.3 Hopf–Rinow Theorem in the setting of a length space

Let us recall the following well-known theorem.

Theorem 5.2.9 (Ascoli–Arselà)**.** *Suppose (X,d) is a compact metric space and let $\gamma_n\colon [a,b] \to X$ be a sequence of paths in X with uniformly bounded length (i.e. there exists a constant $k > 0$ such that $\ell(\gamma) < k$ for all $n \in \mathbb{N}$). Then there exists a subsequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ which is uniformly convergent on $[a,b]$ to a path $\gamma\colon [a,b] \to X$.*

By Ascoli–Arselà, we have the following theorem.

Theorem 5.2.10 (Hopf–Rinow)**.** *Suppose (X,d) is a complete and locally compact (i.e. for all $x \in X$ there exists a ball $B \ni x$ which is compact) length space. Then*

1. *(X,d) is a proper metric space: every closed ball in X is compact;*
2. *(X,d) is a geodesic space: for all $x, y \in X$ there exists a path γ from x to y such that $\ell(\gamma) = d(x,y)$.*

Proof. (1) Let $x \in X$. Then there exists $r > 0$ such that $\bar{B}(x, r)$ is compact. Consider $0 < R = \sup\{r > 0\colon \bar{B}(x, r) \text{ is compact }\} \leq +\infty$. We want to prove that $R = \infty$. Suppose, for a contradiction, that $R < \infty$. We see that $B = \bar{B}(x, R)$ is compact since it is precompact (and the space is complete). Given $\varepsilon > 0$, we find an $\varepsilon/2$-net S for $B' = \bar{B}(x, R - \varepsilon/3)$. For every $y \in B$, since X is a length space, we find $y' \in B'$ such that $d(y, y') < \varepsilon/2$, and there exists $s \in S$ such that $d(y', s) < \varepsilon/2$. Hence, S is an ε-net for B.

(2) Let $x, y \in X$ and $R = d(x, y) + 1$. Then $\bar{B}(x, R)$ is compact, and for every $n \in \mathbb{N}$ there exists a path γ_n from x to y such that $\ell(\gamma_n) \leq d(x, y) + 1/n$ and $(\gamma_n) \subset \bar{B}(x, R)$. We assume that all γ_n are defined on the same interval $[a, b]$. By Ascoli–Arselà, there exists a subsequence $\gamma_{n_k} \to \gamma\colon [a, b] \to X$. By lower semi-continuity of length, $\ell(\gamma) \leq \liminf_{n\to\infty} \ell(\gamma_n) = d(x, y)$ showing that $\ell(\gamma) = d(x, y)$. □

5.3 Lecture Three: Spaces of pointwise-Lipschitz functions

Suppose (X, d) is a metric space. We say that function $f\colon X \to \mathbb{R}$ is Lipschitz if

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < +\infty.$$

Then $\mathrm{Lip}(f)$ is called *the Lipschitz constant* of f.

We define the following function spaces:

$$\mathrm{Lip}(X) = \{f\colon X \to \mathbb{R} \mid f \text{ is Lipschitz}\}$$
$$\text{For a fixed } x_0 \in X \text{ define } \|f\|_{x_0} := |f(x_0)| + \mathrm{Lip}(f)$$
$$\mathrm{Lip}^\infty(X) = \{f\colon X \to \mathbb{R} \mid f \text{ is bounded and Lipschitz}\}$$
$$\|f\|_{\mathrm{Lip}^\infty} := \|f\|_\infty + \mathrm{Lip}(f)$$

$(\mathrm{Lip}(X), \|\cdot\|_{x_0})$ and $(\mathrm{Lip}^\infty(X), \|\cdot\|_{\mathrm{Lip}^\infty})$ are Banach spaces; $(\mathrm{Lip}^\infty(X), \|\cdot\|_{\mathrm{Lip}^\infty})$ is even a dual Banach space.

5.3.1 The Arens–Eells Space

$AE(X)^* = \mathrm{Lip}^\infty(X)$. The Arens–Eells space $AE(X)$ is a canonical predual of $\mathrm{Lip}^\infty(X)$: $\delta_x \in \mathrm{Lip}^\infty(X)^*, \delta_x(f) = f(x)$ for all f. Then

$$\overline{[\delta_x\colon x \in X]}^{\mathrm{Lip}^\infty(X)^*} = AE(X).$$

Reference: N. Weaver: ”Lipschitz Algebra”

Definition 5.3.1. Suppose (X, d) is a metric space, $f\colon X \to \mathbb{R}$ and let $x \in X$. We define

$$\mathrm{Lip} f(x) = \begin{cases} \limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)}, & \text{if } x \text{ is not isolated} \\ 0, & \text{if } x \text{ is isolated.} \end{cases}$$

This defines a function $\mathrm{Lip} f\colon X \to \bar{\mathbb{R}}$. Then let

$$\begin{aligned} D^\infty(X) &= \{f\colon X \to \mathbb{R} \,|\, f \text{ is bounded and } \mathrm{Lip} f \text{ is bounded}\} \\ \|f\|_{D^\infty} &:= \|f\|_\infty + \|\mathrm{Lip} f\|_\infty \end{aligned}$$

Further define

$$D(X) = \{f\colon X \to \mathbb{R} \,|\, \mathrm{Lip} f \text{ is bounded}\}.$$

(The space $D(X)$ does not have a natural norm.)

Remarks 5.3.1.

- $\mathrm{Lip}^\infty(X) \subset D^\infty(X) \subset C(X)$
- *If* $f \in \mathrm{Lip}(x)$ *then* $\mathrm{Lip} f(x) \leq \mathrm{Lip}(f)$ *and* $\|\mathrm{Lip} f\|_\infty \leq \mathrm{Lip}(f)$
- *Suppose* $f \in D(X)$ *and* $x \in X$ *is not isolated. Given* $\varepsilon > 0$ *there exists* $r > 0$ *such that*

$$|f(x) - f(y)| \leq (\mathrm{Lip} f(x) + \varepsilon) d(x, y) \underset{y\to x}{\to} 0 \quad \forall\, y \in B(x, r)$$

Question: Are the two spaces $\mathrm{Lip}^\infty(X)$ and $D^\infty(X)$ equivalent?

In general, $\mathrm{Lip}^\infty(X) \subsetneq D^\infty(X)$.

Lemma 5.3.2. *Suppose* $f \in D^\infty(X)$ *and let* $\gamma\colon [a, b] \to X$ *be a path from* x *to* y. *Then*

$$|f(x) - f(y)| \leq \left| \sup_{z\in \mathrm{Im}\gamma} \mathrm{Lip} f(z) \right| \cdot \ell(\gamma).$$

Proof. Fix $\varepsilon > 0$. For every $t \in [a, b]$ there exists $r_t > 0$ such that

$$|f(\gamma(t)) - f(z)| \leq \underbrace{\big(\mathrm{Lip} f(\gamma(t)) + \varepsilon\big)}_{\leq K+\varepsilon} \cdot d(\gamma(t), z) \quad \forall\, z \in B(\gamma(t), r_t),$$

where we have introduced the notation

$$K := \sup_{z\in \mathrm{Im}\gamma} \mathrm{Lip} f(z).$$

For $t \in [a,b]$ and $r_t > 0$ there exists $I_t = (t-\delta_t, t+\delta_t)$ such that $\gamma(I_t) \subset B(\gamma(t), r_t)$, and $[a,b] \subset I_{t_1} \cup \ldots \cup I_{t_n}$ with $I_{t_j} \cap I_{t_{j+1}} \neq \emptyset$ for all $j = 1, \ldots, n-1$. We can select intermediate points $a \leq t_1 \leq s_1 \leq t_2 \leq s_2 \leq \ldots \leq s_{n-1} \leq t_n \leq b$ where each $s_j \in I_{t_j} \cap I_{t_{j+1}}$ for $j = 1, \ldots, n-1$. The points $a, b, t_1, \ldots, t_n, s_1, \ldots, s_{n-1}$ form a partition of $[a,b]$ which we call $p = \{\tau_i\}_{i=1}^m$. We have

$$\begin{aligned} |f(x) - f(y)| = |f(\gamma(b)) - f(\gamma(a))| &\leq \sum_{i=1}^{m} |f(\gamma(\tau_i)) - f(\gamma(\tau_{i-1}))| \\ &\leq \sum_{i=1}^{m} (K+\varepsilon) d(\gamma(\tau_i) - \gamma(\tau_{i-1})) \leq (K+\varepsilon)\ell(\gamma). \end{aligned}$$

□

As an answer our question, we have the following.

Corollary 5.3.3. *If (X,d) is a quasi-convex metric space, then $\mathrm{Lip}^\infty(X) = D^\infty(X)$ (with equivalent energy norms: there exists $C > 0$ such that $\mathrm{Lip}(f) \leq C \|\mathrm{Lip} f\|_\infty$).*

Proof. Let $x, y \in X$ and $\varepsilon > 0$. By quasi-convexity, there exists a path γ from x to y such that $\ell(\gamma) \leq C d(x,y)$. Suppose $f \in D^\infty(X)$. Then by the previous Lemma,

$$|f(x) - f(y)| \leq \|\mathrm{Lip} f\|_\infty \cdot C d(x,y).$$

From this we see that $f \in \mathrm{Lip}^\infty(X)$ and the Lipschitz constant of f satisfies $\mathrm{Lip}(f) \leq C \|\mathrm{Lip} f\|_\infty$. □

In general, $\mathrm{Lip}^\infty(X) \subsetneq D^\infty(X)$.

Examples 5.3.2.

1. *Let $X = [0,1) \cup (1,2] \subset \mathbb{R}$ with the Euclidean metric. Consider $f = \chi_{[0,1)}$. Then $f \in D^\infty(X)$ since $\mathrm{Lip} f(x) = 0$ for every $x \in X$ but $f \notin \mathrm{Lip}(X)$.*

2. *Example of a compact connected space X with $\mathrm{Lip}^\infty(X) \neq D^\infty(X)$: Let*

$$X = \{(t^3, t^2)\colon -1 \leq t \leq 1\} = \{(x,y) \in \mathbb{R}^2 \colon x^2 = y^3, 0 \leq x \leq 1\} \quad \textit{(cusp)}$$

with the Euclidean metric (see picture).

Consider $f\colon X \to \mathbb{R}$,

$$f(x,y) = \begin{cases} y, & x \geq 0 \\ -y, & x \leq 0. \end{cases}$$

Then $f \in D^\infty(X) \setminus \mathrm{Lip}^\infty(X)$.

3. We may have $D^\infty(X) = \mathrm{Lip}^\infty(X)$ without equivalent energy semi-norms: Let

$$X = \mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty} R_n; \quad R_n = (2n+1, 2n+2) \times (0, n+1)$$

with the Euclidean metric. Note that X is not quasi-convex but is locally 2-quasiconvex (see Example 5.1.3). Let $f \in D^\infty(X)$. If f is not Lipschitz, then there exist $\{x_k\}$ and $\{y_k\}$ such that $|f(x_k) - f(y_k)| \geq K d(x_k, y_k)$ for all $K > 0$, implying that

$$d(x_k, y_k) \leq \frac{|f(x_k) - f(y_k)|}{K} \leq \frac{2\,\|f\|_\infty}{K} \to 0 \quad \textit{as } k \to \infty.$$

Thus, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ there holds $d(x_k, y_k) \leq 1/2$. This means that x_k and y_k belong to the same ball $B(p_k, 1)$, and by Lemma 5.3.2,

$$|f(x_k) - f(y_k)| \leq 2\,\|\mathrm{Lip} f\|_\infty \cdot d(x_k, y_k).$$

By the following theorem, this is a contradiction.

Theorem 5.3.4. *Suppose (X, d) is a complete, connected and locally compact metric space. Then $D^\infty(X) = \mathrm{Lip}^\infty(X)$ with equivalent energy semi-norms if and only if X is quasi-convex.*

Proof. x $\Leftarrow$: This we have already seen.

$\Rightarrow$: Fix $\varepsilon > 0$ and $x \in X$. Let

$U_{\varepsilon,x} := \{y \in X\colon$ there exists an ε-chain $x = z_0, \ldots, z_n = y; d(z_j, z_{j+1}) < \varepsilon \,\forall\, j = 1, \ldots, n-1\}$.

$U_{\varepsilon,x}$ is non-empty, open and closed. So, $U_{\varepsilon,x} = X$. We define

$$d_\varepsilon(x, y) = \inf \left\{ \sum_{j=1}^{n} d(z_j, z_{j-1})\colon z_0, z_1, \ldots, z_n \text{ is an } \varepsilon\text{-chain from } x \text{ to } y \right\};$$

then $d(x, y) \leq d_\varepsilon(x, y)$ by the triangle inequality, and $d(x, y) = d_\varepsilon(x, y)$ if $d(x, y) < \varepsilon$. With the fixed $\varepsilon > 0$ and x, consider $f_{x,\varepsilon} : z \mapsto d_\varepsilon(x, z)$. Then $\mathrm{Lip} f_{x,\varepsilon}(z) \leq 1$ for all $z \in X$ and hence, $\|\mathrm{Lip} f_{x,\varepsilon}\|_\infty \leq 1$. Given $y \in X$, let $N > d_\varepsilon(x, y)$ and consider $f_{x,\varepsilon,N} = \min\{N, f_{x,\varepsilon}\} \in D^\infty(X) = L^\infty(X)$. There exists $C > 0$ such that $\mathrm{Lip} f_{x,\varepsilon,N} \leq C\,\|\mathrm{Lip} f_{x,\varepsilon,N}\|_\infty = C$. We have

$$d_\varepsilon(x, y) = f_{x,\varepsilon}(y) = f_{x,\varepsilon,N}(y) = |f_{x,\varepsilon,N}(y) - f_{x,\varepsilon,N}(x)| \leq C d(x, y).$$

Thus, $d_\varepsilon(x, y) \leq C d(x, y)$ for all x, y and for all $\varepsilon > 0$. Hence, we can obtain a path γ from x to y with $\ell(\gamma) \leq C d(x, y)$, and this shows quasi-convexity. $\square$

5.4 Lecture Four: Newton–Sobolev Spaces of infinity type

5.4.1 Metric measure spaces

Let (X, d) be a metric space. A mapping μ is a Borel regular mapping if

- μ is a outer measure defined on every subset of X;
- every Borel set A in X is measurable, i.e. for all $E \subseteq X$ there holds $\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$;
- every $E \subseteq X$ is contained in some Borel set A such that $\mu(E) = \mu(A)$;
- we also require that for all $x \in X$ and $r > 0$ there holds $0 < \mu(B(x, r)) < \infty$.

Definition 5.4.1. The measure μ is *doubling* if there exists a constant $C \geq 1$ such that for all balls $B(x, r)$ there holds

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

We recall the following well-known lemma.

Lemma 5.4.2. *The measure μ is doubling if and only if there exist constants $C' \geq 1$ and $s > 0$ such that for all $x, y \in X$, $0 < r \leq R$ with $x \in B(y, r)$ there holds*

$$\frac{\mu(B(x, r))}{\mu(B(y, R))} \leq C' \left(\frac{r}{R}\right)^s.$$

5.4.2 Newtonian–Sobolev spaces $N^{1,\infty}(X)$

In classical situation let $\Omega \subseteq \mathbb{R}^n$ open. Then we define

$$W^{1,\infty}(\Omega) = \{u \in L^\infty(\Omega) \colon u \text{ admits a distributional gradient } \nabla u \in L^\infty(\Omega, \mathbb{R}^n)\};$$
$$\text{for all } i = 1, \ldots, n \text{ we have } \frac{\partial u}{\partial x_i} \in L^\infty(\Omega).$$

Recall that

$$D^\infty(\Omega) = \{u \colon \Omega \to \mathbb{R} \mid u \text{ is bounded and } \exists\, L > 0 \text{ s.t. } u \text{ is } L\text{-locally Lipschitz on } \Omega\}.$$

Theorem 5.4.3. *If Ω is an open subset of $\mathbb{R}^n$, then $W^{1,\infty}(\Omega) = D^\infty(\Omega)$.*

Proof. It is enough to consider the case when $\Omega = B$ is a ball.

1. If $u \in W^{1,\infty}(B)$ then $u \in W^{1,p}(B), p > n$, and the Sobolev embedding theorem gives that u is continuous. In the same way we can see that it is Lipschitz.

2. If u is uniformly locally Lipschitz then, by Rademaher theorem, u is differentiable a.e. and the classical gradient ∇u defined a.e. is a distributional gradient. □

Corollary 5.4.4. *If Ω is quasi-convex then* $W^{1,\infty}(\Omega) = \mathrm{Lip}^{\infty}(\Omega)$.

The idea in the following: (Heinonen, Koskela, Shanmugalingam)

If $u\colon \Omega \to \mathbb{R}$ is smooth and $\gamma\colon [a,b] \to \Omega$ is a rectifiable path from x to y, then $u \circ \gamma\colon [a,b] \to \mathbb{R}$ is absolutely continuous, and we may apply the Fundamental Theorem of Calculus: we have that

$$\begin{aligned} |u(y) - u(x)| = |u(\gamma(b)) - u(\gamma(a))| &= \left| \int_a^b (u \circ \gamma)'(t) dt \right| \\ &\leq \int_a^b |\nabla u(\gamma(t))| \cdot |\gamma'(t)| dt = \int_a^b |\nabla u(\gamma(t))| dt \end{aligned}$$

since we may assume that $|\gamma'(t)| = 1$ by considering the arc-length parameterization of γ. Hence,

$$|u(y) - u(x)| \leq \int_\gamma \|\nabla u\|$$

for every rectifiable path γ from x to y.

Next we will generalize the definition of a gradient to measure spaces.

Definition 5.4.5. Let (X, d, μ) be a metric measure space and $u\colon X \to \mathbb{R}$. We say that a non-negative Borel function $g\colon X \to [0,\infty]$ is an upper gradient for u if

$$|u(y) - u(x)| \leq \int_\gamma g$$

for every rectifiable path γ from x to y.

Remarks 5.4.1. 1. *If it happens that $u \in C^{\infty}(\Omega)$ then we may choose $g = |\nabla u|$ in the above definition.*

2. *If g is an upper gradient and $\tilde{g} = g$ a.e. then $\tilde{g}$ may not be an upper gradient. Indeed, we may have that $\tilde{g} \equiv +\infty$ on* $\mathrm{Im}\gamma$.

We denote

$$\Upsilon = \{\gamma\colon [a,b] \to X \mid \gamma \text{ is a non-constant rectifiable path}\}.$$

Definition 5.4.6 (Modulus of a family of paths)**.** Let $\Gamma \subseteq \Upsilon$ be a family of non-constant rectifiable paths on X. For $1 \le p < \infty$ we define

$$\mathrm{Mod}_p(\Gamma) = \inf_\rho \left\{ \int_X \rho^p d\mu \mid \rho\colon X \to [0,\infty] \text{ Borel and } \int_\gamma \rho \ge 1 \text{ for all } \gamma \in \Gamma \right\}.$$

For $p = \infty$ we define

$$\mathrm{Mod}_\infty(\Gamma) = \inf_\rho \left\{ \|\rho\|_{L^\infty} \mid \rho\colon X \to [0,\infty] \text{ Borel and } \int_\gamma \rho \ge 1 \text{ for all } \gamma \in \Gamma \right\}.$$

Remark 5.4.7. Mod_p is an outer measure on Υ. We say that a property holds for p-a.e. path if it holds except for a set of paths with $\mathrm{Mod}_p = 0$.

Lemma 5.4.8. *Suppose $\Gamma \subseteq \Upsilon$. Then the following are equivalent:*

(a) $\mathrm{Mod}_\infty(\Gamma) = 0$

(b) There exists $\rho \in L^\infty(X)$ such that

$$\int_\gamma \rho = +\infty \quad \textit{for all } \gamma \in \Gamma.$$

Proof. $(a) \Rightarrow (b)$: By (a), for all $n \in \mathbb{N}$ there exists $\rho_n \in L^\infty(X)$ such that

$$\int_\gamma \rho_n \ge 1 \quad \text{and} \quad \|\rho_n\|_{L^\infty} < \frac{1}{2^n}.$$

Consider $\rho = \sum_{n=1}^\infty \rho_n$. Then

$$\|\rho\|_{L^\infty(X)} \le \sum_{n=1}^\infty \frac{1}{2^n} = 1,$$

hence $\rho \in L^\infty(X)$, and

$$\int_\gamma \rho = \sum_{n=1}^\infty \int_\gamma \rho_n = +\infty.$$

$(b) \Rightarrow (a)$: For every $n \in \mathbb{N}$, consider $\rho_n = \frac{1}{n}\rho \in L^\infty(X)$ with

$$\|\rho_n\|_{L^\infty} \le \frac{1}{n} \|\rho\|_{L^\infty} \to 0 \quad \text{and} \quad \int_\gamma \rho_n = \frac{1}{n} \int_\gamma \rho = +\infty \ge 1.$$

We see that $\mathrm{Mod}_\infty(\Gamma) = 0$ for such a family. $\square$

Definition 5.4.9. Let (X, d, μ) be a metric measure space and $u\colon X \to \mathbb{R}$. We say that a non-negative Borel function $g\colon X \to [0, +\infty]$ is an (∞)-weak upper gradient for u if

$$|u(y) - u(x)| \leq \int_\gamma g$$

for ∞-a.e. rectifiable path γ from x to y.

Lemma 5.4.10. *If g is an ∞-weak upper gradient and $\varepsilon > 0$ then there exists g_ε upper gradient such that $g \leq g_\varepsilon$ and $\|g - g_\varepsilon\|_{L^\infty} < \varepsilon$.*

Proof. Let Γ be the set of paths for which g is not an upper gradient. Then $\mathrm{Mod}_\infty(\Gamma) = 0$. By Lemma 5.4.8 there exists $\rho \in L^\infty(X)$ with

$$\int_\gamma \rho = +\infty \quad \text{for all } \gamma \in \Gamma.$$

We define

$$g_\varepsilon = g + \frac{\varepsilon\rho}{1 + \|\rho\|_{L^\infty}} \geq g$$

□

Definition 5.4.11. Let (X, d, μ) be a metric measure space. We define

$$\begin{aligned}\tilde{N}^{1,\infty}(X) &= \{u \in L^\infty(X)\colon \exists\ \infty\text{-weak upper gradient } g \in L^\infty(X)\}\\ \|u\|_{\tilde{N}^{1,\infty}} &= \|u\|_{L^\infty} + \inf_g \|g\|_{L^\infty}\end{aligned}$$

where the last term in the summation is called *the energy semi-norm*. We say that $u \sim v$ in $\tilde{N}^{1,\infty}(X)$ if $\|u - v\|_{\tilde{N}^{1,\infty}} = 0$. Note that $\|\cdot\|_{\tilde{N}^{1,\infty}}$ is not a norm and we need to take quotients. Thus, we define

$$\begin{aligned}N^{1,\infty}(X) &= \tilde{N}^{1,\infty}(X)/\sim\\ \|u\|_{N^{1,\infty}} &= \|u\|_{\tilde{N}^{1,\infty}}.\end{aligned}$$

Then $(N^{1,\infty}(X), \|\cdot\|_{N^{1,\infty}})$ is a Banach space (Shanmugalingam).

Definition 5.4.12. For $E \subset X$ we define

$$\mathrm{Cap}_\infty(E) = \inf\{\|u\|_{N^{1,\infty}} : u \in N^{1,\infty}, u|_E \geq 1\}.$$

Cap_∞ is an outer measure on X.

Remarks 5.4.2. *1. It may happen that $u \in N^{1,\infty}$ and $u = v$ a.e. but $v \notin L^{1,\infty}$. Indeed, let $X = [-1.1]$ with the Lebesgue measure. Consider $u \equiv 1 \in N^{1,\infty}$ and*

$$v = \begin{cases} 1, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0. \end{cases}$$

Then $u = v$ a.e. Let us show that $v \notin L^{1,\infty}$. Suppose, for a contradiction, that g is an upper gradient for v. Then we have that

$$|v(0) - v(1)| \leq \int_\gamma g \quad \text{for all paths } \gamma \text{ from } 0 \text{ to } 1.$$

But since $|v(0) - v(1)| = +\infty$, this implies that

$$\int_\gamma g = +\infty \quad \text{and hence } g \notin L^\infty(X).$$

2. But if $u = v$ everywhere except the set of E with $\mathrm{Cap}_\infty(E) = 0$, then $u \in N^{1,\infty}$ if and only if $v \in N^{1,\infty}$.

Proposition 5.4.13. *Let (X, d, μ) be a metric measure space. Then $D^\infty(X) \hookrightarrow N^{1,\infty}(X)$ and $\|\cdot\|_{N^{1,\infty}} \leq \|\cdot\|_{D^\infty}$.*

Proof. Suppose $u \in D^\infty(X)$. Then we see that $g = \mathrm{Lip}u$ is an upper gradient for u. Let γ be a rectifiable path from x to y and suppose γ is parameterized by arc-length (hence, 1-Lipschitz). Then $u \circ \gamma \colon [a, b] \to \mathbb{R}$ is pointwise Lipschitz. By Stepanov differentiation theorem[1], $u \circ \gamma$ is differentiable a.e. on $[a, b]$. Hence,

$$|u \circ \gamma(b) - u \circ \gamma(a)| \leq \int_0^1 |(u\circ)'(t)|dt \leq \int_0^1 \mathrm{Lip}u(\gamma(t))dt = \int_\gamma \mathrm{Lip}u.$$

∎

5.4.3 Sobolev spaces $M^{1,p}(X)$ (Hajtasz)

Definition 5.4.14. Let (X, d, μ) be a metric measure space, and $1 \leq p \leq \infty$. $M^{1,p}(X)$ is the space of all functions $u \in L^p(X)$ such that there exist $N \subseteq X, \mu(N) = 0$, and $0 \leq g \in L^p(X)$ with

$$|u(x) - u(y)| \leq (g(x) + g(y))d(x, y) \quad \text{for all } x, y \in X \setminus N.$$

[1] *Stepanov differentiation theorem:* Let $\Omega \subset \mathbb{R}^n$ be open and $f \colon \Omega \to \mathbb{R}^n$. Then f is differentiable a.e. on $\{x \in \Omega \colon \mathrm{Lip}f(x) < \infty\}$

For such functions u we define

$$\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p} \text{ (where } g \text{ satisfies the above conditions).}$$

Remarks 5.4.3. *1.* $(M^{1,p}(X), \|\cdot\|_{M^{1,p}})$ *is a Banach space (we saw this on Shanmugalingam's course).*

2. For $p = \infty$, $u \in M^{1,\infty}(X)$ *implies that* u *is Lipschitz on* $X \setminus N$ *which is dense in* X*. Thus,* u *can be extended to a Lipschitz function in* X*, and we see that* $\mathrm{Lip}^\infty(X) = M^{1,\infty}(X)$*.*

3. In general, $\mathrm{Lip}^\infty(X) = M^{1,\infty}(X) \subset D^\infty(X) \subset N^{1,\infty}(X)$ *for* $1 \le p < \infty$*. We also have* $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$ *with* $\|\cdot\|_{N^{1,p}(X)} \le 2\|\cdot\|_{M^{1,p}(X)}$*. For example, if* X *is the snowflake then* X *has no rectifiable path and thus,* $M^{1,p}(X) \ne N^{1,p}(X) = L^p(X)$*.*

5.5 Lecture Five: Poincaré inequalities

Definition 5.5.1. We say that (X, d, μ) admits a p-Poincaré inequality (p-PI) for $1 \le p < \infty$ if there exists $C > 0$ and $\lambda \le 1$ such that for all Borel functions $u\colon X \to \mathbb{R}$ and all $g\colon X \to [0, \infty]$ which are upper gradients for u there holds that

$$⨍_B |u - u_B| d\mu \le Cr \left(⨍_{\lambda B} g^p d\mu \right)^{1/p} \quad \forall \text{ balls } B = B(x, r).$$

Here $\lambda B = B(x, \lambda r)$, and as usual

$$f_E = ⨍_E f = \frac{1}{\mu(E)} \int_E f d\mu$$

denotes the integral average of $f \in L^1_{\mathrm{loc}}(x)$ over a measurable set E.

5.5.1 The Euclidean case $\mathbb{R}^n$

For $X = \mathbb{R}^n$ we have

$$⨍_B |u - u_B| d\mathcal{L}^n \le C \mathrm{rad}(B) \left(⨍_{\lambda B} \|\nabla u\|^p d\mathcal{L}^n \right)^{1/p} \quad (\lambda = 1). \tag{5.5.1}$$

Reference: Heinonen: "Nonsmooth calculus" Bull AMS 2007

In case (5.5.1) holds with $\lambda > 0$, we say that the space admits a *weak* p-Poincaré inequality.

Remark 5.5.2. If X is a geodesic space then we can choose $\lambda = 1$.

Cheeger: Suppose X is doubling and admits p-Poincaré inequality for some $1 < p < \infty$. Then X admits a so-called "measurable differentiable structure" with respect to which every Lipschitz function $f\colon X \to \mathbb{R}$ is differentiable a.e. (Rademacher theorem holds). This gives measurable cotangent structure on X: for every $f \in \mathrm{Lip}(X)$ there is df a.e.

S. Keith extended the result of Cheeger (Lip-lip condition). For $f \in \mathrm{Lip}(X)$, let

$$\begin{aligned}
\mathrm{Lip}f(x) &= \limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)} = \limsup_{r\to 0}\left[\sup_{y\in B(x,r)} \frac{|f(x)-f(y)|}{d(x,y)}\right]\\
\mathrm{lip}f(x) &= \liminf_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)}.
\end{aligned}$$

Then X satisfies *the Lip-lip condition* if there exists $K > 0$ such that

$$\mathrm{Lip}f(x) \le K\mathrm{lip}f(x) \quad \text{for e.a. } x \in X, \forall\, f \in \mathrm{Lip}(X).$$

Remark 5.5.3. By Hölders inequality, if X admits p-Poincaré inequality for some $1 < p < \infty$ then X admits g-Poincaré inequality for all $q \ge p$.

Theorem 5.5.4 (Keith–Zhong, 2008)**.** *Suppose (X, d, μ) is doubling and admits p-PI for some $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that X admits $(p-\varepsilon)$-PI.*

Theorem 5.5.5. *Suppose (X, d, μ) is doubling and admits p-PI for some $1 < p < \infty$. Then $M^{1,p}(X) = N^{1,p}(X)$ with equivalent norms.*

In the Euclidean case $X = \mathbb{R}^n$ and with $1 < p < \infty$ there holds $M^{1,p}(\mathbb{R}^n) = N^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Definition 5.5.6. We say that (X, d, μ) satisfies ∞-PI if there exist $C > 0$ and $\lambda \ge 1$ such that for all Borel functions $u\colon X \to \mathbb{R}$ and all $g\colon X \to [0, \infty]$ which are upper gradients for u and all balls B there holds that

$$⨍_B |u - u_B|\,d\mu \le C\mathrm{rad}(B)\,\|g\|_{L^\infty(\lambda B)}\,.$$

Remark 5.5.7. p-PI implies ∞-PI but the reverse implication is not true. Indeed, consider $X = \{(x, y) \in \mathbb{R}^2 \colon -x \le y \le x\}$ with

$$d\mu = e^{-\frac{1}{|x|^2}}\,d\mathcal{L}^2.$$

In this case μ is not doubling.

Open problem 5.5.1. *Is there an equivalence between ∞-PI and the Lip-lip-condition?*

Definition 5.5.8. (X, d, μ) is *thick-quasiconvex* if there exists $C > 0$ such that for all $x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x, y)$ and $E \subset B(x, \varepsilon), F \subset B(y, \varepsilon)$ with $\mu(E)\mu(F) > 0$ there holds that

$$\mathrm{Mod}_\infty(\Gamma(E, F, C)) > 0$$

where $\Gamma(E, F, C) = \{\gamma\colon \gamma \text{ is a path from } p \in E \text{ to } q \in F \text{ with } \ell(\gamma) \leq Cd(p, q)\}$.

Theorem 5.5.9. *Suppose (X, d, μ) is doubling and admits ∞-PI. Then X is thick-quasiconvex.*

Remark 5.5.10. If (X, d, μ) is complete and doubling, then thick-quasiconvexity implies quasiconvexity. To see this, first note that completeness and the doubling property imply that X is proper. Indeed, we notice that $\bar{B}(x, r)$ is precompact: by assuming that it is not, we consider a maximal ε-net in $\bar{B}(x, r)$ which is infinite; then $\mu(\bar{B}(x, r)) = +\infty$ which is a contradiction (since μ is doubling and thereby locally finite).

Then we choose $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$. For all $j \in \mathbb{N}$ we find γ_j from $x_j \in B(x, \varepsilon)$ to $y_j \in B(y, \varepsilon)$ with $\ell(\gamma_j) \leq Cd(x_j, y_j) \leq 2Cd(x, y)$. Since X is proper, by Arzela–Ascoli, there exists $\{\gamma_{i_k}\}$ with $\gamma_{i_k} \to \gamma$ with $\ell(\gamma) \leq 2Cd(x, y)$ implying also that γ is a path from x to y.

Theorem 5.5.11 (Durand, J, Shanmugalingam)**.** *Suppose (X, d, μ) is complete and doubling. Then the following conditions are equivalent.*

- *X admits ∞-PI;*
- *X is thick-quasiconvex;*
- $\mathrm{Lip}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy semi-norms (i.e. there exists $C > 0$ s.t. for all $u \in N^{1,\infty}(x)$ we have*

$$C \inf\{\|g\|_{L^\infty} : g \text{ an upper gradient for } u\} \geq \mathrm{Lip}\, u.)$$

Chapter 6

Magnani

6.1 Lecture 1: Preliminaries

Before introducing stratified groups, let us first introduce the following notion.

6.1.1

We say that W is a **graded vector space** if there exists $W_1, ..., W_\iota$ nontrivial linear subspaces such that $W = W_1 \oplus ... \oplus W_\iota,\ \iota \in \mathbb{N}^+$. This means that any $w \in W$ can be **uniquely** written as $w = w_1 + ... + w_\iota$, where $w_\ell \in W_\ell$ for any $\ell = 1, ..., \iota$. Then the projections $\tilde{H}_{W_\ell}(w) = w_\ell,\ \tilde{H}_{W_\ell} : W \to W_\ell$ are well defined.

- The elements of W_ℓ are called **homogeneous vectors of degree** ℓ.
- We set the integers
$$m_0 = 0 \text{ and } m_\ell = \sum_{s=1}^{\ell} \dim W_s$$
for all $\ell = 1, 2, ..., \iota$ and define
  - The **degree function** $d : \{1, ..., \dim W\} \to \{1, ..., \iota\},\ d_j = i \iff i \in \mathbb{N}^+$ is the unique integer such that $m_{i-1} < j \leq m_i$.
  - a **graded basis** $(e_1, ..., e_n)$ of W is a basis of W such that for all $\ell = 1, ..., \iota$,
$$(e_{m_{\ell-1}+1}, e_{m_{\ell-1}+2}, ..., e_{m_\ell})$$
is a basis of W_ℓ.

6.1.2

We say that a graded vector space $\mathbb{G} = H \oplus ... \oplus H_\iota$ has a **triangular group operation** if there exists a polynomial mapping $p : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that

1. $p(x, y) = \sum_{\ell=1}^{\iota} p_\ell(x, y)$ with $p_\ell \in H_\ell \; \forall x, y \in \mathbb{G}$.
2. $p_\ell(x, y) = p_\ell \left(\sum_{s=1}^{\ell-1} x_s, \sum_{s=1}^{\ell_1} y_s \right)$, with $x_s, y_s \in H_s,\; 2 \le \ell \le \iota,\; 1 \le s \le \iota$.
3. $p(tx, \tau x) = 0 \forall t, \tau \in \mathbb{R} \; \forall x \in \mathbb{G}$, and
4. $x \cdot y = x + y + p(x, y) \; \forall x, y \in \mathbb{G}$ defines a group operation of $\mathbb{G}$.

- **Example 1**
  $\mathbb{G} = H_1,\; p \equiv 0,\; x \cdot y = x + y$ is the trivial triangular group operation, where the group structure coincides with the linear structure of $\mathbb{G}$.
- **Example 2**
  $\mathbb{G} = H_1 \oplus H_2,\; p_2 : H_2 \times H_1 \to H_2$ a bilinear skew-symmetric mapping,
  $p(x, y) = p_2(x_1, y_1),\; x \cdot y = x + y + p_2(x_1, y_1)$ is a triangular group operation.
  - This is indeed the **general model** for any **two step stratified group**.
- **Exercise 1**
  Show in the previous example that $x \cdot y$ is indeed a triangular group operation.
- **Exercise 2**
  Let $\mathbb{G}$ be a graded vector space equipped with a graded basis and a triangular group operation. Let $\ell_x : \mathbb{R}^n \to \mathbb{R}^n,\; \ell_x(z) = x \cdot z$ w.r.t. the basis. Show that the Euclidean Jacobian of ℓ_x is one, namely, the left translation ℓ_x preserves the Lebesgue measure. (Hint: Show that the Jacobian matrix is upper triangular with ones on the diagonal.)

6.1.3 Vector Fields on Vector Spaces and Lie Algebras

Let W be an $n-$dimensional vector space with a basis $(e_1, ..., e_n)$ and the corresponding coordinates $(\xi_1, ..., \xi_n) \in \mathbb{R}^n$ with $x = \sum_{j=1}^{n} \xi_j e_j$. A smooth vector field $X \in \mathfrak{X}(W)$ is $X(x) = \sum_{j=1}^{n} d_j(x) e_j \; d_j \in C^\infty(W)$. We canonically identify X with the **derivation** $X : C^\infty(W) \to C^\infty(W)$ defined as $Xu(x) = \sum_{j=1}^{n} d_j(x) \frac{\partial}{\partial \xi_j} u(x)$, hence $X = \sum_{j=1}^{n} d_j \partial_{\xi_j}$ is an equivalent notion. This correspondence is a linear isomorphism.

6.1.4

For $X = \sum_{j=1}^{n} d_j \partial_{\xi_j}$, and $Y = \sum_{j=1}^{n} b_j \partial_{\xi_j}$,

$$[X, Y] = \sum_{i,j=1}^{n} (d_i \partial_{\xi_i} b_j - b_i \partial_{\xi_i} d_j) \partial_{\xi_j}$$

is a new vector field called a **Lie bracket** of X and Y. It vanishes if and only if the vector fields "commute".

6.1.5 Left Invariant Vector Fields and Lie Algebras

Let $\mathbb{G}$ be a graded linear space equipped with a triangular polynomial group operation. Let $X \in \mathfrak{X}(\mathbb{G})$ be a vector field. We say that X is **left invariant** if for any $x \in \mathbb{G}$

$$X(x) = d\ell_x(0) X(0)$$

where $\ell_x : \mathbb{G} \to \mathbb{G}$, $\ell_x(z) = x \cdot z$ is the **left translation** by x.

Proposition 6.1.1. *If X, Y are left invariant, then so is $[X, Y]$.*

The linear combinations of left invariant vector fields of $\mathbb{G}$ form its Lie algebra, denoted by $\mathcal{G}$.

- **Exercise 3**
  Show that any left invariant vector field satisfies $X(x \cdot y) = d\ell_x(y) X(y)\ \forall x, y \in \mathbb{G}$.

- **Exercise 4**
  Observing that $[X, Y]u = XYu - YXu\ \forall u \in C^\infty(\mathbb{G})$ as a derivation, show the following:

  Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

- **Exercise 5**
  Show that $\dim \mathcal{G} = \dim G$.

- **Exercise 6**
  Let $\mathbb{G}$ be equipped with a triangular group operation. Show that the left invariant vector fields $X_1, ..., X_n$ spanned by a graded basis $(e_1, ..., e_n)$ have the form
  $$X_j(x) = \partial_{\xi_j} + \sum_{i=1}^{n} d_j^i(x)\partial_{\xi_i}$$
  where $d_j^i(x) = \frac{\partial}{\partial \eta_j} p^i(x, \eta)\Big|_{\eta=0}$ and
  $$d_j^i(x) = d_j^i\left(\sum_{d_s \leq d_{\ell-1}} \xi_s e_s\right),$$
  Where $p(x, y) = \sum_{i=1}^{n} p^i(x, y)e_i$.

6.2 Lecture 2: Stratified Groups

6.2.1

Let $S, W \subset \mathcal{G}$ be linear subspaces of the Lie algebras of $\mathbb{G}$ equipped with a triangular group operation. Then we set

$$[S, W] = span\{[X, Y] : X \in S,\ Y \in W\}.$$

Let $\mathbb{G}$ be a graded vector space with grading $\mathbb{G} = H_1 \oplus ... \oplus H_\iota$ and let $\mathbb{G}$ be equipped with a triangular group operation. Define the subspaces

$$V_\ell := \{X \in \mathcal{G} : X(0) \in H\ell\}$$

where $\mathcal{G}$ is the Lie algebra of $\mathbb{G}$. Then $\mathbb{G}$ is stratified if

i. $[V_1, V_\ell] = V_{\ell+1}\ \forall \ell = 1, 2, ..., \iota - 1$.

ii. $[V_1, V_{\iota+1}] = \{0\}$.

iii. it is canonically equipped with dilations

$$\delta_r : \mathbb{G} \to \mathbb{G},\ \delta_r x = \sum_{\ell=1}^{\iota} r^\ell x_\ell\ \forall r > 0.$$

Theorem 6.2.1. *Let $\mathbb{G}$ be a stratified group. Then the dilations $\delta_r 5 : \mathbb{G} \to \mathbb{G}$ are group homomorphisms, namely $\delta(x \cdot y) = (\delta_r x)(\delta_r y)\ \forall r > 0,\ \forall x, y \in \mathbb{G}$.*

Main point of the proof: Observe that the exponential mapping $\exp : \mathcal{G} \to \mathbb{G}$ is $\exp X = X(0)$. We are looking at the group in the best possible system of coordinates.

- **Exercise 7**
Let $\mathbb{G}$ be a stratified group equipped with dilations $\delta_r : \mathbb{G} \to \mathbb{G},\ r > 0$. Let $(e_1, ..., e_n)$ be a graded basis and let d_i be the degree of e_i. Show that $\delta_r x = \sum_{i=1}^n r^{d_i} \xi_i e_i$, where $x = \sum_{j=1}^n \xi_i e_j$. Conclude that

$$J_{\delta_r} = r^{\sum_{i=1}^n d_i} \quad \text{(the Euclidean Jacobian of } \delta_r \text{ seen as a mapping of } \mathbb{R}^n)$$

If $\mathbb{G} = H_1 \oplus ... \oplus H_\iota$, conclude that

$$\sum_{i=1}^n d_i = \sum_{j=1}^\iota j \dim H_j.$$

(This is the homogeneous dimension of $\mathbb{G}$.)

- **Exercise 8**
Let $\mathbb{G}$ be a stratified group with dilations δ_r and triangular group operation given by p. Show that $\delta_r p^j(x, y) = r^{d_j} p(x, y) \quad \forall j = 1, ..., n$. Then conclude that the linear independent vector fields are

$$X_j(x) = \partial_{\xi_j} + \sum_{i: d_i > d_j} d_j^i(x) \partial_{\xi_i}$$

snd $d_j^i(\delta_r x) = r^{d_i - d_j} d_j^i(x)$. Hence

$$d_j^i(x) = d_j^i\left(\sum_{d_p < d_i} \xi_p e_p\right).$$

6.2.2

A **homogeneous norm** $||\cdot|| : \mathbb{G} \to [0, +\infty)$ is a continuous function such that

1. $||x|| = 0$ iff $x = 0$.
2. $||\delta_r x|| = r||x|| \quad \forall x \in \mathbb{G},\ \forall r > 0$.
3. $||x \cdot y|| \leq ||x|| + ||y|| \quad \forall x, y \in \mathbb{G}$.
4. $||x|| = ||x^{-1}|| \quad \forall x \in \mathbb{G}$.

6.2.3

A **homogeneous distance** $d : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ is a continuous distance on $\mathbb{G}$ such that

1. $d(x \cdot y, x \cdot z)d(y, z) \quad \forall x, y, z \in \mathbb{G}$.
2. $d(\delta_r x, \delta y) = rd(x, y) \quad \forall x, y \in \mathbb{G}, \ \forall r > 0$.

- **Exercise 9**
  Show that any homogeneous norm $||\cdot||$ defines a homogeneous distance $d(x, y) := ||x^{-1}y||$ and conversely, and homogeneous distance defines a homogeneous norm $||x|| := d(x, 0)$.

In analogy with linear spaces, where norms are characterized by bounded, symmetric, and convex open sets, homogeneous norms can also be characterized by a special class of “generalized” convex sets”.

6.2.4

We say that $A \subset \mathbb{G}$ is **Minkowski convex** if or any $0 < t < 1, \ x, y \in A \implies (\delta_t x)(\delta_{1-t} y) \in A$. Furthermore, we say that A is **balanced** if $x \in A \implies x^{-1} \in A$.

Theorem 6.2.2 (Hebisch-Sikora, 1990)**.**

a) Let A be an open, bounded, balanced, and Minkowski convex set of $\mathbb{G}$. Then $||x|| = \inf\{t > 0 : \delta_{1/t} x \in A\}$ is a homogeneous norm on $\mathbb{G}$.

b) If $|| \cdot ||$ is a homogeneous norm, then $\{x \in \mathbb{G} : ||x|| < 1\}$ is open, bounded, balanced, and Minkowski convex.

c) In any stratified group there exists an open, bounded, balanced, and Minkowski convex set A such that

   i) ∂A is smooth.

   ii) $\frac{d}{dt}(\delta_t x)\big|_{t=1} \notin T_x \partial A$ for all $x \in \partial A$.

   and this implies that $|| \cdot ||$ is smooth.

Statements a) and b) are simple exercises, whereas c) requires more work, the details are in the original paper.

- **Example 3** The Heisenberg group.
  $\mathbb{H}^n = H_1 \oplus H_2,\ \dim H_1 = 2n,\ \dim H_2 = 1,$
  $x \cdot y = x + y + w(x_1, y_1)$
  $x : H_1 \times H_1 \to H_2 \simeq \mathbb{R}$ skew-symmetric non-degenerate.
  $\delta_r x = r x_1 + r^2 x_2$ dilation.
  Without invoking the general theorem about existence of smooth homogeneous norms, we can construct explicitly an example of a homogeneous norm. We fix an Euclidean norm $|\cdot|$ on $\mathbb{H}^n$ that makes H_1 and H_2 orthogonal, hence $||x|| = \max\{|x_1|, \sqrt{\alpha|x_2|}\}$ where $0 < \alpha \leq \frac{2}{c}$ if $|x(x_1, y_1)| \leq c|x_1||y_1|$.
  Remark 6.2.3. $\mathbb{H}^n$ equipped with the previous group operation and dilations is called the $(2n+1)-$dimensional Heisenberg group.

- **Exercise 10**
  Let $w(x_1, y_1) = \sum_{j=1}^{n} (\xi_j \eta_{j+n} - \xi_{n+j}\eta_j) e_{2n+1}$, where $(e_1, ..., e_{2n})$ is a basis of H_1 and e_{2n+1} spans H_2. $x_1 = \sum_{j=1}^{2n} \xi_j e_j,\ y_1 = \sum_{j=1}^{2n} \eta_j e_j$ with both belonging to H_1.

  a) Compute the vector fields $X_1, ..., X_{2n+1}$ of the lie algebra of $\mathbb{H}^n$, where $X_i(0) = e_i$.
  b) Show that $\mathbb{H}^n$ is a stratified group.
  c) Show that $||x|| = \max\{|x_1|, \sqrt{\alpha|x_2|}\}$, with α as in example 3, is indeed a homogeneous norm.

6.2.5 Geometric Measures on Stratified Groups

Let (X, d) be a metric space, let $\alpha > 0$ and choose $c_\alpha > 0$ as a "normalizing factor", and $t > 0$. Then

$$\mathcal{H}_t^\alpha(E) = \inf \left\{ \sum_{j=1}^{\infty} c_\alpha \left(\frac{diam\ E_j}{2} \right)^\alpha : E \subset \bigcup_{j=1}^{\infty} E_j,\ diam\ E_j \leq t \right\}$$

is the "size $t-$approximating measure", and $\mathcal{H}^\alpha(E) = \sup_{t>0} \mathcal{H}_t^\alpha(E)$ is the $\alpha-$dimensional Hausdorff measure. For $x \in X$ and $r > 0$, define the closed ball $D(x, r) = \{z \in X : d(z, x) \leq r\}$. Clearly $diam\ D(x, r) \leq 2r$, and in general the inequality is strict.

$X = [0, 1]^2$ Euclidean norm	$diam\ D(0, 1) = 1 < 2$

- **Exercise 11**
  Show that in a stratified group $\mathbb{G}$ equipped with a homogeneous norm, $D(x,r) = 2r$

6.2.6

Assume that in (X,d) we have $diam\ D(x,r) = 2r \ \ \forall x \in X,\ \forall r > 0$, define

$$\mathcal{S}^\alpha_t(E) = \inf\left\{\textstyle\sum_{j=1}^\infty c_\alpha r_j^\alpha : E \subset \bigcup_{j=1}^\infty E_j,\ diam\ E_j \le t\right\}$$
$$\mathcal{S}^\alpha = \sup_{t>0} \mathcal{S}^\alpha_t$$

- **Exercise 12**
  Show that $\mathcal{H}^\alpha \le \mathcal{S}^\alpha \le 2^\alpha \mathcal{H}^\alpha$.

$$\text{Hint: } E \subset \bigcup_{j=1}^\infty E_j,\ diam\ E_j \le t\ \mathcal{S}^\alpha_{2t}(E) \le \sum_{j=1}^\infty c_\alpha (diam E_j)^\alpha$$

Proposition 6.2.4. *If $\mathcal{H}^\alpha(E) > 0$, then $\mathcal{H}^\beta(E) = +\infty \ \ \forall \beta < \alpha$, and if $\mathcal{H}^\alpha(E) < +\infty$, then $\mathcal{H}^\beta(E) = 0 \ \ \forall \beta > \alpha$.*

Proof. With $diam\ E_j \le t,\ E \subset \bigcup E_j$, we have

$$\begin{aligned} c_\beta \sum_{j=1}^\infty \frac{(diam\ E_j)^\beta}{2^\beta} &\ge c_\beta t^{\beta-\alpha} \sum_{j=1}^\infty \frac{(diam\ E_j)^\alpha}{2^\beta} \\ &\ge c_\beta t^{\beta-\alpha} \mathcal{H}^\alpha_t(E) \\ \implies \mathcal{H}^\beta_t(E) &\ge t^{\beta-\alpha} \mathcal{H}^\alpha_t(E) \end{aligned}$$

The remaining part is analogous. □

The **Hausdorff dimension** is given by

$$\mathcal{H} - \dim(E) = \inf\{\beta > 0 : \mathcal{H}^\beta(E) = 0\}.$$

Remark 6.2.5. $0 < \mathcal{H}^\beta(E) < +\infty \implies \beta = \mathcal{H} - \dim(E)$.

In the Euclidean space $(X,d) = (\mathbb{R}^k, |\cdot|),\ \alpha = k$, taking $c_k = \mathcal{L}^k(\{x \in \mathbb{R}^k : |x| \le 1\})$ in the definition of $\mathcal{H}^k_{|\cdot|},\ 1 \le k \le n$, the classical area formula yields for a chart $\varphi : A \to U$ of a $k-$dimensional submanifold Σ.

$$\mathcal{H}^k_{|\cdot|}(U) = \int_A J_\varphi(x) d\mathcal{L}^k(x),$$

where $J_\varphi(x) = ||\partial_{\xi_1}\varphi(x) \wedge ... \wedge \partial_{\xi_k}\varphi(x)|| = \sqrt{\det(D\varphi(x)^T D\varphi(x))}$ is the Jacobian of φ. Notice that as a quick exercise, one can check that the Euclidean space and more general finite dimensional Banach spaces coincide with commutative stratified groups, where $\delta_r x = rx$.

6.2.7 General Motivating Question

Dow we have a "nice integral formula" to represent the "natural measure" of submanifolds in stratified groups equipped with their homogeneous distance?
The proof of the classical area formula relies on the property of Lipschitz mappings with respect to the Hausdorff measure, and the fact that surfaces in the Euclidean space, with respect to the induced metric, can be parameterized by Lipschitz mappings.

Lemma 6.2.6 (Vitali)**.** *Let $\mathcal{F}$ be a family of closed balls with uniformly bounded radius. It follows that there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that*

$$\bigcup_{D} \in \mathcal{F} D \subset \bigcup_{D\in\mathcal{G}} 5D$$

The proof is elementary and relies on the Hausdorff principle.

Theorem 6.2.7. *Let(X,d) be a metric space equipped with a Borel regular outer measure that is finite on bounded sets. Let $A \subset X$ and let $0 < \alpha < \beta$ and assume that $\forall x \in A$ we have*

$$\alpha < \lim_{r\searrow 0} \frac{\mu(D(x,r))}{c_q r^q} < \beta.$$

It follows that

$$5^{-q}\alpha\mathcal{S}^q(A) \leq \mu(A) \leq 2^q\beta\mathcal{S}^q(A).$$

Proof. It is not restrictive to assume that both A and V are bounded... $\square$

Chapter 7

Valentino

7.1 Lecture One: Sobolev Spaces

7.1.1 Differentiability and Smoothness in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be open, let $x_0 \in \Omega$ define and let $f : \Omega \to \mathbb{R}$

- Define what it means for f to be **differentiable** at x_0.
- Define what it means for f to be **smooth** at x_0.

We will denote n times differentiable functions by $\mathbb{C}^n(\Omega)$ and smooth functions by $\mathbb{C}^\infty(\Omega)$.

7.1.2 Sobolev spaces in $\mathbb{R}^n$ by completing the smooth functions

Let $p \in (1, \infty)$ and $f \in \mathbb{C}^\infty(X)$, define the **first Sobolev norm** by

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

- Show that $\|\cdot\|_{W_{1,p}}$ is a norm.

The set of smooth functions $\mathbb{C}^\infty$ with the first Sobolev, $\| - \|_{W_{1,p}(\Omega)}$ is a normed vector space.

- Show that the normed vector space $(\mathbb{C}^\infty(\Omega), W^{1,p}$ is not a Banach space.

The completion of the smooth functions on Ω with respect to the norm $\|-\|_{W^{1,p}(\Omega)}$ is called the **first** p **Sobolev space** and is denoted by $W^{1,p}(\Omega)$. It consists of functions in $L^p(\Omega)$ with a gradient in $L^p(\Omega, \mathbb{R}^n)$.

7.1.3 Integration by Parts and Weak Derivatives

If $f \in \mathbb{C}^\infty(\Omega)$ and $\varphi \in \mathbb{C}_c^\infty(\Omega, \mathbb{R}^n)$ then

$$\int_\Omega \langle \varphi(x), \nabla f(x) \rangle dx = -\int_\Omega f(x) \nabla \cdot \varphi(x) dx \tag{7.1.1}$$

by applying integration by parts and the fact that φ has compact support. For $f : \Omega \to \mathbb{R}$ measurable there exists an operator $df : \mathbb{C}_c^\infty(\Omega, \mathbb{R}^n) \to \mathbb{R}$ given by

$$df(\varphi) = -\int_\Omega f(x) \nabla \cdot \varphi(x) dx.$$

The idea is that when f is differentiable we have

$$df = \langle -, \nabla f \rangle,$$

by equation 7.1.1, so this operator is a replacement for derivatives for functions that don't have them.

- What are the functions f such that there exists some $V : \Omega \to \mathbb{R}^n$ such that for all $\varphi :\in \mathbb{C}_c^\infty(\Omega)$ we have

$$df(\varphi) = \int_\Omega \langle V(x), \varphi(x) \rangle ?$$

Functions which have these are called **weakly differentiable**. The following exercise shows that they don't always exist.

- We claim that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 1, & x > 0 \\ 0. & x \leq 0 \end{cases}$$

  does not have such a V.

When a function does have a weak derivative we should observe we can change the definition of V on a set of measure zero and get another weak derivative. The weak derivitive is unique up to a set of measure zero.

7.1.4 Sobolev spaces by existence of weak derivatives

If $f \in L^p$ has a weak derivative ∇f then on the subset of functions which have a weak derivative we can define the first Sobolev norm as

$$\|f|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

The Sobolev Space could then be defined as the set of weak differentiable functions in L^p with bounded first Sobolev norm.

7.1.5 Equivalence of two ways of defining Sobolev spaces

We have given two ways of defining the Sobolev Spaces. The first is as the completion of $\mathbb{C}^\infty(\Omega)$ with respect to $\|-\|_{W^{1,p}}$ and the second as the subspace of $L^p(\Omega)$ of weakly differentiable functions with bounded Sobolev norm.

Theorem 7.1.1 (Serrin)**.** *Both approaches to defining the Sobolev space*

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with compact support. Define $\eta(x) = \psi(|x|)$. For $f \in W^{1,p}(\Omega)$ we can define for every $\varepsilon > 0$

$$(f * \eta_\varepsilon)(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(y)\eta(\frac{x-y}{\varepsilon})dy.$$

By approximations of the identity this converges to f in L^p sense as $\varepsilon \to 0$. Next we have

$$\begin{aligned} \partial_j(f * \eta_\varepsilon) &= f * \partial_j \eta_\varepsilon \\ &= \partial_j f * \eta_\varepsilon \end{aligned}$$

which approaches $\partial_j f$ as $\varepsilon \to 0$ for every j. This gives a sequence of approximations of $f \in W^{1,p}(\Omega)$ by smooth functions.

7.1.6 absolute continuity

A function $f : \Omega \to \mathbb{R}$ is **absolutely continuous** along $\gamma : [0,1] \to \Omega$ provided the following holds:

For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k] \subset [0,1]$ we have

$$\sum_i |b_i - a_i| < \delta \implies \int_{\sum[a_i,b_i]} f \circ \gamma < \varepsilon.$$

7.1.7 convention: parametrizing in terms of arclength

Let $\gamma \subset \Omega$ be a curve of finite length and let $f \in \mathbb{C}^\infty(\Omega)$. We can view γ as $\gamma : [0,1] \to \Omega$. We have the inequality

$$|f(y_\gamma) - f(x_\gamma)| \leq \int_0^1 |\nabla f(\gamma(t))||\gamma'(t)|dt,$$

where $y(\gamma)$ is the end point of the curve and x_γ is the end point of the curve. We will parametrize curves in term of arclength so the the norm of the derivative will be equal to one reducing the above expression to

$$|f(y_\gamma) - f(x_\gamma)| \leq \int_0^1 |\nabla f(\gamma(t))|dt.$$

7.1.8 criteria for absolute continuity

Proposition 7.1.2. *Suppose that $f : \Omega \to \mathbb{R}$ and γ is a rectifiable curve in Ω. Also suppose that there exists some $g : \Omega \to [0, \infty]$ measurable such that*

$$\int_\gamma gds < \infty$$

and such that for all subcurves $\beta \subset \gamma$ we have

$$|f(y_\beta) - f(x_\gamma)| \leq \int_\beta gds.$$

Then, $f \circ \gamma$ is absolutely continuous and $|(f \circ \gamma)'| \leq g \circ \gamma$ pointwise.

- Using the definition of absolute continuity prove the above proposition.

7.1.9 the Poincaré inequality

This number is about the **Poincaré inequality**. For a set A we use the following notation for integral average

$$\fint_A f = \frac{1}{|A|} \int_A f.$$

Proposition 7.1.3 (Poincaré Inequality)**.** *For $f \in \mathbb{C}^\infty(\Omega)$ and $x_0 \in \Omega$ let $r > 0$ be such that $\overline{B(x_0, r))} \subset \Omega$*

$$\fint_{B(x_0,r)} |f(y) - f(x_0)| \leq c_n \int_{B(x_0,r)} \frac{|\nabla f(z)|}{|z - x_0|^{n-1}}.$$

Proof. The left hand side of the above we convert to polar coordinated

$$\frac{1}{a_n r^n}\int_B |f(y)-f(x)|dy \;=\; \frac{1}{a_n r^n}\int_0^r \rho^{n-1}\int_{S^{n-1}}|f(\rho,\theta)-f(x_0)|d\theta d\rho$$

where the volume of the n dimensional ball or radius r in $a_n r^n$ we let $y=(\rho,\theta)$ where $\theta\in S^{n-1}$ and $\rho\in[0,r]$. Since we have

$$|f(\rho,\theta)-f(x_0)| = |\int_0^\rho \nabla f(s,\theta)ds| \le \int_0^\rho |\nabla f(s,\theta)|ds$$

we have

$$\begin{aligned}
\frac{1}{a_n r^n}\int_0^r \rho^{n-1}\int_{S^{n-1}}|f(\rho,\theta)-f(x_0)|d\theta d\rho &\le \frac{1}{a_n r^n}\int_0^r\int_0^\rho\int_{S^{n-1}}\frac{|\nabla f(s,\theta)|}{s^{n-1}}s^{n-1}d\theta ds d\rho\\
&= \frac{1}{a_n r^n}\int_0^r\int_{B(x_0,\rho)}\frac{|\nabla f(z)|}{|z-x_0|}dz d\rho\\
&= \frac{1}{n a_n}\int_0^r \frac{|\nabla f(z)|}{|z-x_0|^{n-1}}dz.
\end{aligned}$$

as an application of Fubini's Theorem. $\square$

Remark 7.1.4. The expression

$$\frac{|\nabla f(z)|}{|z-x_0|^{n-1}}$$

is called the **Reisz Potential** of $|\nabla f|$

7.2 Lecture 2

7.2.1

Last time we showed the Poincare Inequality. This Inequality still holds if we enlarge the Ball we are taking the integral over.

Proposition 7.2.1. *For $x\in B$ and $f\in L^1_{loc}(\Omega)$ we have*

$$\fint_B |f(x)-f(y)| \le C\int_{4B}\frac{|\nabla f(z)|}{|z-x|^{n-1}}dz \tag{7.2.1}$$

.

Consider the operator I_B which takes in functions $V:\mathbb{R}^n\to R$ and is defined by

$$I_B(V)(x) := \int_{4B}\frac{|V(z)|}{|z-x|^{n-1}}dz.$$

Proposition 7.2.2. *The operator I_B is a bounded operator $L^p(4B) \to L^p(4B)$.*

The strategy is to control this operator by the **Hardy-Littlewood maximal function**

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B f(y) dy$$

which is known to be a bounded function. The proof will allso use the Lebesgue differentiation theorem which states that

$$\frac{1}{|B|} f(y) dy \to f(x) \text{ as } |B| \to 0$$

where the balls are contain x and the limit exists almost everywhere.

Proof. Let $B_i = B(x, 2^{-i}r)$ for $i \geq 0$ then

$$\begin{aligned} I_B(V)(x) &= \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{g(z)}{|z-x|^{n-1}} dz \\ &\leq \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{g(z)}{(2^{-i}r)^{n-1}} dz \\ &\leq \sum_{i=0}^{\infty} \frac{1}{(2^{-i}r)^{n-1}} \int_{B_i} g(z) dz \\ &\leq \sum_{i=0}^{\infty} \frac{a_n 2/r}{|B_i|} \int_{B_i} g(z) dz \\ &\leq \sum (Mg)(x) \end{aligned}$$

then use the boundedness of the maximal function. □

$$|f(x) - \fint_{B(x,r)} f(y) dy \leq \fint_{B(x,r)} |f(x) - f(y)| dy$$

$$|f - f_B| \leq C \int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz$$

Let $g \in W^{1,p}$ and take $g_n \in \mathbb{C}^{\infty}(\mathbb{R}^n)$ such that $g_n \to g$ as $n \to \infty$ in the $W^{1,p}$ norm.

Proposition 7.2.3. *For $f \in W^{1,p}(\mathbb{R}^n)$ and B a ball containing x,*

$$|f(x) - f_B| \leq C \int_{4B} \int \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz.$$

Proposition 7.2.4. *Let $f \in W^{1,p}(\mathbb{R}^n)$. For almost every x and y in $\mathbb{R}^n$ we have*

$$|f(x) - f(y)| \leq Cd(x,y)\left[M|\nabla f|(x) + M|\nabla f|(y)\right]$$

Proof. Let $B = B(x, 2d)$ where $d = d(x,y)$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_B| + |f(y) - f_B| \\ &= \leq C\left(\int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dz + \int_{4B} \frac{|\nabla f(z)|}{|z-y|^{n-1}} dz\right) \\ &= C(M|\nabla f|(x) + M|\nabla f|(y)) \end{aligned}$$

□

Remark 7.2.5. If $\Omega = \mathbb{R}^n$ then we need $\frac{1}{8}\operatorname{dist}(x, \partial\Omega) = d(x,y)$

7.2.2

We have the following characterization for $f \in W^{1,p}(\mathbb{R}^n)$

Proposition 7.2.6. *Let $f \in L^p(\mathbb{R}^n)$ measurable. If there exists a $g \in L^p(\mathbb{R}^n)$ non-negative function such that*

$$|f(y) - f(x)| \leq d(x,y)\left[g(x) + g(y)\right]$$

then $f \in W^{1,p}(\mathbb{R}^n)$.

7.2.3

We will now use the result in the previous section to define a new space $M^{1,p}(X)$ where (X, μ, d) is a certain measure space with a metric. Here

- μ is Borel regular.
- All balls have positive measure.
- Bounded sets have finite measure.

Remark 7.2.7. Non σ-finite measures have stupid pieces of the measure space.

Definition 7.2.8. $M^{3p}(X)$ is called the **Hajtasz-Sobolev Space**. It consists of functions $f \in L^p(X)$ such that there exists some $g \geq 0$ which satisifies

$$|f(x) - f(y)| \leq d(x,y)\left[g(x) + g(y)\right].$$

7.2.4

$f : X \to \mathbb{R}$ is in $M^{1,p}$ if and only if f is in L^p and there exists some positive measure L^p such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)).$$

One can check easily that this defined a vectorspace. To do this suppose that For f_1 and f_2 in $M^{p,1}$ with corresponding g_1 and and g_2 then $f_1 + f_2$ has $g_1 + g_2$ as its L^p bounder. We have to show that g_1 and g_2 We can also make it a normed vector space with

$$\|f\| := \|f\|_{L^p} + \inf_{g \in S(f)} \|g\|_{L^p}$$

where $S(f)$ consists all of the possible positive, measurable $g \in L^p$ satisfying the inequality.

7.2.5

We claim that the vector space $M^{1,p}$ is complete. Suppose that f_k is a convergent sequence in $M^{1,p}$. We need to show that $f_k \to f$ such that there exists a g for the f with the appropriate bound. It is that f exists as a function in L^p since the $M^{p,1}$ norm dominates the L^p norm of the sequence of f_k's. What is not clear is that there exist some g which gives the

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \text{ a.e.}$$

This is what we will prove.

Since $\|f_{k+1} - f_k\|_{M^{1,p}}$ can find some $g_{k+1,k}$ with $\|g_{k+1,k}\|_{L^p} \to 0$ as $k \to \infty$. Note that for every n we can write f_{n+1} as a sum of functions

$$f_{n+1} = \sum_{k=1}^{n} (f_{k+1} - f_k) + f_1.$$

By the remark in section 7.2.4 the function $g_{n+1} := \sum_{k=1}^{n} g_{k+1,k} + g_1$ works as a sufficient bounder for the f_{n+1}'s. We can assume that g_{n+1} converges to some g in L^p by taking a subsequence if necessary to make the convergence more rapid.

For every n we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(y) - f(y)| + |f_n(x) - f_n(y)| \\ &\leq \varepsilon_n(x) + \varepsilon_n(y) + d(x, y)(g_n(x) + g_n(y)). \end{aligned}$$

Which implies

$$\begin{aligned}
|f(x) - f(y)| &\leq \inf_n \left(\varepsilon_n(x) + \varepsilon_n(y) + d(x,y)(g_n(x) + g_n(y))\right) \\
&\leq \inf_n \left(\varepsilon_n(x) + \varepsilon_n(y)\right) + d(x,y) \sup_n (g_n(x) + g_n(y)) \\
&\leq d(x,y) \left(\sup_n g_n(x) - \sup_n g_n(y) \right)
\end{aligned}$$

Since $g_n(x)$ is monotonically increasing with n we must have that $\sup_n g_n(x) = \sum_{n=1}^{\infty} g_{n+1,n}(x) + g_1(x)$. This limit exists in L^p which gives the g we were looking for.

7.3 Lecture 3

7.3.1

Let $f \in L^1(X)$ and $g : X \to [0, \infty]$ be a possible unbounded curve. Then for almost every rectifiable γ in X we have

1. $\int_\gamma g ds < \infty$
2. for all $\beta \subset \gamma$
$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta g ds$$

implies that f is absolutely continous on the curve γ with $|f \circ \gamma| \leq g \circ \gamma$ almost everywhere in γ.

7.3.2

Another way to generalize the Sobolev spaces in the metric space setting. For $f \in \mathbb{C}^\infty(X)$ and $g = |\nabla f|$ works to get the above conditions.

For $f \in W^{1,p}(X)$ we can find smooth f_k such that $\|f_k - f\|_{W^{1,p}(X)} \to 0$ as $k \to \infty$. For smooth functions we have

$$|f_k(x_\beta) - f_k(y_\beta)| \leq \int_\beta |\nabla f_k| ds$$

almost everywhere and f_k converges pointwise almost everywhere. Note that convergence could be bad at the endpoints of the curves β. We need to throw a small collection of curves to make the above statement make sense. This boils down to *computing capacities.*

7.3.3

μ is an **exterior measure** provided

- $T_1 \subset T_2$ implies that $\mu(T_1) \leq \mu(T_2)$
- $\mu(\emptyset) = 0$
- For a countable family $T_1, T_2, \ldots$ we have
$$\mu(\bigcup_{i \geq 0} T_i) \leq \sum_{i \geq 0} \mu(T_i)$$

7.3.4

Let T be a family of curves in X. The family of admissible functions is

$$A(T) = \{g : X \to [0.\infty] | \forall \gamma \in T, \int_\gamma g ds \geq 1\}$$

Note that these functions detect the family T in the sense that $\int_\gamma g ds \geq 1$. For $p \in (0, 1)$, we define the p-**modulus** of the family T to be

$$\mu_p(T) = \int_{g \in A(T)} \int_X g(x)^p dx.$$

Lemma 7.3.1. *μ_p is an exterior measure on the collection of curves in X.*

In the above statment we are considering the collection of curves which are arclength parametrized and are of finite length.

- Prove lemma 7.3.1.

7.3.5

Consider a cylindrical family K of line segments in $\mathbb{R}^n$ where $A \subset \{x \in \mathbb{R}^n : x_n = 0\}$ and above every point in A there is a straight line starting at $x_n = 0$ going to $x_n = h$. The family is $C = K \times [0, h]$.

The set $A(T)$ is nonempty. The function

$$g(x_1, \ldots, x_n) := \frac{1}{h} \chi_K(x_1, \ldots, x_n)$$

gives us

$$\begin{aligned}
\mu_p(T) &\leq \int_K g^p dx \\
&= \frac{1}{h^p}\int_K dx \\
&= \frac{1}{h^p}{}_{[0,h]\times A} dydt \\
&= \frac{1}{h^p}\cdot |A| \cdot h
\end{aligned}$$

Let $\rho \in A(T)$. This means that $\int_{L_y} \rho ds \geq 1$ where L_y denotes the fiber above $y \in A$ in K. By Hölder's inequality this is less than or equal to

$$(\int_{L_y} \rho^p ds)^{1/p}(\int_{L_y} 1^{p'})^{1/p'}$$

which tells us that

$$1 \leq \left(\int_{L_y} \rho^p ds\right) h^{p-1}.$$

Computing further we have

$$\begin{aligned}
\int_K \rho^p dx &= \int_{A\times[0,h]} \rho^p dyds \\
&= \int_A \left(\int_0^h \rho^p ds\right) dy \\
&= (\rho^p ds)\,|A| \\
&\geq |A| h^{1-p}
\end{aligned}$$

7.3.6

Let $E \subset X$ and define the family

$$T_E^+ := \{\gamma : |\gamma^{-1}(E)| > 0\}.$$

Lemma 7.3.2. *If $E \subset X$ has measure zero them $\mu_p(T_E^+) = 0$.*

- Show that a function in admissible class is zero.
- If S has measure zero then there exists covers of S which shrink to zero.

Let $E_0 \subset X$ such that $E \subset E_0$ and define

$$g(x) = \infty\chi_{E_0}(x).$$

if $\gamma \in T_E^+$ then

$$\int_\gamma g ds = \infty$$

which means that $g \in A(T_E^+)$ but we have that

$$\int_X g^p dx = 0$$

because g is supported on a set of measure zero.

7.3.7

Lemma 7.3.3 (Fuglede's Lemma)**.** *Let g_k be a sequence of non-negative borel measurable functions on X, $g : X \to (0, \infty]$ borel measurable with*

$$\|g - g_k\|_{L^p(X)} \leq 2^{-(k+1)p}.$$

The set of curves for which $\int_\gamma g_k$ does not approach $\int_\gamma g$ has μ_p measure zero.

We will actually be proving a stronger statement. The family

$$T = \{\gamma : \limsup_{k\to\infty} \int_\gamma |g_k - g| ds > 0\}$$

has $\mu_p(T) = 0$.

7.3.8

Lemma 7.3.4 (Koskela-MacManis)**.** *Let T be a family of curves in X. $\mu_p(T) = 0$ if and only iff there exists a borel measurable $g \in L^p(X)$ such that for all curves γ we have*

$$\int_\gamma g ds = \infty$$

This is a generalization of the above.

7.3.9

Theorem 7.3.5. *$f \in W^{1,p}(X)$ if and only if*

1. *$f \in L^1(X)$*
2. *There exists some $g \in L^p(X)$ a non-negative borel measurable function such that the family of curves T where one of the conditions for absolute continuity fails has measure zero.*

Observe that this theorem fixes exactly what was wrong in the beginning of this lecture.

Proof. For the forward implication observe that $f \in W^{1,p}(X)$ implies that we can find f_k smooth where $f_k \to f$ as functions in $W^{1,p}(X)$ and $\|f_k - f\|_{W^{1,p}} \leq 2^{-(k+2)p}$. Let

$$E = \{x : \lim_k f_k(x) \neq f(x)\}.$$

This set has measure zero which implies that $\mu_p(T_E^+)$ has measure zero.

By Fugleday's lemma we have

$$\mu_p\{\gamma : \lim_{k\to\infty} \int_\gamma ||\nabla f_k| - |\nabla f||ds > 0\} = 0.$$

Observe that $|\nabla f| \in L^p(X)$ does not have to be Borel measurable bu can be modified on a set of measure zero that that it becomes measurable.

Finally, observe that the set of curves on which $\int_\gamma |\nabla f|ds = \infty$ has measure zero.

If γ is not a member of one of these three bad sets of measure zero we have

$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta |\nabla f_k|ds$$

for every subcurve β such that

$$\int_\beta |\nabla f_k|ds \to \int_\beta |\nabla f|ds.$$

The complement of any set of measure zero is dense so we can fine x_β and y_β not in E that give

$$|f(x_\beta) - f(y_\beta)| \leq \int_\beta |\nabla f|ds.$$

We need to show that $E \cap \gamma = \emptyset$. Suppose that $x_0 \notin E$ but $x_0 \in \gamma$. We can choose x_0 as close to y as we want.

$$\lim_{k\to\infty} f_k(y) = f(y)$$

$$f_k(y) - f_k(x_0) = \int_{\beta_y} \nabla f_k ds$$

and since

$$\int_{\beta} |\nabla f_j| ds < \varepsilon$$

we get that

$$|f_k(y) - f_k(x_0)| < \epsilon$$

Use some triple triangle inequality to get what we want. □

The converse depends heavily on the coordinates

- The line segments parallel to the coordinate axes
- get partial derivatives

Remark 7.3.6. The family of integrals $\int_\gamma |\nabla f_j| ds$ are equicontinuous.

7.4 Lecture 4

7.4.1

In the previous section we gave a characterization of Sobolev functions. For X a nice enough metric space we had $f \in W^{1,p}(X)$ if and only if there existed some Borel measurable $g \in L^p(X)$ with a family T such that $\mu_p(T) = 0$ and for all $\gamma \notin T$

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds$$

And recall that by nice enough we means that (X, d, μ) was a triple consisting of a set a metric and a measure where the measure was Borel regular, bounded sets had finite measure and open sets had positive measure.

7.4.2

We define the collection $\widetilde{N}^{1,p}(X)$ to be the set of $f \in L^p(X)$ such that there exists a $g \in L^p(X)$ borel measurable and nonnegative and for almost every γ we have

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds.$$

- Verify that the definition of μ_p did not use any Euclidean structure.

7.4.3

Consider the functions

$$f(x, y) = 0 \text{ and } g(x, y) = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \end{cases}$$

Observe that since these two functions differ only on a set of measure zero we have $g \sim f$ as Sobolev functions.

7.4.4

Let $p \leq 2$ and let $T = \{$ curves going through x_0 $\}$

- Show that $\mu_p(T) = 0$.

  g as in definitino of $\widetilde{N}^{1,p}(X)$ is called a **weak upper gradient**.

  $$\|f\|_{\widetilde{N}^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)}.$$

- Show that $N^{1,p}(X)$ is a vector space.
- Show that $\|-\|_{N^{1,p}}$ is a seminorm (some nonzero elements may have $\|f\|_{N^{1,p}(X)} = 0$.
- Quotient out by the kernel of this semi-norm to get a norm

The **Newton-Sobolev** space is defined by

$$N^{1,p}(X) := \widetilde{N}^{1,p}(X)/\sim$$

where $f \sim g$ if and only if $\|f - g\|_{N^{1,p}} = 0$. The word Newton is because the Fundamental Theorem of Calculus applies.

7.4.5

Theorem 7.4.1. *If X is a subset on $\mathbb{R}^n$ with the induced metric and measure then*

$$N^{1,p}(X) \cong W^{1,p}(X)$$

as Banach Spaces.

Remark 7.4.2.

1. The vector spaces above are not equal as sets since they consist of different collections of equivalence classes.
2. Suppose that f_1 and f_2 have upper gradients g_1 and g_2. It is not the case that $g_1 - g_2$ is a upper gradients.

Proposition 7.4.3. *$N^{1,p}(X)$ is a Banach space.*

Let $\{f_k\}_{k\in\mathbb{N}}$ a sequence of functions in $N^{1,p}(X)$ such that

1. $f_k \to f$ in $L^p(X)$
2. f_k has p-weak upper gradients $g_k \in L^p(X)$ with $g_k \to g$ in $L^p(X)$.

Then there exists some $f_0 : X \to [-\infty, \infty]$ with $f_0 = f$ almost everywhere such that $f_0 \in N^{1,p}(X)$ and a Borel representative g_0 of g is a weak upper gradients of f_0.

By passing to a subsequence if necessary we have

$$\begin{aligned} \|f - f_k\|_{L^p(X)} &< 2^{-kp}, \\ \|g - g_k\|_{L^p(X)} &< 2^{-kp}. \end{aligned}$$

The set

$$A = \{x \in X : \lim_{k\to\infty} f_k(x) \neq f(x) \text{ or } \lim_{k\to\infty} g_k(x) \neq g(x)\}$$

has $\mu(A) = 0$. The implies that $mu_p(T_A^+) = 0$ since the set A has measure zero.

Also, the family

$$T = \{\gamma : \lim_{k\to\infty} \int_\gamma |g_k - g| ds$$

has $\mu_p(T) = 0$ by Fugleday's lemma.

By the K MacManus lemma we have

$$\mu_p\{\gamma : \int_\gamma g_k ds = \infty \text{ for some } k \text{ or } \int_\gamma g ds = \infty\} = 0.$$

Also for each $k \geq 1$ the family

$$T_k = \{\gamma : (f_k.g_k) \text{ do not satisfy } |f_k(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g_k ds\}$$

has $\mu_p(T_k) = 0$.

Now the union of these families

$$S = T_A \cup T \cup \bigcup_{k \geq 1} T_k$$

has $\mu_p(S) = 0$.

For $\gamma \notin S$, we have

$$|f_k(x_\gamma) - f_k(y_\gamma)| \leq \int_\gamma g ds$$

with

$$\int_\gamma g_k ds \to \int_\gamma g_k ds$$

and

$$|f_k(x_\gamma) - f_k(y_k)| \to |f(x_\gamma) - f(y_\gamma)|$$

by finding a dense set of points where this works and the triangle inequality.

7.4.6

$M^{1,p}(X)$ is the set of $f \in L^p(X)$ such that there exists some $g \geq 0$ in $L^p(X)$ with the property that for almost every $x, y \in C$ we have

$$|f(x) + f(y)| \leq d(x,y) \left[g(x) + g(Y) \right].$$

$N^{1,p}(X)$ is the set of $f \in L^p(X)$ such that there exists some $g \geq 0$ where

$$|f(x_\gamma) - f(y_\gamma)| \leq \int_\gamma g ds.$$

We have proved that

$$\begin{aligned} M^{1,p}(\mathbb{R}^n) &= W^{1,p}(\mathbb{R}^n) \text{ for p¿1} \\ N^{1,p}(\Omega) &= W^{1,p}(\Omega) \end{aligned}$$

Proposition 7.4.4. *In general we do not have $M^{1,p}(X) = N^{1,p}(X)$.*

counter-example. Suppose that D is the unit disc in $\mathbb{R}^2$ and consider $X = D \setminus [0,1]$. We have

$$M^{1,p}(D \setminus [0,1]) = M^{1,p}(D)$$

since the bounding g in the definition does not see the deleted line segment. On the other hand

$$W^{1,p}(D \setminus [0,1]) \neq W^{1,p}(D)$$

since $\arg(z) \in W^{1,p}(D \setminus [0,1])$ but not in $W^{1,p}(D)$. □

Remark 7.4.5. The example $D \setminus [0,1]$ is called a **Sobolev extension domain**.

7.4.7

Proposition 7.4.6. $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$

Proof. Given $f \in M^{1,p}(X)$ we can modify it on a set of measure zero to get

$$|f(x) - f(y)| \leq d(x,y)|g(x) + g(y)|,$$

with $g \in L^p(X)$. By the Kaskela-MacManis theorem

$$\mu_p\{\gamma : \int_\gamma g ds = \infty\} = 0.$$

since the inequality fails on a set of measure zero the set of curves which see E as a large set has measure zero,

$$\mu_p(T_E^+) = 0.$$

Let γ be not one of the bad curves and partition it into small pieces γ_j each of length $l(\gamma)/N$ having some $z_j \in \gamma_j$ such that

$$g(z_j) \leq \fint_\gamma g ds.$$

$$
\begin{aligned}
|f(z_1) - f(z_n)| &\leq \sum_i |f(z_i) - f(z_{i+1})| \\
&\leq \sum_i 2\frac{l(\gamma)}{N}[g(z_i) + g(z_{i+1}) \\
&\leq \sum_i 2\frac{l(\gamma)}{N}\left(\frac{1}{l(\gamma_i)}\int_{\gamma_i} g + \frac{1}{l(\gamma_{i+1})}\int_{\gamma_{i+1}} gds\right) \\
&\leq 2\sum_i 2\int_{\gamma_i} gds \\
&\leq 4\int_\gamma gds
\end{aligned}
$$

If f is continuous then we have

$$|f(z_1) - f(z_n)| \to |f(x_\gamma) - f(y_\gamma)|.$$

This show s that

$$M^{1,p}(X) \cap \mathbb{C}(X) \hookrightarrow N^{1,p}(X)$$

is bounded. If we can show that $M^{1,p}(X) \cap \mathbb{C}(C)$ is dense in $M^{1,p}(X)$ then we are done.

We claim that the Lipschitz Functions are dense in $M^{1,p}(X)$.

□

- Show that the Lipschitz functions are dense
- $\mu\{g > \lambda\} \leq \frac{1}{\lambda^p}\|g\|_{L^p(X)}$
- $\mu\{g \leq \lambda \ni x, y$

7.4.8

Proposition 7.4.7. *Let $f \in W^{1,p}(\Omega)$ and $4B \subset \Omega$. Then for all $x \in B$ we have*

$$⨍_B |f(y) - f(x)|dy \leq C\int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}}dz.$$

Remark 7.4.8. The integral on the righthand side above is not a singular integral.

- Show that
$$\int_B \frac{1}{|z-x|^{n-1}} \approx r_B$$

$$\begin{aligned}
\fint_B \fint_B |f(y)-f(x)|dydx &\leq \fint_B C \int_{4B} \frac{|\nabla f(z)|}{|z-x|^{n-1}} dzdx \\
&= C \int_{4B} |\nabla f(z)| \fint_B \frac{1}{|z-x|^{n-1}} dxdz \\
&= C r_B \fint_{4B} |\nabla f(z)| dz
\end{aligned}$$

This give the 1-**Poincaré inequality**:

Proposition 7.4.9. *For $f \in W^{1,p}(\Omega)$, $4B \subset \Omega$ and $x \in B$ we have*

$$\fint_B \fint_B |f(x)-f(y)|dydx \leq C r_B \fint_{4B} |\nabla f(z)| dz \tag{7.4.1}$$

Remark 7.4.10. Using Hölder's inequality on the right hand side we get the p-Poincaré inequality which is weaker.

7.5 Lecture 5

In this section we will compare for four different Sobolev spaces in the Metric Setting:

$$M^{1,p}(X), N^{1,p}(X), P^{1,p}(X), KS^{1,p}(X)$$

7.5.1

The p-**Poincaré Inequality** is satisfied on a space (X, d, μ) if there exists some $\lambda \geq 1$ and some positive constant C such that for all f

$$\fint_B |f - f_B| d\mu \leq C r_B (\fint_{\lambda B} |\nabla f|^p d\mu)^{1/p}.$$

Let $\Omega \subset \mathbb{R}^2$ be rectangle with a deleted center. Recall that balls are of the form $B \cap \Omega$ in this space. The λ is needed in this region to engulf the center.

7.5.2

Proposition 7.5.1. *For $X \subset \mathbb{R}^n$,*

$$\textit{Sobolev} \implies \textit{Haiwatch} .$$

The trick is to consider a telescoping series of balls (binoculars) at two different points.

Proposition 7.5.2. *If X supports a p-Poincare Inequality for upper gradient pairs and $p > 1$ we have*

$$\fint |f - f_B| d\mu \leq C r_B (\int_{\lambda B} |g|^p d\mu)^{1/p}$$

then $M^{1,p}(X) = N^{1,p}(X)$

Suppose that $\mu(x, 2r) \leq C\mu(B(x, r))$

Remark 7.5.3. The property of **asymptotic doubling measures** are enough to show the boundedness of Maximal functions.

$$\lim_{r \to 0} \frac{B(x, r)}{B(x, 2r)} = C(x)$$

exists and is bounded for all x.

Proof. Let $f \in N^{1,p}(X)$. Let E be the set of non-Lesbegue points (where the Mean Values on balls don't converge to the value at the point). By the Lebesgue differentiation theorem $\mu(E) = 0$.

Let $x, y \in X \setminus E$, let $d = d(x, y)$ and define

$$B_i = \begin{cases} B(x, 2^{i-1}d), & i \geq 0 \\ B(y, 2^i d), & i < 0 \end{cases}.$$

Notice that

$$\lim_{i \to \infty} f_{B_i} = f(x) \text{ and } \lim_{i \to -\infty} f_{B_i} = f(y).$$

By a triangle inequality we have

$$|f(x) - f(y)| \leq |\sum_{i=-\infty}^{\infty} |f_{B_i} - f_{B_{i+1}}| = *$$

and since $B_i, B_{i+1} \subset 2B_{i+1}$ we have

$$* \leq \sum |f_{B_i} - f_{2B_{i+1}}| + |f_{2B_{i+1}} - f_{B_{i+1}}|$$

and

$$|f_{B_i} - f_{B_{i+1}}| \leq \int_{B_i} |f - f_{B_{i+1}}| d\mu$$

gives us

$$* \leq 2C \sum_{i=-\infty}^{\infty} \int |f - f_{2B_{i+1}}| d\mu$$

□

Remark 7.5.4. Keith-Zhang have proven for X a doubling space with a p Poincare inequality that there exists some $q \in (1, p)$ where X has a q Poincare inequality. They require the metrix space to be complete.

$$\begin{aligned} |f(x) - f(y)| &\leq C \sum_{i=-\infty}^{\infty} r_{2B_{i+1}} (\fint_{2\lambda B_{i+1}} g^q d\mu)^{1/q} \\ &\leq C \left(\sum_{i \geq 0} (2^{1-i} dM(g^q)(x)^{1/q} + \sum_{i<0} 2^{i+1} M(g^q)(x)^{1/q} \right) \\ &\leq Cd \left(M(g^q)(x)^{1/q} + M(g)(x)^{1/q} \right) \end{aligned}$$

The Hajtase quotient is $M(g^q)^{1/q} \in L^p(X)$

7.5.3

A topological space X is **quasi-convex** if for every $x, y \in X$ there exist some path γ with endpoints x and y such that $l(\gamma) < \infty$ and

$$l(\gamma) \leq Cd(x, y)$$

Remark 7.5.5. Observe that by the triangle inequality $l(\gamma)Cd(x, y)$.

The domain enclosed by the Cardiod with the subspace metric and measure in $\mathbb{R}^2$ is not quasi-convex since there are points arbitrarily close to each other whose minimal path has a positive length.

7.5.4

Proposition 7.5.6. *If X supports a p-Poincaré inequality for (f, g) pairs then X is quasi-convex.*

7.5.5

The $P^{1,p}$-space is the collection of $f \in L^p(X)$ for which there exists a non-negative $g \in L^p(X)$ with the property that for all ball $B \subset X$ we have

$$\fint_B |f - f_B| d\mu \leq r_B (\int_{\lambda B} g^p d\mu)^{1/p}$$

We call this space the $(1, p)$ **Poincaré space**.

Remark 7.5.7. The space $M^{1,p}(X)$ requires

$$|f(x) - f(y)| \leq d(x, y) \left[g(x) + g(y) \right].$$

It does not have the property that if a function is contains along an open set that "its derivative" g must be zero. We say that it fails to have **strong locality** .

The space $N^{1,p}(X)$ requires

$$|f(x) - f(y)| \leq \int_\gamma g ds.$$

this space *does* have the strong locality property.

The space $P^{1,p}(X)$ is somewhere in between. It has **locality**.

7.5.6

Proposition 7.5.8. *If X supports a p-Poincaré inequality then $P^{1,p}(X) = N^{1,p}(X)$*

Proof. We will prove that $P^{1,p}(X) \subset N^{1,p}(X)$ the other direction is left as an exercise. Let $f \in P^{1,p}(X)$. After modifying on a set of measure zero we will get something in the Newton-Sobolev space. We will be using that $N^{1,p}(X)$ is close to a Banach space in the sense that there exists some $f_k \to f$ which $f_k \in P^{1,p}(X)$.

Fix some $\varepsilon > 0$ we can find an ε-**net** which consists of a sequence of points $\{x_i\}_{i \in I}$ such that for all $x \in X$ there exists some $i \in I$ such that $d(x, x_i) < 2\varepsilon$ and the balls $B_j = B(x_j, 2\varepsilon)$ have small overlap:

$$\sum_i \chi_{10B_i} \leq C.$$

Form a Lipschitz partition of unity subordinate to the cover. There are functions φ_i which are $\frac{C}{\epsilon}$-Lipschitz where $\varphi_i : X \to [0,1]$ have $\mathrm{supp}(\varphi_i) \subset 2B_i$ and

$$\sum_i \varphi_i = 1.$$

We define

$$f_\varepsilon(x) = \sum_i f_{B_i} \varphi_i.$$

For each epslion the function is locally Lipschitz and the function

$$\mathrm{Lip} h(x) := \limsup_{y \to x, y \neq x} \frac{|h(x) - h(y)|}{d(x,y)}$$

is an uppergradient for h.

1. $|f_\varepsilon - f| \to 0$ as $\varepsilon \to 0$.
2. $\{\mathrm{Lip} f_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^p(X)$.
3. Since $p > 1$, L^p is reflexive.
4. Look at a convex combination of the f_ε to

□

Remark 7.5.9. The spaces $M^{1,p}$ and $N^{1,p}$ are used in potential theory. The space $P^{1,p}$ is not used so much.

7.5.7

The **Korevaar-Schoen Space** $KS^{1,p}(X)$ consists of $f \in L^p(X)$ such that

$$\int_X \left(\fint_{B(x,\varepsilon)} \left| \frac{f(x) - f(y)}{\varepsilon} \right|^p d\mu(y) \right) d\mu(x) < \infty$$

The inside can be viewed as an integral average of the derivative or the integral average of energy.

Remark 7.5.10. The above space is used (if one looks at the papers closely) in Diffusion processes on Fractals. See papers by Kigami, Kumagai and Strichartz.

Observe that

$$\int_{B(x,\varepsilon)} \left| \frac{f(x) - f(y)}{\varepsilon} \right|^p d\mu(y) \leq \frac{|f(x) - f_B|^p}{\varepsilon^p} + \frac{1}{\varepsilon^p} \int_B |f(y) - f_{B(x,\varepsilon)}|^p d\mu(y)$$

by the triangle inequality.

- Show that all the spaces $N^{1,p}(X), M^{1,p}(X), P^{1,p}(X), KS^{1,p}(X)$ are the same if X supports a Poincare inequality for upper gradient pairs.

Bibliography

[1] Cite reference