Interpretations & Anabelian Geometry

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(preliminary report)

<u>GOAL</u>: To show how

$$G \rightsquigarrow \begin{array}{l} (G, \Theta^{\nabla}) \\ (G, \Theta^{\times}) \\ (G, \Theta^{\times\mu}) \end{array}$$

can be formalized.

G_{group} classical group language

$G_{\mathrm{group},\infty}$ infinitary group language

G^{Γ} Γ-invariant sets $\Gamma = \mathrm{Aut}(G)$

$G^{\Gamma}_{\mathrm{pro}}$ pro-Γ-invariant sets

G^{Γ}_{∞} Γ-invariant sets w/ disjoint unions

$\underline{p}(G)$
$\underline{f}(G)$
$\underline{d}(G)$
$\underline{e}(G)$ $\Big\}$ auxiliary

$\mu_\infty(G)$

$\boxed{\widehat{\underline{k}^\times}(G) = G^{ab}}$

$\boxed{\underline{\mu}_n(G)}$ $\boxed{\underline{\mu}_\infty(G)}$

$\varprojlim \mu_n(G)$

$\boxed{\underline{\Lambda}(G)}$

$\boxed{\underline{\mathcal{O}}^\times(G) = \underline{I}(G)^{ab}}$ $\overline{\mathcal{O}}^\times(G)$

$\boxed{I(G) = \bigcap\{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\}}$

$\boxed{\underline{k}^+_{\log}(G)}$ $\mathcal{O}^{\times\mu}$

$\boxed{\underline{P}(G)}$ $\bigcap\{N \triangleleft G : p \nmid \frac{e(N)}{e(G)}\}$

$\boxed{\mathrm{Fr}}$ $\{\sigma \in G_{res} ; \forall x \in I/P \;\; \sigma(x) = p^{f} x\}$

$\boxed{\underline{G}_{res}(G) = G/\underline{I}(G)}$

$\boxed{\underline{k}^\times(G)}$ $\overline{\underline{k}}^\times(G)$

$\boxed{\underline{\mathrm{ord}}} : \underline{k}^\times(G) \to \underline{\Gamma}(G) = \frac{\underline{k}^\times(G)}{\underline{\mathcal{O}}^\times(G)}$

$\boxed{\underline{\mathcal{O}}^\triangleright(G)}$ $\overline{\mathcal{O}}^\triangleright(G)$

$\boxed{\underline{\mathcal{I}}(G)}$

G^Γ_∞
G^Γ_{pro}
G^Γ
$\mathcal{G}_{group, m}$
$\mathcal{G}_{group}$

Crash Course In Model Theory

Crash Course In Model Theory

$$
\begin{tikzcd}
\text{Signature} \arrow[r] \arrow[d] & \text{Language / Formulas} \arrow[d] \\
\text{Structure} \arrow[d] & \text{Theories} \arrow[dl] \\
\text{Models} &
\end{tikzcd}
$$

Axioms of Groups

First Order Formulas

0) $\underbrace{\cancel{\exists! e \in G}, \forall x \in G}$ $(x * e = e * x = x)$

1) $\underbrace{\forall x \in G, \exists y \in G}$ $(x * y = y * x = e)$

2) $\underbrace{\forall x, y, z \in G}$ $((x * y) * z = x * (y * z))$

Signature $\ni$ constant symbol

Axioms of Groups

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1) $\forall x \in G, \exists y \in G \; (x * y = y * x = e)$

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First Order Formulas

Signature

$$(G, *, \underline{\underline{e}})$$

function symbol

$$* : G \times G \to G$$

constant symbol

From the signature we generate a language

Axioms of Groups

0) $\cancel{\exists! e \in G}, \forall x \in G \; (x * e = e * x = x)$

1) $\forall x \in G, \exists y \in G \; (x * y = y * x = e)$

2) $\forall x, y, z \in G \; ((x * y) * z = x * (y * z))$

$$(G, *, e)$$

Another Example: Fields

$$(F, +, *, 0, 1)$$

Axioms of Groups

$(G, *, e)$

1) $\cancel{\forall x \in G, \exists y \in G \ (x * y = y * x = e)}$

1') $\forall x \in G \ (S(x) * x = x * S(x) = e)$

New Function Symbol: $S : G \to G$

$S(x) = x^{-1}$

Axioms of Groups

Old Signature

$(G, *, e)$

Expansion

Reduct

New Signature

$(G, *, S, e)$

UPSHOT: You can have multiple ways of talking about the same object.

$\boxed{\text{Crash Course In Model Theory}}$

$\sigma_{\text{rings}} \rightsquigarrow L(\sigma_{\text{rings}})$

$\sigma_{\text{groups}} \rightsquigarrow L(\sigma_{\text{groups}})$

$\sigma_{\text{ordered abelian groups}} = (\Gamma, +, 0, \leq) \rightsquigarrow L(\sigma_{\text{ordered abelian groups}})$

$\sigma_{\text{valued fields}} = \big((K, +, -, \cdot, 0, 1), (\Gamma \cup \{\infty\}, +, -, 0, \infty), \text{val}\big]$

$\text{val}: K^\times \longrightarrow \Gamma \cup \{\infty\}$

$\sigma_{\text{ordered abelian groups}} = (\Gamma, +, 0, \leq) \rightsquigarrow L(\sigma_{\text{ordered abelian groups}})$

$\sigma_{\text{valued fields}} = \big((K, +, -, \cdot, 0, 1), (\Gamma \cup \{\infty\}, +, -, 0, \infty), \text{val} \big]$

two "sorts"

<u>Definable sets</u>:

$X \subseteq K^n \times \Gamma^m$

Crash Course In Model Theory

$\sigma_{\substack{\text{valued} \\ \text{fields}}} = \left((K, +, -, \cdot, 0, 1), (\Gamma \cup \{\infty\}, +, -, 0, \infty), \mathrm{val} \right]$

Similar to Valued Fields: $(G, \mathcal{O}^{\triangleright})$ or $(G, \mathcal{O}^{\times})$

seems multi-sorted.

$\boxed{\text{Crash Course In Model Theory}}$

Signature for Categories:

$$(C_0, C_1, \underbrace{s, t}_{\text{function symbols}}, \circ)$$

Relation on $C_1 \times C_1 \times C_1$

$f = g \circ h$

Sort for objects

Sort for morphisms

$s : C_1 \to C_0$ (source)

$t : C_1 \to C_0$ (target)

Relation Principle:

Functions, constants, and sorts can be encoded as relations.

UPSHOT: When making abstract definitions we only need ~~to~~ worry how it behaves with respect to relations.

Definable Sets

Definable Sets

Fix a signature σ

Fix a σ-structure M

Formula $\phi(x_1, \ldots, x_n)$ with n free variables,

$$M \longmapsto \underbrace{\{(a_1, \ldots, a_n) \in M^n : \phi(a_1, \ldots, a_n) \text{ is TRUE}\}}_{\text{Definable Set}}$$

EXAMPLE

$M \longmapsto \{(a_1, \ldots, a_n) \in M^n : \phi(a_1, \ldots, a_n) \text{ IS TRUE}\}$

$\sigma \leftarrow$ Language of Fields

$M \leftarrow$ An actual field

$\phi(x_1, \ldots, x_n) \leftarrow [\exists y\ (x^2+y^2=1)] = \phi(x)$

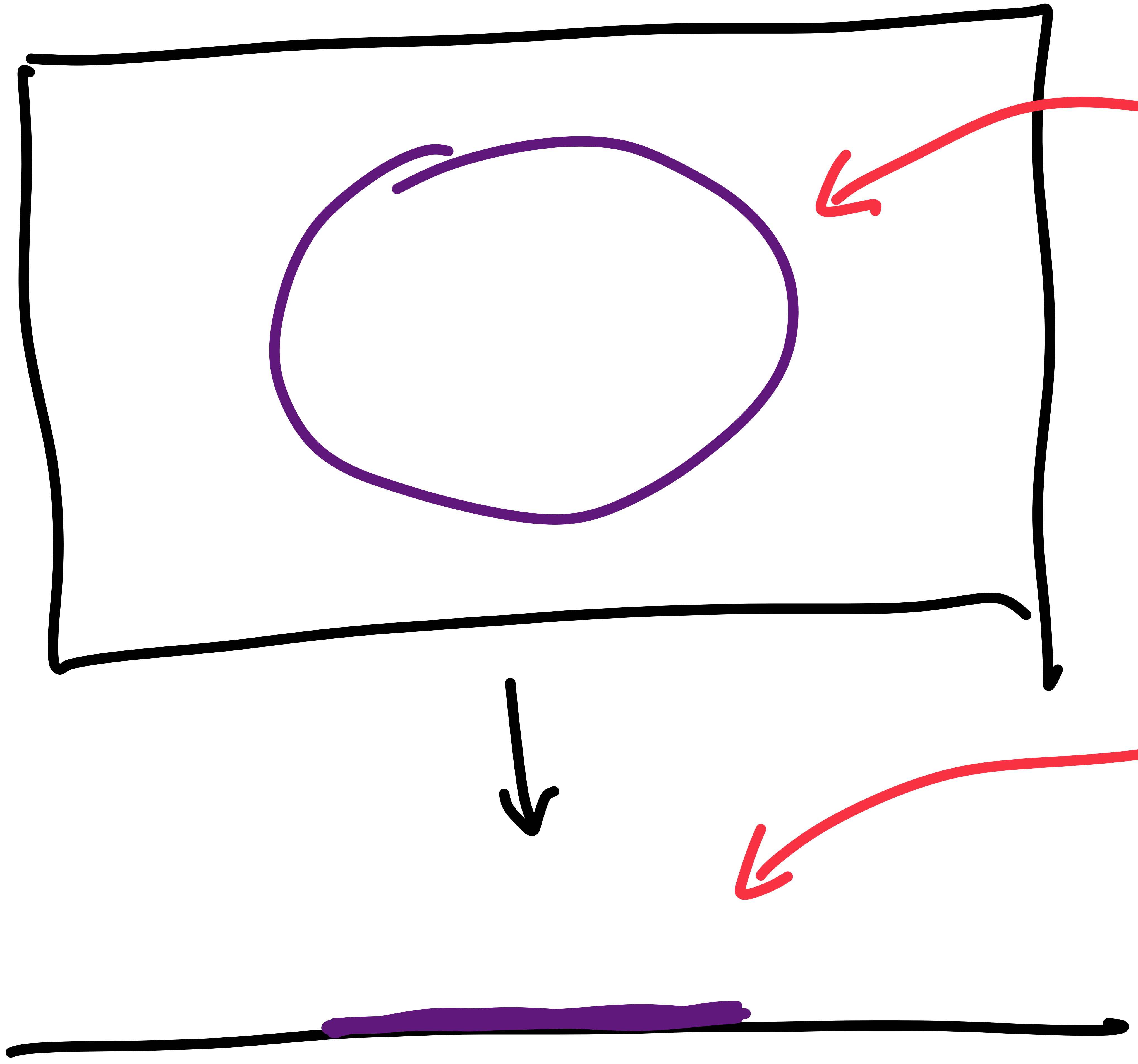
$\boxed{\text{Definable Sets}}$

Definable Sets

EXAMPLE

M An actual field

$M \longmapsto \{(a_1,\dots,a_n)\in M^n : \phi(a_1,\dots,a_n) \text{ is TRUE}\}$



$\{(x,y)\in M^2 : x^2+y^2=1\}$

$\{x\in M : \exists y\,(x^2+y^2=1)\}$

$[\exists y\,(x^2+y^2=1)] = \phi(x)$

Definable Sets

Two PESPECTIVES

1) Functor of Points

$M \longmapsto \{(a_1, \ldots, a_n) \in M^n : \phi(a_1, \ldots, a_n) \text{ is TRUE}\}$

2) Subset $X \subseteq M^n$ for fixed M.

$\{(a_1, \ldots, a_n) \in M^n : \phi(a_1, \ldots, a_n) \text{ is TRUE}\}$

Definable Sets

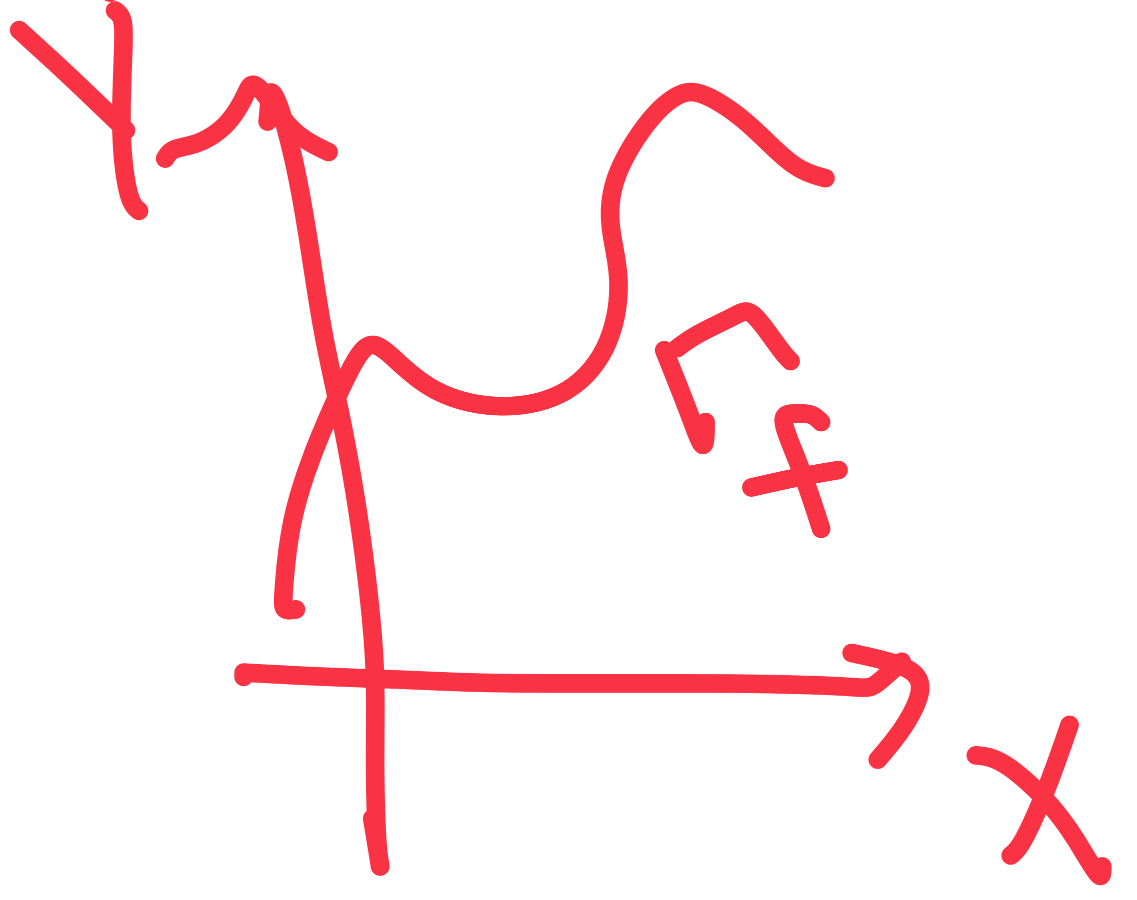
Def_M = (Category of Definable Sets in a Structure M)

Objects: Definable Sets $X \subseteq M^n$

Morphism: Definable Morphisms

$f: X \to Y$

Graph is Definable



Interpretations

$\boxed{\text{Interpretations}}$

M, N structures.

An <u>interpretation</u> of N in M is

$$\begin{array}{l} I : X \twoheadrightarrow N \\ \quad\ \ \cap| \\ \quad\ \ M^m \end{array}$$

such that the inverse image of every definable set is definable.

<u>Think:</u>

$M = \mathbb{R}$

$N = \mathbb{C}$

$$I : \underset{\substack{\cap \\ M^m}}{X} \xrightarrow{\text{SURJECTIVE}} \!\!\!\!\rightarrow N$$

Definable Subset of M^m.

Such that the inverse image of every definable set is definable.

$$Y \subseteq N^n, \quad I^{-1}(Y) \stackrel{def}{=} (\underbrace{I \times \cdots \times I}_{n\text{-times}})^{-1}(Y)$$

$\boxed{\text{Interpretations}}$

$\boxed{\text{Interpretations}}$ M, N structures.

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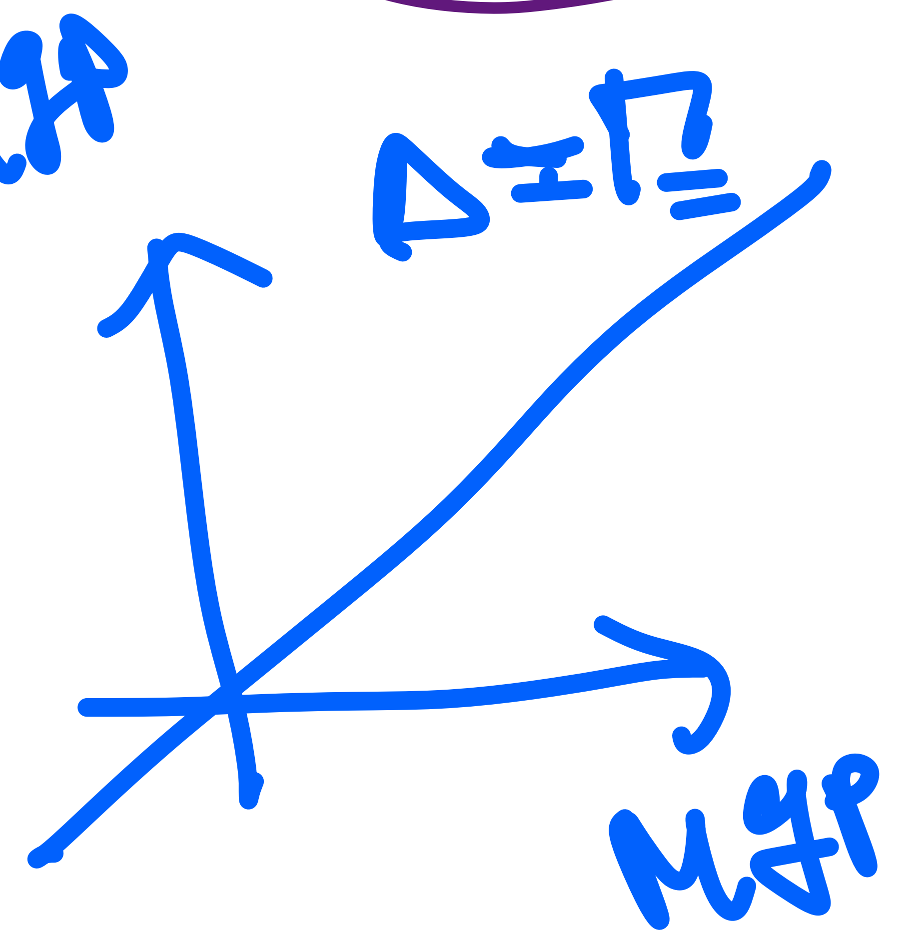
such that the inverse image of every definable set is definable.

$I : M^2 \twoheadrightarrow M^{gp}$ $I(a,b) = a-b$.   EXAMPLE

Definable set for M^{gp}

$\Gamma_= = \begin{pmatrix}\text{graph} \\ \text{of equality} \\ \text{in } M^{gp}\end{pmatrix} \subseteq M^{gp} \times M^{gp}$



$$I(a_1,b_1) = I(a_2,b_2) \iff a_1 - b_1 = a_2 - b_2$$
$$\iff \underbrace{a_1 + b_2 = a_2 + b_1}$$

$$I^{-1}(\Gamma_=) = \{(a_1,b_1,a_2,b_2) \in M^4 : a_1+b_2 = a_2+b_1\}$$

Definable Set on Target

Inverse Image

$\Gamma_=$ $\quad I(a_1,b_1) = I(a_2,b_2) \quad a_1+b_2 = a_2+b_1$

Γ_+ $\quad I(a_3,b_3) = I(a_1,b_1) + I(a_2,b_2) \quad a_3+b_1+b_2 = a_1+a_2+b_3$

EXAMPLE

Lemma: To check that

$$I : \underset{\substack{\cong \\ M^n}}{X} \longrightarrow N$$

is an interpretation it suffices to check the graph of all the relations.

EXAMPLE $(G, *, e)$ interprets $(G, *, S, e)$

proof. • $G \xrightarrow{id} G$, we just check $\Gamma_S \subseteq G \times G$

• $\Gamma_S = \{(x,y) \in G^2 : y = S(x)\}$

$= \{(x,y) \in G^2 : x * y = e\}$, ← definable in $(G, *, e)$. //

EXAMPLE:

$(G, *, e, \underbrace{\in N}_{\text{new relation symbol}})$

G group

$N \trianglelefteq G$ normal subgroup.

G with "$\in N$" interprets G/N.

~~proof~~ idea $G \to G/N$, $g \longmapsto [g] = gN$.

$[g_1] = [g_2] \iff g_1 N = g_2 N \iff \underbrace{g_2^{-1} g_1 \in N}_{\text{Definable in enhanced signature.}}$

R, C structures.

$\underline{I} : X \to C$ interp.

$X \subseteq R^n$

Notation: $\underline{C}(R) = I^{-1}(C)$

structure constructed from R.

Notation:

Write $C \leq R \iff R$ interprets C.

R, C structures.

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Notation: $\underline{C}(R) = I^{-1}(C)$
structure constructed from R.

Notation:
Write $C \leq R \iff R$ interprets C,

Composition: $R_1 \leq R_2$ & $R_2 \leq R_3$ implies $R_1 \leq R_3$
$\rightsquigarrow R_3 \longmapsto \underline{R_1}(R_2(R_3))$ is the interp.

Defn. A history of computations is a sequence $R_1 \leq R_2 \leq \cdots \leq R_n$

Representations: $\mathrm{Aut}(R) \xrightarrow[\rho_I]{} \mathrm{Aut}(C)$ as above.

Defn. I is multiradical $\iff \rho_I$ surjective.

R, C structures.

$I: X \to C$ interp.

$X \subseteq R^n$

Notation: $\underline{C}(R) = I^{-1}(C)$ structure constructed from R.

Notation: Write $C \leq R \iff R$ interprets C.

Composition: $R_1 \leq R_2$ & $R_2 \leq R_3$ implies $R_1 \leq R_3$

$\rightsquigarrow$ $R_3 \mapsto \underline{R_1}(\underline{R_2}(R_3))$ is the interp.

Representations: $\mathrm{Aut}(R) \xrightarrow[\rho_I]{} \mathrm{Aut}(C)$ as above.

Defn. A *history of computations* is a sequence $R_1 \leq R_2 \leq \cdots \leq R_n$

Defn. I is *multiradical* $\iff$ ρ_I surjective.

Functorality: The interpretation I induces a functor $\mathrm{Def}_C \longrightarrow \mathrm{Def}_R$ in the category of definable sets

<u>Functorality</u>: The interpretation I induces a functor $\mathrm{Def}_C \to \mathrm{Def}_K$ in the category of definable sets

- One can characterize such functors
- One can prove <u>converses</u>!

Functorality: The interpretation I induces a functor $\mathrm{Def}_C \longrightarrow \mathrm{Def}_R$ in the category of definable sets

Theorem. Any Boolean logical Functor $\mathrm{Def}_C \longrightarrow \mathrm{Def}_R$ is induced by an interpretation.

Defn. C and R are **bi-interpretable** iff $C \leq R$ and $R \leq C$ such that the associated functors induce an equivalence of categories. *

Remark: A formal justification for replacing Frobenius structures with $(G_v, \mathcal{O}^\diamond)$ could be achieved by specifying & proving bi-interpretability.

$\boxed{\text{Pre-Topos Completion}}$

Given interpretation of A in B: $\begin{array}{c} X \xrightarrow{I} A \\ \cap\!| \\ B^n \end{array}$

$\rightsquigarrow$ expansion of B: (B, A, r)

new sort (under A)

relation on $B^n \times A$ (under r)

$r(\vec{b}, a) \iff I(\vec{b}) = a.$

$$\boxed{\begin{array}{l} \underline{\text{UPSHOT}}: \ \mathrm{Aut}((B,A)) = \mathrm{Aut}(B) \times \mathrm{Aut}(A) \\ \qquad\qquad\ \ \mathrm{Aut}((B,A,r)) = \mathrm{Aut}(B) \end{array}}$$

"on/off switch" for interpretation

Pre-Topos Completion

Given interpretation of A in B: $\begin{array}{ccc} X & \xrightarrow{I} & A \\ \cap\!| & & \\ B^n & & \end{array}$

$\rightsquigarrow$ expansion of B: (B, A, r)

new sort

relation on $B^n \times A$

$r(\vec{b}, a) \iff I(\vec{b}) = a$.

(Pre-Topos Completion) $\leftarrow$ (Add a sort for <u>every</u> interpretable equivalence relation.)

Applications to IUT

Fundamental Groups

Object	Description	Interpretation
Π	$\cong \pi_1(Z, \overline{z})$	Given
$\mathbf{p}(G)$	$\mathrm{char}(\mathcal{O}_K/\mathfrak{m}_K)$	the unique prime l such that for all other primes $l' \neq l$ we have $\dim_l(\Pi) - \dim_{l'}(\Pi) > 0$.
$\mathbf{d}(G)$	$[K : \mathbb{Q}_p]$	$\dim_{\mathbf{p}(\Pi)}(\Pi) - \dim_l(\Pi)$ for any prime $l \neq \mathbf{p}(\Pi)$
$\mathbf{\Delta}(\Pi)$	$\cong \pi_1(Z_{\overline{K}})$	as in Table 5 or $\mathbf{\Delta}(\Pi) = \bigcap\{\Pi_0 \subset \Pi$ clopen : $\dim_{\mathbf{p}(\Pi)}(\Pi) - \dim_l(\Pi_0) = \mathbf{d}(\Pi)[\Pi : \Pi_0]\}$ where l is any prime not equal to $\mathbf{p}(\Pi)$.
$\mathbf{G}(\Pi)$	$\cong G_K$	as in Table 5
D_I for $I \in \mathrm{Cusp}(\Pi)$	$\cong D_{\overline{z}/z}$, $I_{\overline{z}/z}$ inertial group of a cusp	$N_\Pi(I)$
$\overline{\mathbf{\Delta}}(\Pi)$	$\cong (\pi_1(\overline{Z}_{\overline{K}}))$	$\mathbf{\Delta}/J(\Pi)$ where $J(\Pi) \subset \mathbf{\Delta}(\Pi)$ smallest open normal containing I for $I \in \mathrm{Cusp}(\Pi)$
$\Lambda(\Pi)$	$\cong \widehat{\mathbb{Z}}(1)$ as G_K-module.	$H^2(\overline{\Delta}, \widehat{\mathbb{Z}})^* = \mathrm{Hom}(H^2(\overline{\mathbf{\Delta}}(\Pi), \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}))$
$\lvert\Pi\rvert$ (SBT)	$\cong \lvert Z\rvert$	approximation by elements of $\mathrm{NF}(\Pi)$ [Moc05, Lemma 3.1.i.iv]

$\mathrm{div}(\mathcal{O}^\times(\Pi))$	$\cong \mathrm{div}(\mathcal{O}(Z)^\times)$	$\ker(\alpha) \cap \deg^{-1}(0)$ where $\alpha \colon \mathbb{Z}^{\oplus \mathrm{Cusp}(\Pi)} \to H^1(\mathbf{G}(\Pi), \overline{\mathbf{\Delta}}(\Pi)^{\mathrm{ab}})$ given by $\sum_{I \in \mathrm{Cusp}(\Pi)} n_I I \mapsto \sum_{I \in \mathrm{Cusp}(\Pi)} n_I s^{\mathrm{ab}}_{D_I}$. See prose at beginning of this appendix.
$\mathcal{O}^\times(\Pi)$	$\cong \mathcal{O}^\times(Z)$	$p^{-1}(\mathrm{div}(\mathcal{O}(\Pi)^\times))$
$\mathcal{O}^\times(\Pi \setminus S)$ (SBT)	$\cong \mathcal{O}^\times(Z \setminus S)$	$\mathcal{O}^\times(\pi_1(\Pi \setminus S))$
$\mathbf{K}^\times(\Pi)$ (SBT)	$\cong \kappa(Z)^\times$	$\varinjlim \mathcal{O}^\times(\Pi \setminus S)$
$\mathbf{k}^\times(\Pi)$	$\cong K^\times$	$\mathcal{O}^\times(\overline{\Pi})$
$\mathbf{K}_0^{\mathrm{geom}\times}(\Pi)$ (SBT)	$\cong \kappa((Z_0)_{\overline{K_0}})^\times$	$\{\eta \in \mathbf{K}^\times(\Pi) : \exists n \in \mathbb{N}, \exists D \in \lvert\Pi\rvert_{\mathrm{NF}} \ \eta^n\vert_D = 1\}$ see [Moc08, 1.8.i]
$\overline{\mathbf{k}}_0^\times(\Pi)$ (SBT)	$\cong \overline{K_0}^\times$	image of $\mathbf{K}_0^{\mathrm{geom}\times}$ and $\lvert\Pi\rvert_{\mathrm{NF}}$ under evaluation; compose with synchronization. [Moc08, 1.8.ii]
$\mathbf{ord}_I : \mathbf{K}^\times(\Pi) \to \mathbb{Z}$, $I \in \mathrm{Cusp}(S)$ $S \in \mathrm{Open}_{\mathrm{NF}}(\Pi)$ (SBT)	$\mathrm{ord}_s : \kappa(Z)^\times \to \mathbb{Z}$, $s \in \lvert Z_0\rvert$	$\kappa_f\vert_I \in H^1(I, \Lambda(\Pi))$
$\mathrm{Div}(\lvert\overline{\Pi}\rvert_{\mathrm{NF}})$ (SBT)	$\cong \mathrm{Div}((\overline{Z}_0)_{\overline{\mathbb{Q}}})$	$\mathbb{Z}^{\oplus\lvert Z\rvert_{\mathrm{NF}}}$
$H^0(\mathcal{O}_\Pi(D))$ $D \in \mathrm{Div}(\lvert\overline{\Pi}\rvert_{\mathrm{NF}})$ (SBT)	$\cong H^0(\overline{Z}, \mathcal{O}_Z(D)) = \{f \in \kappa((Z_0)_{\overline{\mathbb{Q}}})^\times : \mathrm{div}(f)\vert_Z + D\vert_Z \geq 0\}$ for $D \in \mathrm{Div}((\overline{Z}_0)_{\overline{\mathbb{Q}}})$	$\{f \in \mathbf{K}_0^{\mathrm{geom}}(\overline{\Pi}) : \mathrm{div}(f) + D \geq 0\}$ here $D \in \mathrm{Div}(\lvert\overline{Z}\rvert_{\mathrm{NF}})$
$\overline{\mathbf{k}_0}(\Pi)^{\mathrm{Kum}}$ (SBT)	$\cong K_0$ as a field	Uchida trick/fundamental theorem of projective geometry (see §??)
$\overline{\mathbf{K}_0^{\mathrm{geom}}}(\Pi)^{\mathrm{Kum}}$ (SBT)	$\cong \kappa(Z_{\overline{K}_0})$ as a field	$\overline{\mathbf{K}_0^{\mathrm{geom}}}(\Pi)^{\mathrm{Kum}} = \overline{\mathbf{K}_0^{\mathrm{geom}}}^\times(\Pi)^{\mathrm{Kum}} \cup \{0\}$ and the field structure is induced by the injection into $\bigoplus_{x \in \mathrm{NF}(\Pi)} \kappa(x)$

Cyclotomic Synchronizations

Object	Description	Interpretation
MT	monotheta environment	given
G	$\cong G_K$	given
$\overline{M}$	$\cong \mathcal{O}^{\triangleright}_{\overline{K}}$	given
Π	$\cong \pi_1(Z)$, Z hyperbolic curve	given
I $\mathrm{sync}^{\overline{M}}_G : \Lambda(\overline{M}) \to \Lambda(G)$	$\cong \widehat{\mathbb{Z}}(1)$, inertia subgroup of $\pi_1(\mathbb{Z}_{\overline{K}})$ Brauer synchronization	given $\mathbf{inv}_G \circ H^2(G, \mathrm{sync}^{\overline{M}}_G) = \mathbf{inv}_{(\overline{M},G)}$
$\mathrm{sync}^\Pi_G : \Lambda(\Pi) \to \Lambda(G)$	bilinear synchronization	The unique element of $\mathrm{Hom}(\Lambda(\Pi), \Lambda(G)) \cap P$ where P is the positive rational structure. [Moc07a]
$\mathrm{sync}^\Pi_I : \Lambda(\Pi) \to I$	cuspidal synchronization	$d_2^{1,0}(\mathrm{id}_I)$; This is the map on the second page of the spectral sequence associated to the exact sequence $1 \to I \to \mathbf{\Delta}^{\mathrm{cc}}(X) \to \overline{\mathbf{\Delta}}(X) \to 1$. One computes $H^0(\overline{\mathbf{\Delta}}(X), H^1(I, I)) = \mathrm{Hom}(I, I)$ and $H^2(\overline{\mathbf{\Delta}}(\Pi), H^0(I, I)) = \mathrm{Hom}(\Lambda(\Pi), I)$.
$\mathrm{sync}^{\mathrm{ext}}_{\mathrm{int}} : \Lambda(\mathrm{MT})^{\mathrm{ext}} \to \Lambda(\mathrm{MT})^{\mathrm{int}}$	monotheta cyclotomic synchronization	$s - \mathbf{s}^{\mathrm{taut}}$

Table 4: A table of cyclotomic synchronizations.

Galois Groups

Object	Description	Interpretation
G	$\cong G_K$ as topological groups	given
$\mathbf{p}(G)$	$\mathrm{char}(k)$	$l = \mathbf{p}(G) \Longleftrightarrow l$ prime and $\log_l(\#G^{\mathrm{ab/tors}}/lG^{\mathrm{ab/tors}}) \geq 2$
$\mathbf{f}(G)$	$[k : \mathbb{F}_p]$	$\log_{\mathbf{p}(G)}(1 + \#((G^{\mathrm{ab/tors}})^{\mathbf{p}(G)}))$
$\mathbf{d}(G)$	$[K : \mathbb{Q}_p]$	$\log_{\mathbf{p}(G)}(\#G^{\mathrm{ab/tors}}/lG^{\mathrm{ab/tors}}) - 1$ (for any $l \neq \mathbf{p}(G)$)
$\mathbf{e}(G)$	Inertia degree	$\mathbf{d}(G)/\mathbf{f}(G)$
$\widehat{\mathbf{k}^\times}(G)^{\mathrm{LCFT}}$	$\varprojlim K^\times/K^{\times n}$	G^{ab}
$\widehat{\overline{\mathbf{k}}^\times}(G)^{\mathrm{LCFT}}$	$\varinjlim \varprojlim K_i^\times/K_i^{\times n}$	$\varinjlim_{G_0 \subset G} \widehat{\mathbf{k}}^\times(G_0)$, this limit varies over open subgroups and the transition maps are given by the transfer maps
$\mu_n(G)$	$\cong \mu_n(\overline{K})$ as G_K-modules	n-torsion of $\overline{\mathbf{k}}^\times(G)$
$\Lambda(G)$	$\cong \widehat{\mathbb{Z}}(1)$ as G_K-modules	$\varprojlim \mu_n(G)$
$\mathcal{O}^\times(G)$	$\cong \mathcal{O}_K^\times$	$\ker(G \to \mathbf{G}_{\mathrm{res}}(G))$
$\overline{\mathcal{O}}^\times(G)$	$\cong \mathcal{O}_{\overline{K}}^\times$	$\varinjlim_{G_0 \subset G} \mathcal{O}^\times(G)$
$\mathbf{k}^+_{\log}(G)$	$\cong K^+_{\log}$ as topological abelian groups	$\mathcal{O}^\times(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ (same as monoid perfection)

Applications to IUT

$[K:\mathbb{Q}_p] < \infty$ finite extn of $\mathbb{Q}_p$

Galois Groups

$G = \left(\text{Absolute Galois Group of } K \right)$

$= G(\overline{K}|K)$

Object	Description	Interpretation
G	$\cong G_K$ as topological groups	given
$\mathbf{p}(G)$	$\mathrm{char}(k)$	$l = \mathbf{p}(G) \Longleftrightarrow l$ prime and $\log_l(\#G^{\mathrm{ab/tors}}/lG^{\mathrm{ab/tors}}) \geq 2$
$\mathbf{f}(G)$	$[k : \mathbb{F}_p]$	$\log_{\mathbf{p}(G)}(1 + \#((G^{\mathrm{ab/tors}})^{\mathbf{p}(G)}))$
$\mathbf{d}(G)$	$[K : \mathbb{Q}_p]$	$\log_{\mathbf{p}(G)}(\#G^{\mathrm{ab/tors}}/lG^{\mathrm{ab/tors}}) - 1$ (for any $l \neq \mathbf{p}(G)$)
$\mathbf{e}(G)$	Inertia degree	$\mathbf{d}(G)/\mathbf{f}(G)$
$\widehat{\mathbf{k}^\times}(G)^{\mathrm{LCFT}}$	$\varprojlim K^\times/K^{\times n}$	G^{ab}
$\overline{\mathbf{k}}^\times(G)^{\mathrm{LCFT}}$	$\varinjlim \varprojlim K_i^\times/K_i^{\times n}$	$\varinjlim_{G_0 \subset G} \widehat{\mathbf{k}}^\times(G_0)$, this limit varies over open subgroups and the transition maps are given by the transfer maps
$\mu_n(G)$	$\cong \mu_n(\overline{K})$ as G_K-modules	n-torsion of $\overline{\mathbf{k}}^\times(G)$
$\Lambda(G)$	$\cong \widehat{\mathbb{Z}}(1)$ as G_K-modules	$\varprojlim \mu_n(G)$
$\mathcal{O}^\times(G)$	$\cong \mathcal{O}_K^\times$	$\ker(G \to \mathbf{G}_{\mathrm{res}}(G))$
$\overline{\mathcal{O}}^\times(G)$	$\cong \mathcal{O}_{\overline{K}}^\times$	$\varinjlim_{G_0 \subset G} \mathcal{O}^\times(G)$
$\mathbf{k}_{\log}^+(G)$	$\cong K_{\log}^+$ as topological abelian groups	$\mathcal{O}^\times(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ (same as monoid perfection)

$\underline{p}(G)$

$\underline{f}(G)$

$\underline{d}(G)$

$\underline{e}(G)$

[G, G] definable

$\boxed{\widehat{\underline{k}^\times}(G) = G^{ab}}$

$\mu_n(G)$ $\mu_\infty(G)$

$\Lambda(G)$ $\varprojlim \mu_n(G)$

$\underline{O}^\times(G) = \underline{I}(G^{ab})$

$I(G) = \bigcap\{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\}$

$\underline{k}^+_{\log}(G)$ $\quad O^{\times\mu}$

$\underline{P}(G)$

Fr $\quad \{\sigma \in G_{res} : \forall x \in I/p \;\; \sigma(x) = p^{f_x}\}$

$\underline{G}_{res}(G)$ $\quad G/\underline{I}(G)$

$\underline{k}^\times(G)$ $\quad \bar{k}^\times(G)$

ord $\quad \underline{k}^\times(G) \to \underline{\Gamma}(G) = \dfrac{\underline{k}^\times(G)}{\underline{O}^\times(G)}$

$\underline{O}^\triangleright(G)$

$\underline{I}(G)$

Galois Groups

$\boxed{\text{Interpretation of Torsion}}$

A abelian group

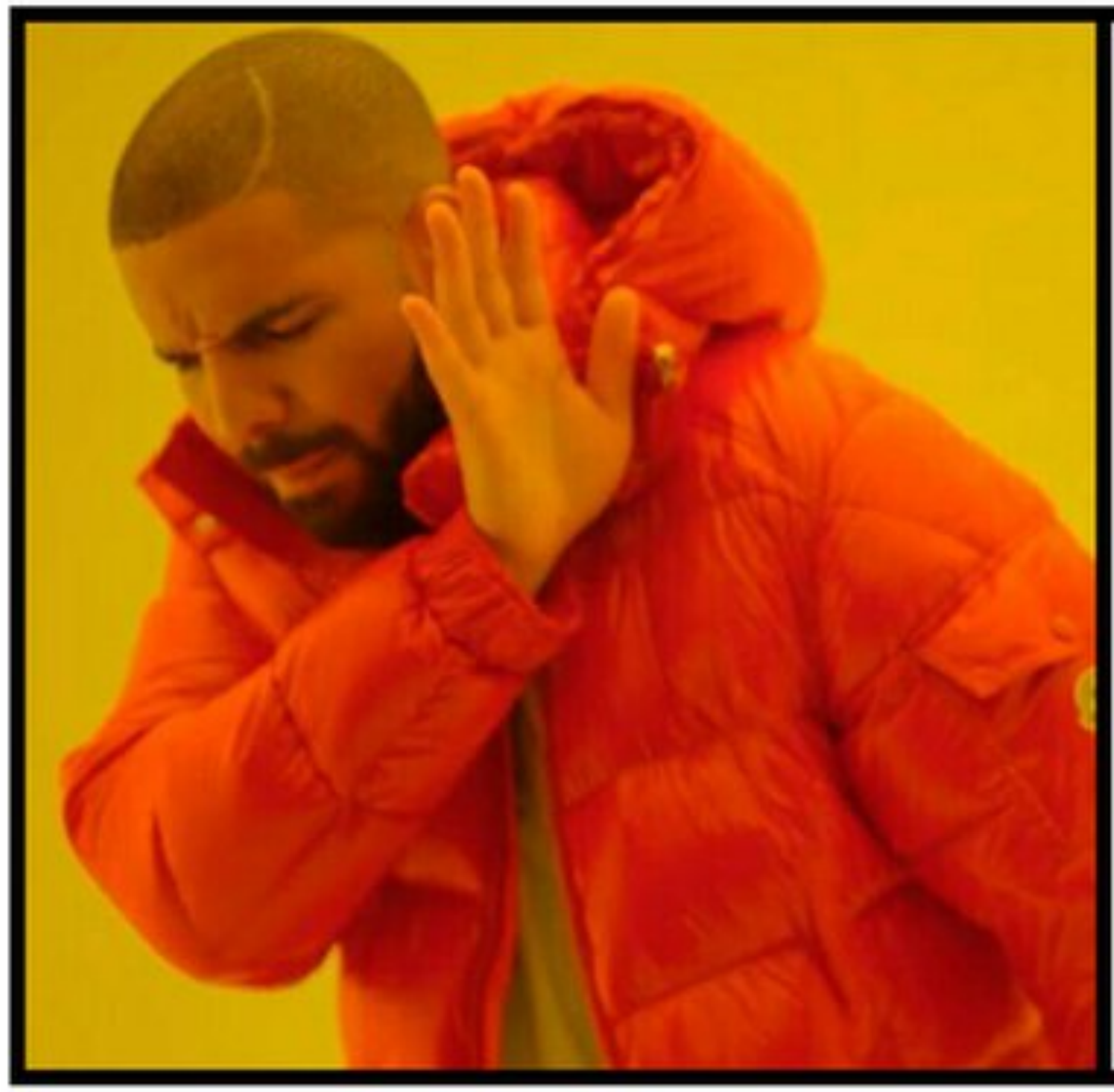
$$a \in A \text{ torsion} \iff \underbrace{\exists n \in \mathbb{N}}_{\text{Not in signature}}, \quad \underbrace{n}_{} a = 0$$

Interpretation of Torsion

A abelian group

$\exists n \in \mathbb{N}$

$na = 0$



$$na = 0$$

$$\underbrace{a + a + \cdots + a}_{n\text{-times}} = 0$$

A abelian group

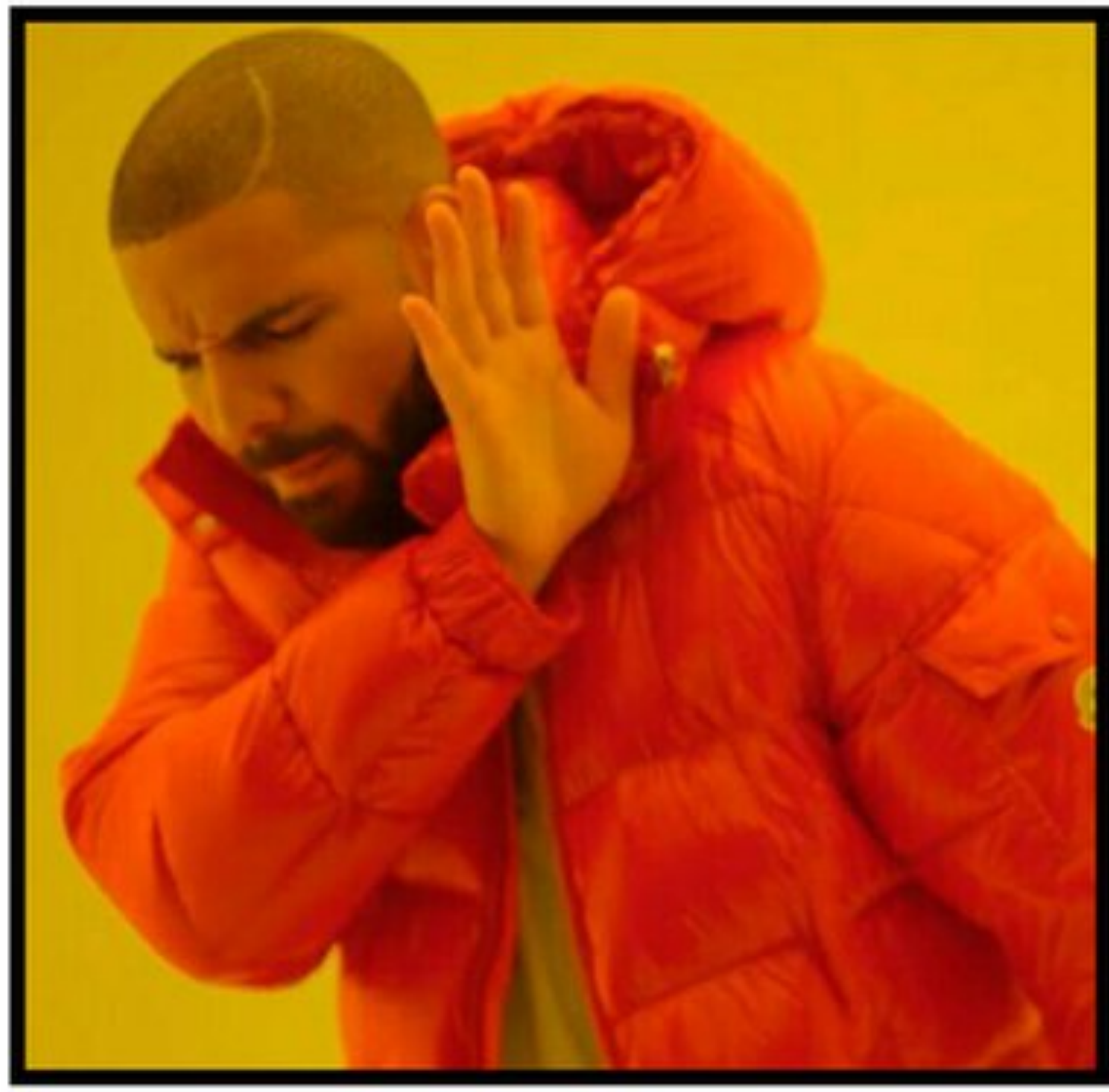
Interpretation of Torsion

$\exists n \in \mathbb{N}$ $na = 0$

$\underbrace{a + a + a + \cdots + a}_{n\text{-times}} = 0$

$\phi_n(a)$

$\exists n \in \mathbb{N},\ \phi_n(a)$



$\exists n \in \mathbb{N},\ \phi_n(a)$

$\phi_1(a) \vee \phi_2(a) \vee \phi_3(a) \vee \dots$

Interpretation of Torsion

A abelian group

$\exists n \in \mathbb{N}$ $\quad$ $na = 0$

$$\underbrace{a + a + a + \cdots + a}_{n\text{-times}} = 0$$

$\exists n \in \mathbb{N},\ \phi_n(a)$ $\quad$ $\phi_n(a)$

$\phi_1(a) \vee \phi_2(a) \vee \phi_3(a) \vee \cdots$

A abelian group

Interpretation of Torsion

$$\exists n \in \mathbb{N} \quad na = 0 \iff \underbrace{\phi_1(a) \vee \phi_2(a) \vee \phi_3(a) \vee \cdots}_{\text{First Order Formula!}}$$

$L_{\omega_1,\omega}$

(Commutator subgroups can be handled similarly)

$\boxed{\text{Interpretation of Torsion}}$

UPSHOT: We will need infinitary logic

First Order Categorical Logic
Model-Theoretical Methods in the Theory of Topoi and Related Categories

Michael Makkai and Gonzalo E. Reyes

Things Get Worse

Things Get Worse

G

$\rightsquigarrow$

$p(G) = \left[\text{unique } \ell \in \mathbb{N} \text{ such that} \right.$

1) ℓ prime

OK

$\left. \text{2) } \log_\ell\left(\#\left[G^{ab/tors} / \ell G^{ab/tors}\right]\right) \geq 2 \right]$

G

$\boxed{\text{Things Get Worse}}$

$$\underline{p}(G) = (\ell \text{ prime}) \;\&\; \left(\log_{\ell}\left(\# G^{ab/tors} / \ell G^{ab/tors} \right) \geq 2 \right)$$

$$\underline{f}(G) = \log_{\underline{p}(G)}\left(1 + \# \left(G^{ab/tors}\right)^{\underline{p}(G)} \right)$$

$$\underline{d}(G) = \log_{\underline{p}(G)}\left(\# G^{ab/tors} / \ell G^{ab/tors} \right) - 1$$

$$\underline{e}(G) = \underline{d}(G) / \underline{f}(G)$$

Things Get Worse

$\underline{p}(G) = (\ell \text{ prime})\ \&\ \left(\log_\ell\left(\#G^{ab/tors}/\ell G^{ab/tors}\right) \geq 2\right)$

$\underline{f}(G) = \log_{\underline{p}(G)}\left(1 + \#(G^{ab/tors})\,\underline{p}(G)\right)$

$\underline{d}(G) = \log_{\underline{p}(G)}\left(\#G^{ab/tors}/\ell G^{ab/tors} - 1\right)$

$\underline{e}(G) = \underline{d}(G)/\underline{f}(G)$

$$G \rightsquigarrow \begin{aligned} \underline{\underline{I}}(G) &= \bigcap \{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\} \\ \underline{G}_{res}(G) &= G/\underline{\underline{I}}(G) \\ \underline{\Theta}^{\times}(G) &= \varinjlim \underline{\Theta}^{\times}(G_0) \end{aligned}$$

Things Get Worse

$$\underline{I}(G) = \bigcap \{ N \trianglelefteq G : \underline{e}(N) = \underline{e}(G) \}$$

$$\underline{G}_{res}(G) = G / \underline{I}(G)$$

$$\underline{\Theta}^x(G) = \varinjlim \underline{\Theta}^x(G_0)$$

$$\underline{p}(G) = (\ell \text{ prime}) \ \& \ \left(\log_\ell \left(\# G^{ab/tors} / \ell G^{ab/tors} \right) \geq 2 \right)$$

$$\underline{f}(G) = \log_{\underline{p}(G)} \left(1 + \# (G^{ab/tors})_{\underline{p}(G)} \right)$$

$$\underline{d}(G) = \log_{\underline{p}(G)} \left(\# G^{ab/tors} / \ell G^{ab/tors} \right) - 1)$$

$$\underline{e}(G) = \underline{d}(G) / \underline{f}(G)$$

Even using infinitary logic it is unclear how to formalize!

- topology
- direct limits
- weird constants

Return To Structures

Return To Structures

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, January 1996

ZARISKI GEOMETRIES

EHUD HRUSHOVSKI AND BORIS ZILBER

1. Introduction

Let k be an algebraically closed field. The set of ordered n-tuples from k is viewe
as an n-dimensional space; a subset described by the vanishing of a polynomial,
a family of polynomials, is called an *algebraic set*, or a *Zariski closed set*. Algebra

LECTURES ON THE AX–SCHANUEL CONJECTURE

BENJAMIN BAKKER AND JACOB TSIMERMAN

Abstract. Functional transcendence results have in the last decade found a number of important applications to the algebraic and arithmetic geometry of varieties X admitting flat or hyperbolic uniformizations: Pila and Zannier's new proof of the Manin–Mumford conjecture, the proof of the André–Oort conjecture for A_g, and the generic Shafarevich conjecture for hypersurfaces

Return To Structures

$\mathbb{R}$ structure $\rightsquigarrow$ Collections of subsets $\underbrace{S_n}_{\text{definable sets.}} \subseteq \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_{\geq 1}$.

S_n closed under
- finite union
- finite intersect
- complement

$(S_n)_{n=1}^{\infty}$ closed under
- finite products
- coordinate projections.

Defn An <u>abstract structure</u> is

Return To Structures

collections of subsets $S_n \subseteq \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_{\geq 1}$, such that:

1) S_n closed under
- finite union
- finite intersect
- complement

2) $(S_n)_{n=1}^{\infty}$ closed under
- finite products
- coordinate projections.

Return To Structures

Classical First Order Structure $\xleftrightarrow{\text{equivalent}}$

Defn An <u>abstract structure</u> is collections of subsets $S_n \subseteq \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_{\geq 1}$, such that:

1) S_n closed under
   - finite union
   - finite intersect
   - complement

2) $(S_n)_{n=1}^{\infty}$ closed under
   - finite products
   - coordinate projections.

Return To Structures

Let G be a set and let $\Gamma \leq \mathrm{Perm}(G)$ be a subgroup.

Defn. The <u>Γ-structure</u> is the Abstract structure (S_n) where $S_n \subseteq \mathcal{P}(G^n)$ is the collection of Γ-invariant sets.

<u>Defn</u> An <u>abstract structure</u> is collections of subsets $S_n \subseteq \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_{\geq 1}$, such that:

1) S_n closed under
   - finite union
   - finite intersect
   - complement

2) $(S_n)_{n=1}^{\infty}$ closed under
   - finite products
   - coordinate projections.

G^{Γ}

Return To Structures

let G be a set and let $\Gamma \subseteq \mathrm{Perm}(G)$ be a subgroup.

<u>Defn</u>. The <u>Γ-structure</u> is the Abstract structure (S_n) where $S_n \subseteq \mathcal{P}(G^n)$ is the collection of Γ-invariant sets.

<u>Application</u>: Take $G = \mathrm{Gal}(\overline{K}/K)$, $[K:\mathbb{Q}_p] < \infty$

Take $\Gamma = \mathrm{Aut}(G)$, (topological)

<u>Defn</u> An <u>abstract structure</u> is collections of subsets $S_n \subseteq \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_{\geq 1}$. such that:

1) S_n closed under
   - finite union
   - finite intersect
   - complement

2) $(S_n)_{n=1}^{\infty}$ closed under
   - finite products
   - coordinate projections.

Application: Take $G = \mathrm{Gal}(\bar{K}/K)$, $[K:\mathbb{Q}_p] < \infty$
Take $\Gamma = \mathrm{Aut}(G)$, (topological)

$$G_{\text{group}} \qquad\qquad G_{\text{group},\infty} \qquad\qquad G^{\Gamma}$$

$$\begin{pmatrix} \text{usual} \\ \text{group} \\ \text{language} \end{pmatrix} \subseteq \begin{pmatrix} \text{infinitary} \\ \text{group} \\ \text{language} \end{pmatrix} \subseteq \begin{pmatrix} \text{language of} \\ \Gamma\text{-invariant} \\ \text{sets} \end{pmatrix}$$

projectivus

$\underline{p}(G)$

$\underline{f}(G)$

$\underline{d}(G)$

$\underline{e}(G)$

$\boxed{\widehat{\underline{k}}^{\times}(G) = G^{ab}}$

$[G,G]$ definable

$\mu_n(G)$ $\quad \mu_\infty(G)$

$\varprojlim \mu_n(G)$

$\underline{\Lambda}(G)$

$\underline{O}^{\times}(G) = \underline{I}(G)$

$\cap\{N \trianglelefteq G : \underline{e}(N) = \underline{e}(G)\}$

$\underline{k}^{+}_{\log}(G)$ $\quad O^{\times,\mu}$

$\underline{I}^{p}(G)$

Fr $\quad \{r \in G_{res} : \forall x \in I/p,\ r(x) = p^{f_x}\}$ $\quad G_{group}$

$\underline{G}_{res}(G)$ $\quad G/\underline{I}(G)$

$G_{group,p}$

$\overline{\underline{k}}^{\times}(G)$

$\underline{k}^{\times}(G)$

$\underline{ord}$ $\quad \underline{k}^{\times}(G) \to \underline{\Gamma}(G) = \dfrac{\underline{k}^{\times}(G)}{\underline{O}^{\times}(G)}$

G^{Γ}

$\underline{O}^{p}(G)$

$\underline{I}(G)$

$\underline{p}(G)$
$\underline{f}(G)$
$\underline{d}(G)$
$\underline{e}(G)$

auxillary

$[G, G]$ definable

$\widehat{\underline{k}^{\times}}(G) = G^{ab}$

$\underline{\mu}_n(G)$ $\quad$ $\underline{\mu}_\infty(G)$

$\varprojlim \mu_n(G)$

$\Lambda(G)$

$\underline{O}^{\times}(G) = \underline{I}(G)^{ab}$

$I(G) = \bigcap\{N \trianglelefteq G : \underline{e}(N) = \underline{e}(G)\}$

$\underline{k}^{+}_{\log}(G)$ $\quad$ $O^{\times\mu}$

$\underline{I}(G)$

$\{\sigma \in G_{res} : \forall x \in I/p,\ \sigma(x) = p^{f_x}\}$ $\quad$ G_{group}

Fr

$\underline{G}_{res}(G)$ $\quad$ $G/\underline{I}(G)$ $\quad$ G_{group}

$\underline{k}^{\times}(G)$ $\quad$ $\underline{k}^{\times}(G)$

$\underline{ord}$ $\quad$ $\underline{k}^{\times}(G) \to \underline{\Gamma}(G) = \dfrac{\underline{k}^{\times}(G)}{\underline{O}^{\times}(G)}$

$\underline{O}^{\triangleright}(G)$

$\underline{I}(G)$

$G^{\ulcorner}$

$$I(G) = \bigcap \underbrace{\{ N \trianglelefteq G : \underline{e}(N) = \underline{e}(G) \}}_{\Gamma\text{-invariant collection}}$$

Definition: A **Γ-invariant collection** is some

$$\mathcal{A} \subseteq \mathcal{P}(G^n)$$

such that $\mathcal{A}$ itself is invariant under Γ.

Lemma: If $\mathcal{A}$ is Γ-invariant then

1) $\bigcap_{A \in \mathcal{A}} A$ is G^Γ-definable

2) $\bigcup_{A \in \mathcal{A}} A$ is G^Γ-definable.

$$I(G) = \bigcap \{ N \trianglelefteq G : \varrho(N) = \varrho(G) \}$$

Definition: A Γ-invariant collection is some $\mathcal{A} \subseteq \mathcal{P}(G^n)$ such that $\mathcal{A}$ itself is invariant under Γ.

$\underline{p}(G)$

$\underline{f}(G)$

$\underline{d}(G)$

$\underline{e}(G)$

auxillary

$[G,G]$ definable

$\widehat{\underline{k}^{p\times}}(G) = G^{ab}$

$\underline{\mu}_n(G)$ $\quad \mu_\infty(G) \xleftarrow{\lim} \mu_n(G)$

$\Lambda(G)$

$\underline{\mathcal{O}}^\times(G) = \underline{I}(G^{ab})$ $\quad \bar{\mathcal{O}}^\times(G)$

$I(G) = \bigcap\{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\}$

$\underline{k}^{+}_{\log}(G)$ $\quad \mathcal{O}^{\times\mu}$

$\underline{P}(G)$

Fr $\quad \{\sigma \in G_{res} : \forall x \in I/p \quad \sigma(x) = p^{f_x}\}$ $\quad G_{group}$

$\underline{G_{res}}(G)$ $\quad G/\underline{I}(G)$ $\quad G_{group}$

$\underline{k}^\times(G)$ $\quad \bar{k}^\times(G)$

$\underline{ord}$ $\quad \underline{k}^\times(G) \to \underline{\Gamma}(G) = \dfrac{\underline{k}^\times(G)}{\underline{\mathcal{O}}^\times(G)}$

$\underline{\mathcal{O}}^\triangleright(G)$ $\quad \bar{\mathcal{O}}^\triangleright(G)$ $\quad G^{\ulcorner}$

$\underline{I}(G)$

G_∞^Γ) New "fragment" for dealing with this

$$\underline{\mathcal{O}}^\times(G) = \varinjlim \underline{\mathcal{O}}^\times(G_0)$$

$$\overbrace{\left(\coprod_{G_0 \leq G} \underline{\mathcal{O}}^\times(G_0)\right)}^{} \Big/ \sim$$

an infinite disjoint union!

<u>WANT</u>: $\coprod_{G_0 \leq G} \underline{\mathcal{O}}^\times(G_0) \xrightarrow{I} \varinjlim \underline{\mathcal{O}}^\times(G_0)$

M a structure.

Definable sets: $S_1 \subseteq \mathcal{P}(M)$, $S_2 \subseteq \mathcal{P}(M^2)$, $S_3 \subseteq \mathcal{P}(M^3)$, ...

(classical)

In powers of M.

Definable sets: $S_\gamma \subseteq \mathcal{P}(M^\gamma)$ $\quad \gamma < \kappa$

ordinal

(infinitary)

M a structure.

Definable sets: $S_\gamma \subseteq \mathcal{P}(M^\gamma)$ $\quad \gamma < \underline{\kappa}$
ordinal

(infinitary)

$\underline{\text{Definable sets}}$: $S_v \subseteq \mathcal{P}(F_v(M))$, where

(what we need)

$v \in V$ a collection

contains $\coprod_{\gamma<\kappa} M$

<u>Defn</u> We define the collection $\mathcal{G}_\infty^{\Gamma}$ as the set of $(S_v)_{v \in V}$, where $S_v \subseteq \mathcal{P}(F_v(G))$ is the collection of Γ-invariant sets.

G_{group} classical group language

$G_{\text{group}, \infty}$ infinitary group language

G^{Γ} Γ-invariant sets $\Gamma = \mathrm{Aut}(G)$

G^{Γ}_{pro} pro-Γ-invariant sets

G^{Γ}_{∞} Γ-invariant sets w/ disjoint unions

$\underline{p}(G)$

$\underline{f}(G)$

$\underline{d}(G)$

$\underline{e}(G)$

auxillary

$\mu_\infty(G)$

$\widehat{\underline{k}^{\times}}(G) = G^{ab}$

$\underline{\mu}_n(G)$

$\mu_\infty(G)$

$\varprojlim \mu_n(G)$

$\underline{\Lambda}(G)$

$\underline{\mathcal{O}}^\times(G) = \underline{I}(G)^{ab}$

$\overline{\mathcal{O}}^\times(G)$

$\underline{I}(G) = \bigcap\{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\}$

$\underline{k}^+_{\log}(G)$

$\mathcal{O}^{\times\mu}$

G^{Γ}_{∞}

$\underline{I}(G)$ $\quad \bigcap\{N \triangleleft G : \underline{e}(N) = \underline{e}(G)\}$

Fr $\quad \{\sigma \in G_{res} : \forall x \in \underline{I}/\underline{P},\ \sigma(x) = p^{f_x}\}$

G^{Γ}_{pro}

$\underline{G}_{res}(G) = G/\underline{I}(G)$

G^{Γ}

$\underline{k}^\times(G)$

$\overline{\underline{k}}^\times(G)$

$G_{group, m}$

$\underline{ord} : \underline{k}^\times(G) \to \underline{\Gamma}(G) = \dfrac{\underline{k}^\times(G)}{\underline{\mathcal{O}}^\times(G)}$

$\underline{\mathcal{O}}^\triangleright(G)$

$\overline{\mathcal{O}}^\triangleright(G)$

G_{group}

$\underline{I}(G)$

THANK YOU!

BONUS: Species & Mutations.

Signature for Categories:

$$(C_0, C_1, s, t, \circ)$$

Sort for objects

Sort for morphisms

function symbols

Relation on $C_1 \times C_1 \times C_1$

$f = g \circ h$

$s : C_1 \to C_0$ (source)

$t : C_1 \to C_0$ (target)

$\boxed{\text{Interpreted Categories}}$

$(C_0, C_1, s, t, \circ) = C$

$$\left.\begin{array}{ll} I_0 : \underset{\underset{M^{n_0}}{\cap\!\shortparallel}}{X_0} \to C_0, & I_1 : \underset{\underset{M^{n_1}}{\cap\!\shortparallel}}{X_1} \to C_1 \end{array}\right\} \text{Interpretation of } C \text{ in } M$$

(Graph of Source/Target) $\Gamma_s, \Gamma_t \subseteq C_1 \times C_0$

(Graph of Composition) $\Gamma_\circ \subseteq C_1 \times C_1 \times C_1$

$$\begin{tikzcd} I_0 : X_0 \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & C_0 \\ M^{n_0} \end{tikzcd} \qquad X_0 = \{ \vec{m} \in M^{n_0} : \Phi_0(\vec{m}) \}$$

$$\begin{tikzcd} I_1 : X_1 \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & C_1 \\ M^{n_1} \end{tikzcd} \qquad X_1 = \{ \vec{m} \in M^{n_1} : \Phi_1(\vec{m}) \}$$

$$\Gamma_t \quad \Gamma_s \subseteq C_1 \times C_0 \qquad \Gamma_0 \subseteq C_1 \times C_1 \times C_1$$

$$\Gamma_s \subseteq C_1 \times C_0 \rightsquigarrow I^{-1}(\Gamma_s) = X_s \subseteq X_1 \times X_0$$

given by formula Φ_s

$$\Gamma_t \subseteq C_1 \times C_0$$

$$\Gamma_0 \subseteq C_1 \times C_1 \times C_1 \rightsquigarrow I^{-1}(\Gamma_0) \subseteq X_1 \times X_1 \times X_1$$

$$f \circ g = h$$

unravel this definition

$$X_0 = \{\vec{m} \in M^{n_0} : \Phi_0(\vec{m})\} \quad X_1 = \{\vec{m} \in M^{n_1} : \Phi_1(\vec{m})\}$$

Definition 3.1.

(i) A 0-*species* $\mathfrak{S}_0$ is a collection of conditions given by a *set-theoretic formula*

$$\Phi_0(\mathfrak{E})$$

involving an ordered collection $\mathfrak{E} = (\mathfrak{E}_1, \dots, \mathfrak{E}_{n_0})$ of sets $\mathfrak{E}_1, \dots, \mathfrak{E}_{n_0}$ [which we think of as "indeterminates"], for some integer $n_0 \geq 1$; in this situation, we shall refer to $\mathfrak{E}$ as a *collection of species-data* for $\mathfrak{S}_0$. If $\mathfrak{S}_0$ is a 0-species given by a set-theoretic formula $\Phi_0(\mathfrak{E})$, then a 0-*specimen* of $\mathfrak{S}_0$ is a *specific* ordered collection of n_0 sets $E = (E_1, \dots, E_{n_0})$ in some *specific* ZFC-model that satisfies $\Phi_0(E)$. If E is a 0-specimen of a 0-species $\mathfrak{S}_0$, then we shall write $E \in \mathfrak{S}_0$. If, moreover, it holds, in any ZFC-model, that the 0-specimens of $\mathfrak{S}_0$ form a *set*, then we shall refer to $\mathfrak{S}_0$ as 0-*small.*

(ii) Let $\mathfrak{S}_0$ be a 0-species. Then a 1-*species* $\mathfrak{S}_1$ *acting on* $\mathfrak{S}_0$ is a collection of

(iii) A *species* $\mathfrak{S}$ is defined to be a pair consisting of a 0-species $\mathfrak{S}_0$ and a 1-species $\mathfrak{S}_1$ acting on $\mathfrak{S}_0$. Fix a species $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$. Let $i \in \{0, 1\}$. Then we shall refer to an i-specimen of $\mathfrak{S}_i$ as an i-*specimen of* $\mathfrak{S}$. We shall refer to a 0-specimen (respectively, 1-specimen) of $\mathfrak{S}$ as a *species-object* (respectively, a *species-morphism*) of $\mathfrak{S}$. We shall say that $\mathfrak{S}$ is i-*small* if $\mathfrak{S}_i$ is i-small. We shall refer to a species-morphism $F : E \to E'$ as a *species-isomorphism* if there exists a species-morphism $F' : E' \to E$ such that the composites $F \circ F'$, $F' \circ F$ are *identity* species-morphisms; in this situation, we shall say that E, E' are *species-isomorphic*. [Thus, one verifies immediately that *composites* of *species-isomorphisms* are species-isomorphisms.] We shall refer to a species-isomorphism whose domain and codomain are equal as a *species-automorphism.* We shall refer to as *model-free* [cf. Remark 3.1.1 below] an i-specimen of $\mathfrak{S}$ equipped with a description via a *set-theoretic formula* that is "*independent* of the ZFC-model in which it is given" in the sense that for any pair of universes V_1, V_2 of some ZFC-model such that $V_1 \in V_2$, the set-theoretic formula determines the *same* i-specimen of $\mathfrak{S}$, whether interpreted relative to the ZFC-model determined by V_1 or the ZFC-model determined by V_2.

$\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$. If, in some ZFC-model, $E, E' \in \mathfrak{S}_0$, and F is a *specific* ordered collection of n_1 sets that satisfies the condition $\Phi_1(E, E', F)$, then we shall refer to the data (E, E', F) as a 1-*specimen* of $\mathfrak{S}_1$ and write $(E, E', F) \in \mathfrak{S}_1$; alternatively, we shall denote a 1-specimen (E, E', F) via the notation $F : E \to E'$ and refer to E (respectively, E') as the *domain* (respectively, *codomain*) of $F : E \to E'$.

(b) $\Phi_{1\circ 1}$ is a set-theoretic formula

$$\Phi_{1\circ 1}(\mathfrak{E}, \mathfrak{E}', \mathfrak{E}'', \mathfrak{F}, \mathfrak{F}', \mathfrak{F}'')$$

involving three collections of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$, $\mathfrak{F}' : \mathfrak{E}' \to \mathfrak{E}''$, $\mathfrak{F}'' : \mathfrak{E} \to \mathfrak{E}''$ for $\mathfrak{S}_1$ [i.e., the conditions $\Phi_0(\mathfrak{E})$; $\Phi_0(\mathfrak{E}')$; $\Phi_0(\mathfrak{E}'')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$; $\Phi_1(\mathfrak{E}', \mathfrak{E}'', \mathfrak{F}')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}'', \mathfrak{F}'')$ hold]; in this situation, we shall refer to $\mathfrak{F}''$ as a *composite of* $\mathfrak{F}$ *with* $\mathfrak{F}'$ and write $\mathfrak{F}'' = \mathfrak{F}' \circ \mathfrak{F}$ [which is, *a priori*, an abuse of notation, since there may exist *many* composites of $\mathfrak{F}$ with $\mathfrak{F}'$ — cf. (c) below]; we shall use similar terminology and notation for 1-specimens in specific ZFC-models.

(c) Given a pair of 1-specimens $F : E \to E'$, $F' : E' \to E''$ of $\mathfrak{S}_1$ in some ZFC-model, there *exists a unique composite* $F'' : E \to E''$ of F with F' in the given ZFC-model.

(d) Composition of 1-specimens $F : E \to E'$, $F' : E' \to E''$, $F'' : E'' \to E'''$ of $\mathfrak{S}_1$ in a ZFC-model is *associative.*

(e) For any 0-specimen E of $\mathfrak{S}_0$ in a ZFC-model, there exists a [necessarily unique] 1-specimen $F : E \to E$ of $\mathfrak{S}_1$ [in the given ZFC-model] — which we shall refer to as the *identity* 1-*specimen* id_E of E — such that for any 1-specimens $F' : E' \to E$, $F'' : E \to E''$ of $\mathfrak{S}_1$ [in the given ZFC-model] we have $F \circ F' = F'$, $F'' \circ F = F''$.

Definition 3.3. Let $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$ be *species*.

(i) A *mutation* $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ is defined to be a collection of *set-theoretic formulas* Ψ_0, Ψ_1 satisfying the following properties:

(a) Ψ_0 is a set-theoretic formula

$$\Psi_0(\mathfrak{E}, \underline{\mathfrak{E}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{E}}$ for $\underline{\mathfrak{S}}_0$; in this situation, we shall write $\mathfrak{M}(\mathfrak{E})$ for $\underline{\mathfrak{E}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{E} \in \underline{\mathfrak{S}}_0$ such that $\Psi_0(E, \underline{E})$ holds; in this situation, we shall write $\mathfrak{M}(E)$ for $\underline{E}$.

(b) Ψ_1 is a set-theoretic formula

$$\Psi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$ for $\mathfrak{S}_1$ and a collection of species-data $\underline{\mathfrak{F}} : \underline{\mathfrak{E}} \to \underline{\mathfrak{E}}'$ for $\underline{\mathfrak{S}}_1$, where $\underline{\mathfrak{E}} = \mathfrak{M}(\mathfrak{E})$, $\underline{\mathfrak{E}}' = \mathfrak{M}(\mathfrak{E}')$; in this situation, we shall write $\mathfrak{M}(\mathfrak{F})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $(F : E \to E') \in \mathfrak{S}_1$, then we require that there *exist a unique* $(\underline{F} : \underline{E} \to \underline{E}') \in \underline{\mathfrak{S}}_1$ such that $\Psi_0(E, E', F, \underline{F})$ holds; in this situation, we shall write $\mathfrak{M}(F)$ for $\underline{F}$. Finally, we require that the assignment $F \mapsto \mathfrak{M}(F)$ be compatible with *composites* and map *identity* species-morphisms of $\mathfrak{S}$ to identity species-morphisms of $\underline{\mathfrak{S}}$. In particular, if one fixes a ZFC-model, then $\mathfrak{M}$ determines a *functor* from the category determined by $\mathfrak{S}$ in the given ZFC-model to the category determined by $\underline{\mathfrak{S}}$ in the given ZFC-model.

(iv) Let $\vec{\Gamma}$ be an *oriented graph*, i.e., a graph Γ, which we shall refer to as the *underlying graph* of $\vec{\Gamma}$, equipped with the additional data of a total ordering, for each edge e of Γ, on the set [of cardinality 2] of *branches* of e [cf., e.g., [AbsTopIII], §0]. Then we define a *mutation-history* $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ *[indexed by* $\vec{\Gamma}$*]* to be a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a *species* $\mathfrak{S}^v$;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *mutation* $\mathfrak{M}^e : \mathfrak{S}^{v_1} \rightsquigarrow \mathfrak{S}^{v_2}$.

In this situation, we shall refer to the vertices, edges, and branches of $\vec{\Gamma}$ as vertices, edges, and branches of $\mathfrak{H}$. Thus, the notion of a "mutation-history" may be thought of as a *species-theoretic* version of the notion of a "diagram of categories" given in [AbsTopIII], Definition 3.5, (i).

(ii) Let $\mathfrak{M}, \mathfrak{M}' : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be *mutations*. Then a *morphism of mutations* $\mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}'$ is defined to be a *set-theoretic formula* Ξ satisfying the following properties:

(a) Ξ is a set-theoretic formula

$$\Xi(\mathfrak{E}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{F}} : \mathfrak{M}(\mathfrak{E}) \to \mathfrak{M}'(\mathfrak{E})$ for $\underline{\mathfrak{S}}_1$; in this situation, we shall write $\mathfrak{Z}(\mathfrak{E})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{F} \in \underline{\mathfrak{S}}_1$ such that $\Xi(E, \underline{F})$ holds; in this situation, we shall write $\mathfrak{Z}(E)$ for $\underline{F}$.

(b) Suppose, in some ZFC-model, that $F : E_1 \to E_2$ is a species-morphism of $\mathfrak{S}$. Then one has an equality of composite species-morphisms $\mathfrak{M}'(F) \circ \mathfrak{Z}(E_1) = \mathfrak{Z}(E_2) \circ \mathfrak{M}(F) : \mathfrak{M}(E_1) \to \mathfrak{M}'(E_2)$. In particular, if one fixes a ZFC-model, then a morphism of mutations $\mathfrak{M} \to \mathfrak{M}'$ determines a *natural transformation* between the functors determined by $\mathfrak{M}$, $\mathfrak{M}'$ in the ZFC-model — cf. (i).

BONUS: Infinitary languages.

The Language $L_{\kappa,\lambda}(\tau)$ κ, λ ordinals

FORMULAS: • $\varphi(x_i : i<\gamma)$, $\gamma<\kappa$

• $\bigwedge_{\alpha<\beta} \varphi_\alpha(\vec{x})$, $\beta<\lambda$ [same for $\bigvee$]

• Formulas have less than κ many free variables (projections of κ-many vars)

• Subsets of M^α for $\alpha<\kappa$

• Intersection of λ many sets

Bonus: Lascar Types

$\boxed{\text{Types}}$

- Let A be a structure.
- The *type* of $\vec{a} \in A^n$ is the collection $\phi \in \mathcal{L}_A$ with one free variable such that $\phi(\vec{a})$,
- $S_n(A) = $ (stone-space of types)

example. F field
$\bar{F}$ alg closed field.

$A = (\bar{F}, +, \cdot, 0, 1, \underbrace{(a)}_{\text{constant symbol for every } a \in F}{}_{a \in F})$

type $b \in \bar{F}$
determined by minimal poly $m_b(x) \in F[x]$.

points of same type are Galois conjugates.

Types

Example. F field

$\bar{F}$ alg closed field.

$A = (\bar{F}, +, \cdot, 0, 1, \underbrace{(a)_{a \in F}}_{\text{constant symbol for every } a \in F})$

- $S_n(A) = (\text{stone-space of types}) = \operatorname{Spec}(F[x_1, \ldots, x_n])$

Types

B structure.

A substructure of B.

$\mathrm{Aut}(B/A) = \{ \sigma \in \mathrm{Aut}(B) : \forall a \in A, \ \sigma(a) = a \}$.

Defn. The Loskar types (of $\vec{b} \in B^n$) are the set

$$B^n / \mathrm{Aut}(B/A)$$

Example: $G = \mathrm{Gal}(\bar{K}/K)$, $[K : \mathbb{Q}_p] < \infty$, $M = \bar{\mathcal{O}}^{\triangleright}, \bar{\mathcal{O}}^{\times}, \bar{\mathcal{O}}^{\times\mu}$

$\mathrm{Aut}(G,M) \to \mathrm{Aut}(G)$, $B = (G,M)$, $A = G$

Losar types then are pairs (g,m) where m is up to indeterminacy.

BONUS: QUESTIONS

Question: What is the interpretation-theoretic content of the main thms of Geometry of Frobeniods?

THE GEOMETRY OF FROBENIOIDS I: THE GENERAL THEORY

Shinichi Mochizuki

June 2008

Abstract. We develop the theory of *Frobenioids*, which may be regarded as a category-theoretic abstraction of the theory of divisors and line bundles on models of finite separable extensions of a given function field or number field. This sort of abstraction is analogous to the role of *Galois categories* in Galois theory or *monoids* in the geometry of log schemes. This abstract category-theoretic framework preserves many of the important features of the classical theory of divisors and line bundles on models of finite separable extensions of a function field or number field such as the *global degree* of an arithmetic line bundle over a number field, but also exhibits interesting new phenomena, such as a *"Frobenius endomorphism"* of the Frobenioid associated to a number field.

Question: What do Lascar types have to do with indeterminacies?

Question: Can you find a classical first order example of non-anabelian transport?

Question:

Why is it the case that when the absolute Grothendieck conjecture holds for Π_X then Π_X interprets a field? Is this necessary?

BONUS: Complete Theories.

- Structures are theories over the empty set.

$$\emptyset \models M \text{ for all } M \in Str_\sigma$$

$\rightsquigarrow Def_\sigma(T) = Def_\sigma(\emptyset) \implies$ functor $Def_\sigma(\emptyset) \to Sets$ are structures.

- $T = Th(M) = \left(\begin{array}{c} \text{All Formulas} \\ \text{That Are True} \\ \text{For } M. \end{array} \right)$

Bonus: Functorial Algorithms

INTER-UNIVERSAL TEICHMÜLLER THEORY II: HODGE-ARAKELOV-THEORETIC EVALUATION

Shinichi Mochizuki

December 2020

Abstract. In the present paper, which is the second in a series of four papers, we study the **Kummer theory** surrounding the Hodge-Arakelov-theoretic evaluation — i.e., evaluation in the style of the **scheme-theoretic Hodge-Arakelov theory** established by the author in previous papers — of the [reciprocal of the l-th root of the] **theta function** at **l-torsion points** [strictly speaking, shifted by a suitable 2-torsion point], for $l \geq 5$ a prime number. In the first paper of the series, we studied *"miniature models of conventional scheme theory"*, which we referred to as $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated to certain data, called *initial* Θ*-data*, that includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. The underlying Θ-Hodge theaters of these $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters were *glued* to one another by means of *"*Θ*-links"*, that identify the [reciprocal of the l-th root of the] *theta function* at primes of bad reduction of E_F in one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater with [$2l$-th roots of] the *q-parameter* at primes of bad reduction of E_F in another $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater. The theory developed in the present paper allows one to construct certain new versions of this "Θ-link". One such new version is the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$**-**

Example 1.7. Radial and Coric Data I: Generalities.

(i) In the following discussion, we would like to consider a certain *"type of mathematical data"*, which we shall refer to as **radial data**. This notion of a "type of mathematical data" may be *formalized* — cf. [IUTchIV], §3, for more details. From the point of view of the present discussion, one may think of a "type of mathematical data" as the input or output data of a "functorial algorithm" [cf. the discussion of [IUTchI], Remark 3.2.1]. At a more concrete level, we shall assume that this "type of mathematical data" gives rise to a *category*

$$\mathcal{R}$$

— i.e., each of whose *objects* is a specific collection of radial data, and each of whose *morphisms* is an isomorphism. In the following discussion, we shall also consider another "type of mathematical data", which we shall refer to as **coric data**. Write

$$\mathcal{C}$$

for the category obtained by considering specific collections of coric data and isomorphisms of collections of coric data. In addition, we shall assume that we are given a *functorial algorithm* — which we shall refer to as **radial** — whose *input data* consists of a collection of radial data, and whose *output data* consists of a collection of coric data. Thus, this functorial algorithm gives rise to a *functor* $\Phi : \mathcal{R} \to \mathcal{C}$. In the following discussion, we shall assume that this functor is *essentially surjective*. We shall refer to the category $\mathcal{R}$ and the functor Φ as *radial* and to the category $\mathcal{C}$ as *coric*. Finally, if I is some *nonempty index set*, then we shall often consider collections

$$\{\Phi_i : \mathcal{R}_i \to \mathcal{C}\}_{i \in I}$$

of copies of Φ and $\mathcal{R}$, such that the various copies of Φ have the *same codomain* $\mathcal{C}$ — cf. Fig. 1.1 below. Thus, one may think of each $\mathcal{R}_i$ as the category of radial data *equipped with a label* $i \in I$, and isomorphisms of such data.

(ii) We shall refer to a triple $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ [or to the triple consisting of

BONUS : Pre-topics Completion

- finite limits
- stable finite sops
- stable images
- stable disj union,

$\boxed{\text{pretopos}}$

small $\rightsquigarrow$

- small $\cdot \rightsquigarrow$
- stable small
-
-

$\boxed{\text{Groth topos}}$

- $T = \mathrm{Th}(M)$, $\quad \mathrm{Def}(T) \simeq \mathrm{Def}_M$

- $\underbrace{\mathrm{Def}(T)}_{\text{topos theory}} = \mathrm{Def}(T^{eq})$

Shelah's construction.

- Abstract Bi-interpretations, give an equivalence of categories between pre-topos completions.
- $\mathrm{Def}(T) \not\simeq \mathrm{Def}(T^{eq})$ in general.

$$I_1 : X_1 \to N \qquad I_2 : X_2 \to N$$

$$X_1 \subseteq M^{n_1} \qquad X_2 \subseteq M^{n_2}$$

$$I_1 \sim I_2 \iff Eq(I_1, I_2) = \{ (\vec{m_1}, \vec{m_2}) \in X_1 \times X_2 : I(\vec{m_1}) = I(\vec{m_2}) \}$$

definable.

A **Bi-interpretation** is I_1, I_2 such that

$I_1 I_2 \sim id$

$I_2 I_1 \sim id$.