Model Theory Notes

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Abstract

These notes are based on Thomas Scanlon's talks at CUNY during the summer of 2010. They only go through the first two days. The main point is that if you know that a certain set is definable and bounded you can should that the family is uniformly bounded.

Contents

1 Signatures, Languages, Theories and Models

1.1 What is a Signature?

A **signature** is a tuple $\sigma = (C, F, R, \text{arity}_F, \text{arity}_C)$ consisting of a set of constant symbols C, a set of functions symbols F and a set of relations R along with arity functions $\text{arity}_F : F \to \mathbb{Z}_+$ and $\text{arity}_R : R \to \mathbb{Z}_+$ which tell how many inputs each function and relation take.

Example 1. Consider the signature for the Language of ordered rings.

$$C = \{0, 1\},$$
$$F = \{+, -, *\}$$
$$R = \{\leq\}.$$

where arity(+) = arity(∗) = 2,(−)1, and arity(≤) = 2.

Example 2. The signature for the Language of Groups:

$$Constants = \{1\}$$
$$Relations = \emptyset$$
$$Functions = \{\text{inverse}, *\}$$

Arity of inverse = 2, Arity(*)=2.

Example 3. Language of graphs

$$\text{Constants} = \emptyset$$
$$\text{Functions} = \emptyset$$
$$\text{Relations} = \{e\}$$

with arity(e) = 2. Suppose the proposition $\theta(v) = [\text{indeg}(v) = \text{outdeg}(v)]$

1.2 What is the Language of a Signature?

From a signature we build up a collection of strings known as the **Language of a signature**. The language will of a signature will have no attached meaning to them. It is just a collection of strings from a family of symbols. To create this language, We first chose a set of variable symbols $\{x_i : i \in \mathbb{N}\}$.

- Next we construct the **terms** of our signature $T(\sigma)$ *inductively*. Initially we will only have constants and variables being terms. Then we say that for and $f \in F_\sigma$ and $t_1, \ldots, t_{\text{arity}(f)} \in T(\sigma)$ then $f(t_1, \ldots, f_{\text{arity}(f)}) \in T(\sigma)$.
- From our terms we define our **formula** (*again inductively*). The base of our inductive definition of the formulae are the **atomic formula**. There are of the form $t = s$ where $t, s \in T(\sigma)$ or of the form $R(t_1, \ldots, t_{\text{arity}(r)})$ for $t_i \in T(\sigma)$ and $R \in R_\sigma$. The inductive step provides **general formula**: Given any two formula P and Q the following strings are also formula:
  - $!P$ (not P)
  - $P \wedge Q$
  - $P \vee Q$

  At this stage the formula we have created can incorperate the x_i in them. These are called **Free Variables** in a formula. Whenever we have a formula $P = P(c_1, \ldots, c_r, x_1, \ldots, x_s)$ involving constants c_i and variables x_j we can choose to bound any of the variables with quantifiers to get a new formula:

- $\exists x_1, \exists x_2, \dots, \exists x_k P(c_1, \dots, c_r, x_1, \dots, x_s)$
- $\forall x_1, \forall x_2, \dots, \forall x_k P(c_1, \dots, c_r, x_1, \dots, x_s)$

Variables which have been quantified are called **bound variables**.

1.3 What is a structure/interpretation?

An $L(\sigma)$-**structure** (definition by example), is a tuple $\mathfrak{M} = (M, C, R, F)$ together with "interpretations". It provides an interpretation $L(\sigma)$-sentences so that they me evaluated as true or false. By interpretation we mean that we decode every string $w \in L(\sigma)$ using our particular structure.

Example 4. Consider the signature for ordered sets OS. There are no constants, no functions and the only relation is $\sim$. Here are three sentences in $L(\mathsf{OS})$:

(R) $\forall x \quad x \sim x$.

(T) $\forall x \forall y \forall z \quad x \sim y \wedge y \sim z \implies z \sim z$.

(S) $\forall x \forall y \quad x \sim y \implies y \sim x$.

To specify a structure we just need to give an example of an object in which the relation can be interpreted. We could take $\mathfrak{M} = (\mathbb{R}, \emptyset, \emptyset, \{<\})$. In this interpretation only the transitive property would hold. If we replace 'less than' with 'less than or equal to' then the transitive and symmetric properties would hold. On the other hand if we used a set with an equivalence relation on it we know that in this interpretation all of these sentences would be true.

An $L(\sigma)$-structure consists of

- a set M (called the **universe** of the model/interpretation)
- a collection of constants $C \subset M$ which is in bijection with the symbols C_σ
- a collection of function F with domain some power of M and range M which are in bijections with F_σ whose number of inputs match the arity
- a collection of relations R on M in bijection with R_σ defined on the appropriate number of variables.

1.4 What is a Theory and what is a Model?

A **theory** T is just a subset of $L(\sigma)$ (the set of all formulas) that contains no free variables.

Let $\mathfrak{M}$ be an $L(\tau)$-structure and P a bounded $L(\tau)$ formula.

$$\mathfrak{M} \models P \iff \text{“}P \text{is true with interpretation} \mathfrak{M}''.$$

If T is a theory we say $\mathfrak{M}$ is a **model** of T write $\mathfrak{M} \models T$ if and only if for all $P \in T$ we have $\mathfrak{M} \models P$.

Example 5. The theory of groups T, would include the following sentences

Ass $\forall x \forall y \forall z \quad *(*(x,y), z) = *(x, *(y,z))$.

Ident $\forall x \quad *(x, 1) = *(1, x) = x$.

Invers $\forall x, \exists y, *(x,y) = *(y,x) = 1$.

A group like $(\mathbb{Z}/10\mathbb{Z}, 0, +)$ would provide a model $\mathfrak{M}$ so $\mathfrak{M} \models T$.

1.5 What is a Complete and Consistent Theory?

An $L(\sigma)$-theory T is **complete** if and only if for every $L(\sigma)$ formula P and every model $\mathfrak{M}$ we have $\mathfrak{M} \vDash P$ or $\mathfrak{M} \vDash !P$. [If true in one of the models, then true in all of the models]

We can't find models for which [Explain Consistency, we need this for the completeness theorem for the other proof of the compactness theorem]

2 Definable Sets and The Compactness Theorem

2.1 What is a Definable Set?

As we vary over $\mathfrak{M}$, which are $L(\tau)$-structures we can consider families of sets $S \subset M^n$ which are **first order definable** or **definable**. Such families are ones for which there exists some $P(x_1, x_2, \ldots, x_n)$ an $L(\tau)$-formula with free variables $x_1, x_2, \ldots, x_n$ such that for every $\mathfrak{M}$,

$$S = \{(a_1, \ldots, a_n) \in M^n : \mathfrak{M} \vDash P(a_1, \ldots, a_n)\} := P(\mathfrak{M}).$$

Observe that we are defining this uniformly over ALL structures and not just for a fixed one. We want to avoid coincidences.

The idea here that some propositions (characterized by membership of a particular set) can't be written down as first order formula (are not definable sets). This is analogous to the fact that there are certain problems in computatbility that exists but have no algorithm: Problems are phrased in terms languages (which are collections of strings) for which you seek a turning machine that outputs "YES!" to every element in the language. These are not in the restricted class of Turing Decidable languages. [1]

Example 6. The set of squares in a ring is definable:

$$\{r \in R : \exists s \in R \; r = s^2\}.$$

2.2 A Reason For Algebraic Geometers to care about definable sets

Proposition 1. *The definable sets in the language of algebraically closed fields are the Zariski constructible sets.*

Proof. Recall that a constructable set is just a finite union of open and closed subsets.

Suppose that $K = \bar{K}$. Every closed set $Z \subset K^n$ is Zariski definable as

$$Z = \{(a_1, a_2, \ldots, a_n) \in K^n : f_1(\vec{a}) = 0 \wedge f_2(\vec{a}) = 0 \wedge \ldots \wedge f_m(\vec{a}) = 0\}$$

where the f_i generate the ideal corresponding to Z.

Similarly the negation of the proposition shows that open sets are definable. Also since unions correspond to taking $\vee$s and intersections correspond to taking

[1] Question: Can you use these crazy ultra product constuctions to get produce new turing machines? Are these projective limits?

$\wedge$'s we have that the finite intersections and unions of open and closed sets are definable. This shows that constructible sets are a subset of definable sets.

Conversely, let $S = \phi(K)$ be a definable subset of K^n. We will show it is constuctable. We will prove it inductively.

First it is true for atomic formula: the onle atomic formulae in the language are those which involve the function symbols (addition, multiplication, subtraction) and "=". These define Zariski closed sets.

Now the general formulas obtained from the atomic formula by taking negation (completmenting), conjunction (intersecting) and disjunction (unioning). Any finite combo of these operations will produce a constructible set. This shows that the definable sets are contained in the constructible sets. □

2.3 What does the Compactness theorem Say and How can you use it?

The idea of the next theorem the following: **In first order logic you can test satisfiability of theories by testing finite subsets of the theory**.

Theorem 1 (Compactness Theorem)**.** *Let $T \subset L(\tau)$ be a theory. If for every finite $T_0 \subset T$ there exists some $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \models T_0$ then there exists some $\mathfrak{M}$ such that $\mathfrak{M} \models T$.*

Before proving the theorem we will look at some applications:

Proposition 2. *Determining if a given element of a group is in the cyclic group generated by another element.*

Proof. To our normal language of groups we adjoin two parameters to specify marked elements. Let $\theta = \theta(x, y)$ be the proposition that y is in the cyclic group generated by x. Let

$$T = \{\theta\} \cup \{x^n \neq y^m : n, m \in \mathbb{N}\}.$$

Observe that y being in the cyclic group generated by x and the propositions on the left hand side contradict each other. We will show that there exists a model for this theory which is a contradiction. This in turn will imply that θ can't be written down as a first order sentence.

We will show that every finite subcollection of propositions from T has a model. By the compactness theorem this implies that there exists a model for T: If T_0 is a subset of T then there exists some M and N such that if $m > M$ and $n > N$ the propositions $x^n \neq y^m$ are not in T_0. Let G_0 be a cyclic group of large order L WAY WAY bigger than M and N. Let x be it's generator and let y be the $(L-1)$st power of x. This models T_0. □

□

Proposition 3. *The set of vertices in a directed graph which have the same in and out degree is not definable.*

This is an example where it is easy to see you really need to consider the "definable sets" as families within every model: If Γ is the directed graph with two nodes and two directed edges then the proposition $x = x$ defines a subset of vertices with the same in and out degree. The same proposition does not define the subset of vertices with the same in and out degree for an *aribitrary* directed graph.

Proof. For a given vertex v convince yourself that one can define the propositions using first order logic:

- v has at least one in edge and out edge.
- v has least n in edges and n out edges.
- v has less than n in edges and n out edges.
- v has exactly n in edges and n out edges.

After you have convinced yourself the above work we continue as follows: Let $\theta(v)$ be the proposition that v has the same in and out degree. Let $\psi_n(v)$ to be the proposition that v has the same in and out degree which is equal to n. Consider the theory

$$T = \{\theta\} \cup \{!\psi_n : n \in \mathbb{N}\}.$$

We will apply the compactness theorem to show that T has a model. The propositions in the language contradict each other. We conclude that θ is not first order.

For any finite subset $T_0 \subset T$ there is a maximal n such that $!\psi_n \in T_0$. Let Γ be any graph such that is in degree is the same as the out degree on every vertex but that the degree is so high at each vertex that all that $!\psi_n$ propositions are satisfied. This show that $\Gamma \models T_0$. □

3 Ultra-Filters, Ultra-Products, New Models from Old Models, and Los's Tautology

We will look at two proofs of the compactness theorem. The one that Tom presented was using Los's theorem. There is a second proof suggested by Michelle Manns that uses Gödel's completeness theorem.

To begin we will need ultra products.

3.1 Ultrafilters give Ultraproducts

Let I be some set and let $\mathcal{P}(I)$ denote it's power set. An **ultra filter** is a special collection of subsets $U \subset \mathcal{P}(I)$. They are special in the sense that $U = \mu^{-1}(1)$ where $\mu : \mathcal{P}(I) \to \{0, 1\}$ is a measure.

Let S_i be a family of sets indexes by I. The **ultra product** is the following quotient

$$\frac{\prod_{i \in I} S_i}{\sim}.$$

The definition equivalence: $(a_i) \sim_U (b_i)$ if and only if the places where the sequences agree is a set of measure one, that is $\mu\{i \in I : a_i = b_i\} = 1$.

The following propositions are nice exercises that show the definition of an Ultrafilter in terms of a measure gives you a definition of an ultra product in terms the wacky definition.

Proposition 4 (Measure Defn implies Crazy Defn)**.** *Let $\mu : \mathcal{P}(I) \to \{0, 1\}$ with $\mu \neq 0$. (Read the claims and proofs one at a time... actually it'll probably be easier to prove these yourself read the proofs.)*

1. *No two disjoint sets both have measure 1.*
2. *Either a set or its compliment have measure 1.*
3. *If two sets have measure 1 and have a nonempty intersection, Then their intersection has measure 1.*

Proof. 1. Suppose that $A \cap B = \emptyset$ and that they both have measure 1. The implies $\mu(A \cup B) = \mu(A) + \mu(B) = 2$. A Contradiction.

2. $1 = \mu(I) = \mu(A \cup A^c) = \mu(A) + \mu(A^c)$.

3.

$$\begin{aligned} \mu(A \cap B) &= \mu((A^c \cup B^c)^c) \\ &= 1 - \mu(A^c \cup B^c) \\ &\geq 1 - (\mu(A^c) + \mu(B^c)) \\ &= 1. \end{aligned}$$

4.

□

3.2 Products and Ultraproducts of $L(\tau)$-structures

This is what we are aiming for:

Lemma 1 (Los's Tautology)**.** *If ϕ is any $L(\tau)$-sentence then*

$$\frac{\prod_i \mathfrak{M}_i}{U} \vDash \phi \iff \{i \in I : \mathfrak{M}_i \vDash \phi\} \in U$$

We are currently lacking definitions.

- What does "$\prod_i \mathfrak{M}_i / \sim$" means as a model?
- What does is mean for "$\prod_i \mathfrak{M}_i / \sim$" to prove a proposition?
- How is What does "$\prod_i \mathfrak{M}_i / \sim$" even and $L(\tau)$-structure?
- Given $\{\mathfrak{M}_i\}_{i \in I}$ a family of $L(\tau)$-structures does $\prod_i \mathfrak{M}_i$ have an $L(\tau)$-structure?
- Does it the expression $\prod_{i \in I} \mathfrak{M}_i$ even make sense?

Recall that an $L(\tau)$-structure consists of three things: constants C_τ, relations R_τ and functions F_τ. Actually I lied, because implicitly for every $\mathfrak{M}$ there is a universe (aka plain old set) M on which the constants, functions and relations are defined. These are what we need to check.

Example 7 (Products of Structures)**.** Let's look at the simple case of the product of two models: suppose $\mathfrak{M}$ and $\mathfrak{N}$ are $L(\tau)$-structures with universes M and N respectively. Then in order to make an $L(\tau)$-structure $\mathfrak{M} \times \mathfrak{N}$ we need to examine the constants, functions and relations. These should exists on the universe for $\mathfrak{M} \times \mathfrak{N}$ which should be $M \times N$.

constants are pairs of constants. If $c \in C_\tau$ and m represents c in M and n represents c in N then (m, n) represents c in $M \times N$.

functions: given f in F_τ represented by f_M and f_N respectively then $f_M \times f_N : M \times N \to M \times N$ will represent f in $\mathfrak{M} \times \mathfrak{N}$.

relations: The product of a relation is a relation: If $R_M \subset M \times M$ and $R_N \subset N \times N$ then $(R_M \times R_N) \subset (M \times M \times N \times N)$ but by reordering we have $R_M \times R_N \subset (M \times N) \times (M \times N)$. There amounts to $(a, b) R_{M\times N} (a', b')$ if and only if $a R_M a'$ and $b R_N b'$.

[Check universal property] [Check for infinite products] [Not every Quotient of a $L(\tau)$ structures is an $L(\tau)$-structure.]

3.2.1 What is the stucture associated the the Ultra-Product of structures?

To make sense of $\prod_i \mathfrak{M}_i / \sim$ we need to make the constants,functions and relations all descend to quotients by the equivalence relations. The universe for our new model will be $\prod_i M_i / \sim$. Elements of the product will be written as $a = (a_i)_{i\in I} = (a_i)$. And recall that on the above set we had $(a_i) \sim (b_i)$ if and only if $\{i \in I : a_i = b_i\}$ has measure one. We will denote equivalence classes in the product by $[(a_i)] = [a_i]$.

Constants We are given $c_i \in M_i$ representing some $c \in C_\sigma$ for each $\mathfrak{M}_i$. The constant in the ultra-product model will be $[c_i]$.

Functions Suppose we have a model and we just want to take the quotient of said model by some arbitrary relation. If f_M represents $f \in F_\tau$ then we need to check that $\bar{f}_M$ defined by

$$\bar{f}_M([-]) := [f_M(-)]$$

is well defined. This means that if $m_1, m_2 \in M$ with $[m_1] = [m_2]$ then $[f_M(m_1)] = [f_M(m_2)]$. [2]

Let $M = \prod_i M_i$, with $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$. Suppose that $\mu(S) := \mu\{i \in I : a_i = b_i\} = 1$ then we need to show that $[(f_i(a_i))_{i\in I}] = [(f_i(b_i))_{i\in I}]$ meaning $\mu(T) := \mu\{i \in I : f_i(a_i) = f_i(b_i)\} = 1$.

Since $S \subset T$ this implies that $\mu(S) \leq \mu(T)$, but $\mu(S) = 1$ which implies that $1 \leq \mu(T)$ which implies that $\mu(T) = 1$ since the measure is 0,1 valued.

Relations We will take our definition to be the following:

$$[(a_i)]\bar{R}[(b_i)] \iff \mu\{i \in I : a_i R_i b_i\} = 1.$$

I have no good reason why one should define the relation like this but it makes it work. We need to show this relation is well-defined on equivalence classes. Suppose that $[a_i] = [b_i]$ that $[a'_i] = [a_i]$ and $[b'_i] = [b_i]$ then we need to show that $[a'_i]R[b'_i]$.

[2] This provides lots of examples of why we are using ultra-filters instead of some random equivalence relation. Closer examination this condition will either show that there exists other quotients (equivalence relations) that give well defined models or that all good quotients come from Ultra-Filters.

Unraveling the definitions we are given

$$\mu\{i \in I : a_i = a_i'\} = 1, \quad \mu\{i \in I : b_i = b_i'\} = 1$$

and we need to show $(a_i')R(b_i')$ which mean

$$\mu\{i \in I : a_i' R_i b_i'\} = 1.$$

Now we can understand how to interpret a sentence in our ultraproduct structure: It starts from understanding interpretations of atomic formula. If we can understand what

$$\frac{\prod_i \mathfrak{M}_i}{\sim} \models \phi$$

means for ϕ an atomic formula then we can understand what it means for general formula.

WEEEELLL, there are only two types of atomic formula: 1. equations involving terms (which is just equality in the set $M = \prod_i M_i/\sim$) and relations (which would have the interpretation in the ultra product relation definition we just defined).

Check Los' Tautology for the simple cases of Atomic Formulae. It should now appear to be a tautology.

Corollary 1. *You can have a bunch of models $\mathfrak{M}_i$ all of which don't prove a theory whose ultra product does model a theory!*

4 Proof Compactness Theorem via Los's Tautology

Lemma 2 (Los's Tautology)**.** *If ϕ is any $L(\tau)$-sentence then*

$$\frac{\prod_i \mathfrak{M}_i}{U} \models \phi \iff \{i \in I : \mathfrak{M}_i \models \phi\} \in U$$

We will prove the compactness theorem from Los's Tautology and using existence of an ultra filter on the collection of finite subsets of the theory T satisfying a particular property. This is weird to parse because our index set I is a collection of finite subsets (a set of sets) and an ultrafilter on I is a measure on the powerset of I (the collection of subsets of a set of sets).

The property that we want is that for any $T' \subset T$ finite the collection of sets that contain this set has measure 1:

$$\forall T' \subset T \text{ finite} \quad \mu\{S \in \mathcal{P}(T) : S \supset T'\} = 1 \tag{1}$$

Tainted Proof of Compactness Theorem. By the hypothesis of the compactness theorem for every finite subset of the theorey $S \subset T$ there exists some $\mathfrak{M}_S$ such that $\mathfrak{M}_S \models S$. We will show

$$\frac{\prod_S \mathfrak{M}_S}{\sim} \models T.$$

For every $\phi \in T$, if $\phi \in S$ then the model that exists for S proves ϕ. In the language of sets we have

$$\{S \in I : \mathfrak{M}_s \models \phi\} \subset \{S \in I : \{\phi\} \subset S\}.$$

Taking measures

$$\mu\{S \in I : \mathfrak{M}_s \models \phi\} \geq \mu\{S \in I : \{\phi\} \subset S\} = 1$$

where the right hand side follows from the hypotheses on the measure. But not

$$\mu(S \in I : \mathfrak{M}_S \models \phi\}) = 1 \iff \prod_S \mathfrak{M}_S / \sim \models \phi$$

by Los Theorem. So we are done.

□

4.0.2 proving the existence of the measure we used

Lemma 3. *Suppose T is an $L(\tau)$-theory and for every finite subset S of T there exists some $\mathfrak{M}_S \models S$. Then there exists some measure $\mu : \mathcal{P}(T)_{finite} \to \{0,1\}$ such that for all $S \subset T$ finite*

$$\mu\{S' \subset T : S \subset S'\} = 1.$$

Lemma 4 (Zorn's Lemma)**.** *Let $(P, \leq)$ be a partially ordered set. If every increasing sequence has a maximal element then there exists a maximal element of P.*

We will define a partial ordering on the set of measures:

$$\mu \leq \nu \iff \text{supp}(\mu) \subset \text{supp}(\nu).$$

The support $\{^{\{0,1\}}\}$-values function. We need to check that every increasing sequence of measures has a maximal element. Suppose that $\ldots \leq \mu_{i-1} \leq \mu_i \leq \mu_{i+1} \leq \ldots$ then let

$$\text{supp}(mu) := \bigcup_i \text{supp}(\mu_i)$$

this sure as hell defines a function but we need to check that it is a measure.

Measure of Full Set $\mu(I) = 1$ by subset property of and nontriviality of μ_i.

Additivity If $A \cap B = \emptyset$. We will analyze this by cases.

If both $\mu(A)$ and $\mu(B)$ are zero then we are ok.

Suppose that $\mu(A) = 1$, if $A \in \text{supp}(\mu_i)$ then $A \in \text{supp}(\mu_j)$ for every $j \geq i$. Since the μ_j are measures we have that $B \notin \text{supp}(\mu_j)$.

Suppose that $\mu(A) = 0$ [Fill in cases]

5 Definability of Finiteness Gives Uniform Bounds

Theorem 2 (Definability and Uniform Bounds)**.** *Let $L(\tau)$ be a first order language, $\phi(x, y)$ be a sentence is that language and T a theory in that Language.*

$$\{b \in M : \#\phi(\mathfrak{M}, b) < \infty\} \textit{ definable}$$

if and only if

There exist some constant $C > 0$ such that for all models and all parameters

$$\#\phi(\mathfrak{M}, b) < \infty \implies \#\phi(\mathfrak{M}, b) < \infty.$$

Proof. Suppose no such uniform bound exist and that there exists some θ such that $\theta(b)$ is true if and only if

$$\#\phi(\mathfrak{M}, b) := \#\{a \in M : \phi(a, b)\} < \infty$$

We will derive a contradition. Since there is no uniform bound there exists a sequence of models and parameters $\{(\mathfrak{M}_n, b_n)\}_{n=1}^{\infty}$ that violate the boundedness... say we have them doing this:

$$n < \#\phi(\mathfrak{M}_n, b_n) < \infty$$

as $n \to \infty$.

We will show that θ is not definable by application of the compactness theorem:

$$T \cup \{\theta\} \cup \{\text{“ at least } n \text{ distinct elts satisfy } \phi(x, b)'' : n \in \mathbb{N}\}$$

Our models $\mathfrak{M}_n$ establish this for finite subset of this theory (they are finite, and we can finite one with m distict elements for every m. Which means that there exists a model for the entire theory. This in turn means that there are more than n distinct elements satisfying the propositions for every n while at the same time being finite. This is a contradiction.

Conversely suppose that such a bound exists. We need to show that we can test for finiteness. This can be done by seeing if $C + 1$ elements define the proposition,

$$\psi(b) := \exists x_1, \ldots, x_{C+1} \quad (\bigwedge_{i<j} x_i \neq x_j) \wedge (\bigwedge_i \phi(x_i, b))$$

$$\psi(b) \iff \#\{a \in M : \phi(a, b)\} < \infty.$$

□

5.1 Is there a variation?

The proposition above states ”if finiteness is definable then you whenever the set of interest is bounded it has a uniform bound”. Here’s what I was thinking you could do: if every model $\mathfrak{M}$ has as a measure on μ on M one could try to replace cardinality hypotheses with measure hypothesis.

Conjecture 1 (Definability and Uniform Bounds (Measure Theoretic Version))**.** *Let $L(\tau)$ be a first order language, $\phi(x, y)$ be a sentence is that language and T a theory such that every model $\mathfrak{M}$ is a measure space with measure μ.*

$$\{b \in M : \mu(\phi(\mathfrak{M}, b)) < \infty\} \textit{ definable}$$

if and only if

There exist some constant $C > 0$ such that for all models and all parameters

$$\mu(\phi(\mathfrak{M}, b)) < \infty \implies \mu(\phi(\mathfrak{M}, b)) < \infty.$$

According to a math overflow question I asked this is false since Measure theory is a second order theory and this doesn’t make sense. Henry Towsner appears to have some sort or replacement for this theory. Also, for those who are interested in relating measure theory to Model Theory some of the papers of Anand Pillay seem to be a good place to start.