“Linearity” in Arithmetic Differential Equations

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UCSD Number Theory Seminar

Part 1: Origins/History

Part 2: Arithmetic Differential Equations

Part 3: "Linear" Diff'l Eqns

PART 1

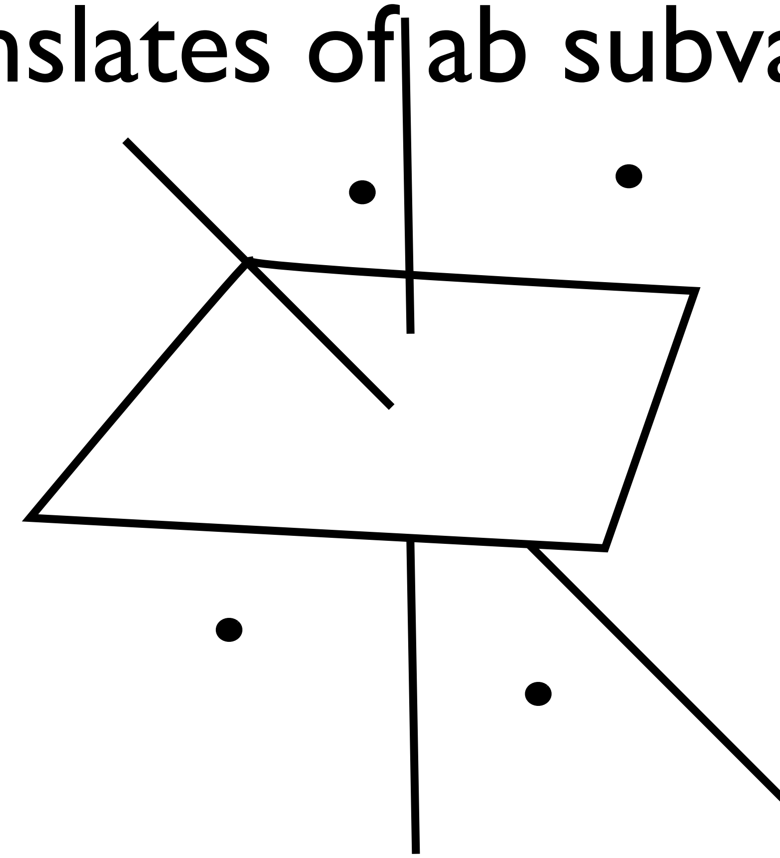
Origins of Arithmetic Differential Algebra

Diophantine Geometry

A/K abelian variety

configuration = finite union of translates of ab subvar

= $\bigcup_{i=1}^{n}(A_i + a_i)$



$A_i \leq A$

$a_i \in A_i(K)$

$A_i = A_i(\overline{K})$

configurations form a basis of closed sets for a topology on $A(\overline{K})$ called the configuration topology

$S \subset A(\overline{K})$

Defn. S f.r. if contained in f.r. subgroup

Mordell-Lang Problem

$$S \text{ f.r.} \implies S^{\mathrm{Conf}} = S^{\mathrm{Zar}}$$

Mordell-Lang Problem

$$S \text{ f.r. } \implies S^{\mathrm{Conf}} = S^{\mathrm{Zar}}$$

Example: (Mordell Problem)

$$C/\mathbf{Q}$$

$$g(C) \geq 2$$

$$\mathrm{Jac}(C) = A$$

$$C \hookrightarrow \mathrm{Jac}(C)$$

$$C(\mathbf{Q})^{\mathrm{Conf}} = C(\mathbf{Q})^{\mathrm{Zar}} \subset C(\overline{K})$$

$$C(\mathbf{Q}) \subset \mathrm{Jac}(C)(\mathbf{Q})$$

$$\#C(\mathbf{Q}) < \infty$$

example. (Manin-Mumford Problem)

C/K

K

$g(C) \geq 2$

$A = \mathrm{Jac}(C).$

$$S = C(\overline{K}) \cap A(\overline{K})_{tors} \subset A(\overline{K})_{tors}$$

$$S^{\mathrm{Zar}} \subset C(\overline{K}).$$

Mordell-Lang Problem

S f.r. $\implies S^{\mathrm{Conf}} = S^{\mathrm{Zar}}$

History of Mordell-Lang Problem

- Manin - Mordell and Manin-Mumford when K = function field.
- Raynaud - Manin-Mumford for number fields
- Faltings, Coleman, Parshin - Lang Conjecture in General
- Buium - Lang Conjecture over function fields (using differential algebra), char 0
- Hrushovski - Relative Lang, char p

Weil Cohomology Theory

$$H^\bullet : \{ \text{ Smooth Varieties}/ \; k \; \} \to \{ \text{ Graded } K\text{-algebras}\}$$

$$H^\bullet(X) = \bigoplus_{w=0}^{2\dim(X)} H^w(X)$$

Finiteness $h^i(X) < \infty$

Vanishing $h^i(X) = 0$ for $i > 2n$ or $i < 0$

Trace Map $\mathrm{Tr : H^{2n} \cong K}$

Poincare Duality $H^i \cong H^{2n-i*}$ via cup product

Kunneth $H^\bullet(X \times Y) = H^\bullet(X) \otimes H^\bullet(Y)$

Lefschetz Axioms

Cycle Map $Z^i(X) \to H^{2i}$ has some nice properties

Weak Lefschetz Re: pullback of hyperplane sections

Hard Lefschetz $H^i \to H^{i+2}$, $\xi \mapsto \xi \smile \omega$
where $\omega = [H]$, H a hyperplane section.

Setup for Desired Category

$$- \otimes_{\mathbf{F}_1} \mathbf{Z} : \mathsf{Sch}_{\mathbf{F}_1} \to \mathsf{Sch}_{\mathbf{Z}}.$$

$\mathrm{Spec}(\mathbf{Z})_{\mathbf{F}_1}$ in the category $\mathsf{Sch}_{\mathbf{F}_1}$ to get $\overline{\mathrm{Spec}(\mathbf{Z})_{\mathbf{F}_1}}$

Deninger Cohomlogy Theory

$$H_D^\bullet : \mathsf{Sch}_{\mathbf{F}_1} \to \text{ Graded } \mathbf{R}\text{-algebras}$$

Deninger Cohomlogy Theory

$$H_D^\bullet : \mathsf{Sch}_{\mathbf{F}_1} \to \text{ Graded } \mathbf{R}\text{-algebras}$$

Frechetness The spaces $H^w(X_D, j_*\mathcal{C})$

Vanishing $H^w(X) = 0$ for $w > 2n$ or $w < 0$

Trace Map $\mathrm{Tr} : \mathrm{H}^{2n} \cong \mathrm{K}$

Poincare Duality $H^i \cong H^{2n-i*}$ via cup product

Kunneth $H^\bullet(X \times Y) = H^\bullet(X) \otimes H^\bullet(Y)$

Lefschetz Axioms

Hodge $*$ $* : H^w \cong H^{2n-w}$

Real Action $\phi : \mathbf{R} \times X_D \to X_D$, which should be thought of as a replacement of the

ϕ^{*t} (induced action on cohomology)

$\theta = \lim_{t \to 0} \frac{\phi^{t*} - I}{t}$

Deninger Says:

$$\widehat{\zeta}(s) \overset{\text{thm}}{=} \frac{\prod_{\widehat{\zeta}(\rho)=0} \frac{s-\rho}{2\pi}}{\frac{s}{\pi} \cdot \frac{s-1}{\pi}}.$$

$$\overset{\text{guess}}{=} \frac{\det_\infty(\frac{s-\theta}{2\pi}|H^1)}{\det_\infty(\frac{s-\theta}{2\pi}|H^0)\det_\infty(\frac{s-\theta}{2\pi}|H^2)}.$$

- θ acts as zero on H^0
- θ acts as identity on H^2

Conditional Proof

Pairing: $\langle f, g\rangle := \mathrm{Tr}(\mathrm{f}_1 \smile *\mathrm{f}_2)$

How derivatives act:

$$\theta(f \smile g) = \theta(f) \smile g + f \smile \theta(f).$$

Get antisymmetry property:

$$\theta \text{ acts trivially on } H^2.$$

$f_1, f_2 \in H^1$ $\qquad$ $f_1 \cup *f_2 \in H^2$

$$\begin{aligned} f_1 \smile *f_2 &= \theta(f_1 \smile *f_2) \\ &= \theta(f_1) \smile *f_2 + f_1 \smile \theta(*f_2) \\ &= \theta(f_1) \smile *f_2 + f_1 \smile *\theta(f_2) \end{aligned}$$

$$
\begin{aligned}
\Longrightarrow &\quad \theta - 1/2 \text{ is antisymmetric on } H^1 \\
\Longrightarrow &\quad \sigma(\theta - 1/2|H^1) \subset i\mathbf{R} \\
\Longrightarrow &\quad \sigma(\theta|H^1) \subset 1/2 + i\mathbf{R}
\end{aligned}
$$

Remarks

Cyclic Cohomology (NCG)

Topological Hochschild Cohomology

De Rham Witt

- Consistent with random matrix theory statistics
- $(H^1(\overline{\mathrm{Spec}(\mathbf{Z})_{\mathbf{F}_1}}), *)$ Algebra with involution

θ

$*$

Noncommutative Geometric Object?

Prototype = BC system ?

Borger-Buium Descent

CLASSICAL

X/K smooth projective

$K = \overline{K} \qquad \mathrm{char}(K) = 0$

$\delta : K \to K$

$$K^\delta = \{r \in K : \delta(r) = 0\}$$

Theorem

T.F.A.E.

1. $\mathrm{KS}(\delta) = 0$
2. $J^1(X) \cong TX$ as schemes over X
3. $\exists X'/K^\delta$ such that $X' \otimes_{K^\delta} K \cong X$

Descent to the constants

$R = W(\overline{\mathbf{F}}_p)$
X/R

Theorem (D.)

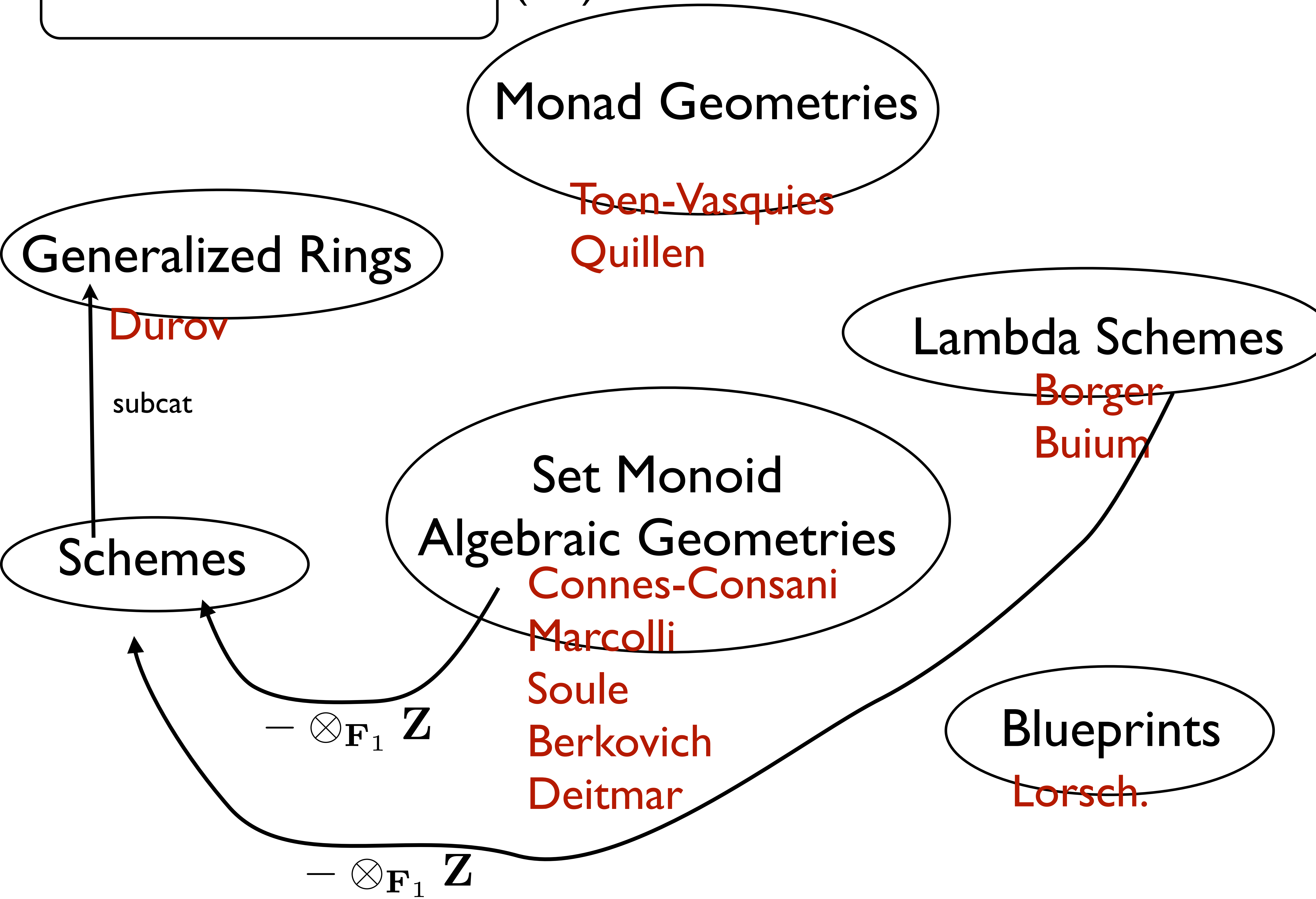
(*) $\Longrightarrow$

"Canonically lifted" Frobenius Tangent Bundle $(FT_{X_n})_{n\geq 0}$

$J^1(X)_n$ torsor under FT_{X_n}

1. $DI_n(X/R) = 0$
2. $J^1(X)_n \cong FT_{X_n}$ as torsors
3. X_n/R_n admits a lift of the Frobenius.

Sketch of $\mathbf{F}_1$ Land (???)



PART 2

Wifferential Algebra

What is a p-derivation?

Fermat's Little Theorem

$\forall n \in \mathbb{Z}, \forall p$ prime

$$n \equiv n^p \mod p$$

$$n - n^p = p \cdot \text{CRAP}$$

This is a p-derivation

$$\text{CRAP} = \frac{n - n^p}{p}$$

$$\delta_p(n) = \frac{n - n^p}{p}$$

Zero mod p

$$\delta_p(n) = \frac{n - n^p}{p}$$

Zero mod p

Product Rule

$$\delta_p(ab) = \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b)$$

Sum Rule

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j} b^j$$

non-linear

Kills Unit

$$\delta(1) = 0$$

derivations ⟷ ring homomorphisms

$\delta : A \to A$ ⟷ $f : A \to A[\varepsilon]/\langle \varepsilon^2 \rangle$

"dual numbers"

"infinitesimals"

p-derivations ⟷ ring homomorphisms

$\delta_p : A \to A$ ⟷ $f : A \to W_1(A)$

"Witt vectors"

"wittfinitesimals"

"Wittferentiation"

Witt Vectors.

$$(x_0, x_1)(y_0, y_1) = (x_0 y_0, x_1 y_0^p + y_1 x_0^p + p x_1 y_1)$$

$$(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 + C_p(x_0, y_0))$$

$$C_p(X, Y) = \frac{X^p + Y^p - (X+Y)^p}{p} \in \mathbf{Z}[X, Y]$$

$$\delta_p(a) := \frac{\phi(a) - a^p}{p}$$

$\phi : A \to B$

$\phi(a) \equiv a^p \mod p$

p-torsion free

lifts of the Frobenius $\approx$ p-derivations

$\delta_p : A \to B$ + rules

$$\phi(a) := a^p + p\delta_p(a)$$

Always an A-algebra

Defn. (Buium, Joyal 90's)

A **p-derivation** is a map of sets $\delta_p : A \to B$ such that

$\forall a, b \in A$

$$\delta_p(ab) = \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b)$$

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j}b^j$$

EXAMPLES

example. $R = \mathbf{Z}_p$

$$\delta_p(x) = \frac{x - x^p}{p}$$

example. $R = \mathbf{Z}_p[\zeta]$

ζ = root of unity coprime to p

$$\delta_p(x) = \frac{\phi(x) - x^p}{p}$$

$\phi(x) =$ unique lift of the frobenius

$$= \begin{cases} \zeta \mapsto \zeta^p, & \text{on roots of unity} \\ identity, & \text{else} \end{cases}$$

example $R = \mathbf{Z}_p^{ur}$

$$= \mathbf{Z}_p[\zeta : \zeta^n = 1, p \nmid 1]$$

$$\delta_p(x) = \frac{\phi(x) - x^p}{p}$$

Decreases valuation:

$$\delta_p(p) = \frac{p - p^p}{p}$$
$$= 1 - p^{p-1}$$

$$\delta_p(p^m) = p^{m-1} \cdot \ (\text{ unit mod } p \ )$$

Decreases valuation:

$$\delta_t = \tfrac{d}{dt}$$
$$\delta_t(t^n) = n \cdot t^{n-1}$$

Constants

Constants of a derivation:

(K, δ) (ring with der)

$K^\delta = \{c \in K : \delta(c) = 0\}$ (subring)

Constants of a p-derivation:

$R = \widehat{\mathbf{Z}}_p^{\mathrm{ur}}$ (ring with p-der)

$R^\delta = \{r \in R : \delta_p(r) = 0$ (submonoid)

PART 2

Linear Arithmetic Differential Equations

The poor man's model companion:

$$R = \widehat{\mathbf{Z}}_p^{\mathrm{ur}}$$

R WILL ALWAYS MEAN THIS RING IN THIS TALK

Properties:

1) unramified CDVR, residue field $\overline{\mathbf{F}}_p$

2) $\widehat{\mathbf{Z}}_p^{\mathrm{ur}} = \mathbf{Z}_p[\zeta; \zeta^n = 1, p \nmid n]\widehat{\ }$

3) unique lift of the Frobenius: $\exists! \phi : R \to R$

$$\phi(\zeta) = \zeta^p$$

Simplest Possible Equation:

$x = (x_{ij})$

$\alpha \in \mathrm{GL}_n(R)$

$$\delta x = \alpha x^{(p)}$$

$x^{(p)} = (x_{ij}^p)$

Remarks.

$x \mapsto \delta(x)(x^{(p)})^{-1}$ almost a cocycle

Theorem (existence and uniqueness)

$\alpha \in \mathfrak{gl}_n(R)$, $u_0 \in \mathrm{GL}_n(R)$

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \mod p \end{cases}$$

has a unique solution

$\epsilon = 1 + p\alpha$

proof.

$$\delta u = \alpha u^{(p)} \iff \phi(u) = \epsilon u^{(p)}$$

Contraction mapping: $f : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R)$

$$f(x) = \phi^{-1}(\epsilon x^{(p)})$$

matrix norms

$$|x - y|_p \leq 1 \implies |f(x) - f(y)|_p \leq \frac{1}{p}|x - y|_p$$

$$u = \lim_{n \to \infty} f^n(u_0)$$

Theorem (coeffs in CDVR)

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \mod p \end{cases}$$

$\mathcal{O}$ = Complete Discrete Valuation Subring

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O})$$

Proof. $\mathcal{O} = R^{\phi^\nu}$ (characterization)

$\epsilon = 1 + p\alpha$

$$\phi^\nu(u_0) = u_0, \quad \phi^\nu(\alpha) = \alpha, \quad \phi^\nu(\epsilon) = \epsilon$$

$$\phi(u) = \epsilon u^{(p)}$$

$$\phi^{\nu+1}(u) = \phi^\nu(\epsilon u^{(p)})$$

$$\phi(\phi^\nu(u)) = \epsilon(\phi^\nu(u))^{(p)}$$

Theorem (coeffs in valuation delta subring)

$$\begin{cases} \delta u = \alpha u^{(p)} \\ u \equiv u_0 \mod p \end{cases}$$

$\mathcal{O}$ = valuation delta-subring

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

$\mathcal{O}'/\mathcal{O}$ finite extension of delta-subrings

Strategy: reformulate as dynamics problem!!

Proof. $u_0, \alpha, \epsilon = 1 + p\alpha \in \mathfrak{gl}_n(\widehat{\mathcal{O}})$ ← complete

CDVR regularity Lemma $\implies \exists \nu \geq 0, \phi^\nu(u) = u$

Theorem (coeffs in valuation delta subring)

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

$\mathcal{O}'/\mathcal{O}$ finite extension of delta-subrings

Strategy: reformulate as dynamics problem!!

$$\begin{aligned} u &= \phi^{\nu}(u) \\ &= \phi^{\nu-1}(\epsilon u^{(p)}) \\ &= \cdots \\ &= \phi^{\nu-1}(\epsilon)(\phi^{\nu-2}(\epsilon)(\cdots(\epsilon u^{(p)})\cdots)^{(p)})^{(p)} \end{aligned}$$

$w \in \mathrm{GL}_n(R)$ $\qquad$ $w_{(m)}$ = pick out mth column

properties

1) $(w^{(p)})_{(m)} = (w_{(m)})^{(p)}$

2) $(vw)_{(m)} = (vw_{(m)})$

Theorem (coeffs in valuation delta subring)

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

Strategy: reformulate as dynamics problem!!

$$\begin{aligned} u &= \phi^{\nu}(u) \\ &= \phi^{\nu-1}(\epsilon u^{(p)}) \\ &= \cdots \\ &= \phi^{\nu-1}(\epsilon)(\phi^{\nu-2}(\epsilon)(\cdots(\epsilon u^{(p)})\cdots)^{(p)})^{(p)} \end{aligned}$$

properties

$$(w^{(p)})_{(m)} = (w_{(m)})^{(p)}$$

$$(vw)_{(m)} = (vw_{(m)})$$

$$u_{(m)} = \epsilon_{\nu-1}(\epsilon_{\nu-2}(\cdots(\epsilon u_{(m)}^{(p)})\cdots)^{(p)})^{(p)}$$

$$\varphi : \mathbf{A}^n_F \to \mathbf{A}^n_F$$

$$\varphi(\eta) = \epsilon_{\nu-1}(\epsilon_{\nu-2}(\cdots(\epsilon \eta^{(p)})\cdots)^{(p)})^{(p)}$$

$u_{(m)}$ **fixed point of** $\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a)$

Theorem (coeffs in valuation delta subring)

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

$$u_{(m)} = \epsilon_{\nu-1}(\epsilon_{\nu-2}(\cdots(\epsilon u_{(m)}^{(p)})\cdots)^{(p)})^{(p)}$$

Dynamics! $\varphi : \mathbf{A}_F^n \to \mathbf{A}_F^n$ $\qquad F = \mathrm{Frac}(\mathcal{O})$

$$\varphi(\eta) = \epsilon_{\nu-1}(\epsilon_{\nu-2}(\cdots(\epsilon\eta^{(p)})\cdots)^{(p)})^{(p)}$$

Lemma.

fixed points of

$\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a) < \infty$

Theorem (coeffs in valuation delta subring)

$$u_0 \in \mathrm{GL}_n(\mathcal{O}) \text{ and } \alpha \in \mathfrak{gl}_n(\mathcal{O}) \implies u \in \mathrm{GL}_n(\mathcal{O}')$$

Lemma.

\# fixed points of

$$\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a) < \infty$$

general principle:

$$\varphi/F \implies \text{fixed points of } \varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a)$$
$$\subset \mathbf{A}^n(F^a)$$

$\implies$ columns are in finite algebraic extensions

Lemma. (Fornaes and Sibonay)

fixed points of

$\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a) < \infty$

proof. $\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$

homogenize

φ built from $\eta \mapsto e_j \eta$, $\eta \mapsto \eta^{(p)}$ $\rightsquigarrow$ $Y : \quad \varphi_j(x_1, \ldots, x_n) - x_0^{d-1} x_j = 0$

Claim

$Y \cap \{x_0 = 0\} = \emptyset$

$\implies Y$ zero dimensional in $\mathbf{P}^n$

$\implies$ fixed points of $\varphi : \mathbf{A}^n(K^a) \to \mathbf{A}^n(K^a)$ ={ (1,u) }

Galois Theory

$\mathcal{O} \subset R$, $\alpha \in \mathfrak{gl}_n(\mathcal{O})$

$$\delta u = \alpha u^{(p)}$$

$\mathcal{O}[u] = \mathcal{O}[u_{ij}]$ = Picard-Vessiot ring

= Ring obtained by adjoining entries of u

Galois Group

=

$$\{c \in \mathrm{GL}_n(\mathcal{O}) : \exists \sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]), \sigma \circ \delta = \delta \circ \sigma, \sigma(u) = uc\}$$

$$G_{u/\mathcal{O}}$$

Galois Group of $\delta u = \alpha u^{(p)}$

=

$$\{c \in \mathrm{GL}_n(\mathcal{O}) : \exists \sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]), \sigma \circ \delta = \delta \circ \sigma, \sigma(u) = uc\}$$

$$G_{u/\mathcal{O}}$$

Alternative Descriptions:

1) $\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$

$$\Gamma_{u/\mathcal{O}} \cong G_{u/\mathcal{O}}$$

2) $$0 \to I_{u/\mathcal{O}} \to \mathcal{O}[x, 1/\det(x)] \to \mathcal{O}[u] \to 0$$

$$\mathrm{Stab}_{\mathrm{GL}_n(R)}(I_{u/\mathcal{O}}) \cong G_{u/\mathcal{O}}$$

$$\delta u = \alpha u^{(p)}$$

Question: For what $c \in \mathrm{GL}_n(R)$ do we get

$$\delta(uc) = \alpha(uc)^{(p)} \text{ ?}$$

$$\delta(uc) = u^{(p)}\delta(c) + \delta(u)c^{(p)} + p\delta(u)\delta(c) + \{u, c\}^*$$

$$\{u, c\}^* = \frac{u^{(p)}c^{(p)} - (uc)^{(p)}}{p}$$

$\delta(uc) = u^{(p)}\delta(c) + \delta(u)c^{(p)} + p\delta(u)\delta(c) + \{u,c\}^*$ $\{u,c\}^* = \frac{u^{(p)}c^{(p)} - (uc)^{(p)}}{p}$

Claim.

1) $\{u,c\}^* = 0$
2) $\delta c = 0$

$\Longrightarrow \quad \delta(uc) = \alpha (uc)^{(p)}$

Main Example:

subgroup of $\mathrm{GL}_n(\mathbf{F}_1^a)$

T = maximal torus of diagonals

W = permutation matrices

$N = WT = TW$

G^δ = matrices with roots of unity entries

$N^\delta = T^\delta W = WT^\delta$ = permutation matrices with roots of unity entries

Theorem.

$\mathcal{O} \subset R$ delta subring

$\mathcal{O} \subset R^{\phi^\nu}$ $\qquad u \in \mathcal{O} \implies G_{u/\mathcal{O}}$ finite

$$\Gamma_{u/\mathcal{O}} = \{\sigma \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}[u]); \sigma \circ \delta = \delta \circ \sigma \in \mathrm{GL}_n(\mathcal{O})\}$$

$= G_{u/\mathcal{O}}$

finite to begin with

Theorem A.

$\exists \Omega \subset \mathbf{Q}^2$ "thin set"

$\forall \alpha \in \mathbf{Z}^2 \setminus \Omega$, $\exists u \in \mathrm{GL}_n(R)$ $\delta u = \alpha u^{(p)}$

$G_{u/\mathcal{O}}$ = finite group containing W

$\Omega \subset \mathbf{Z}^{n^2}$

Theorem B.

$$X = \{u \in \mathrm{GL}_n(R); u \equiv 1 \mod p\}$$

= ball around identity

$\exists \Omega \subset X$ of the second category

$\forall u \in X \setminus \Omega$, $\forall \mathcal{O} \subset R$ δ-closed subring

1) $\delta(u)(u^{(p)})^{-1} \in \mathfrak{gl}_n(\mathcal{O})$
2) $R^\delta \subset \mathcal{O}$

$$\Longrightarrow \quad G_{u/\mathcal{O}} = N^\delta$$

Theorem C.

$\exists \Omega \subset \mathrm{GL}_n(K^a)$ Zariski closed

$\forall u \in \mathrm{GL}_n(R) \setminus \Omega$

$$u' = \alpha u^{(p)}$$

$\mathcal{O} \ni \alpha$ δ-closed subring of R

$$\dim((Z \cdot G_{u/\mathcal{O}})^{\mathrm{Zar}}) \leq n$$

PART 4

Remarks on Picard-Fuchs and Painleve 6 and the Frobenius

Manin Map:

E/K elliptic curve

$K =$ function field

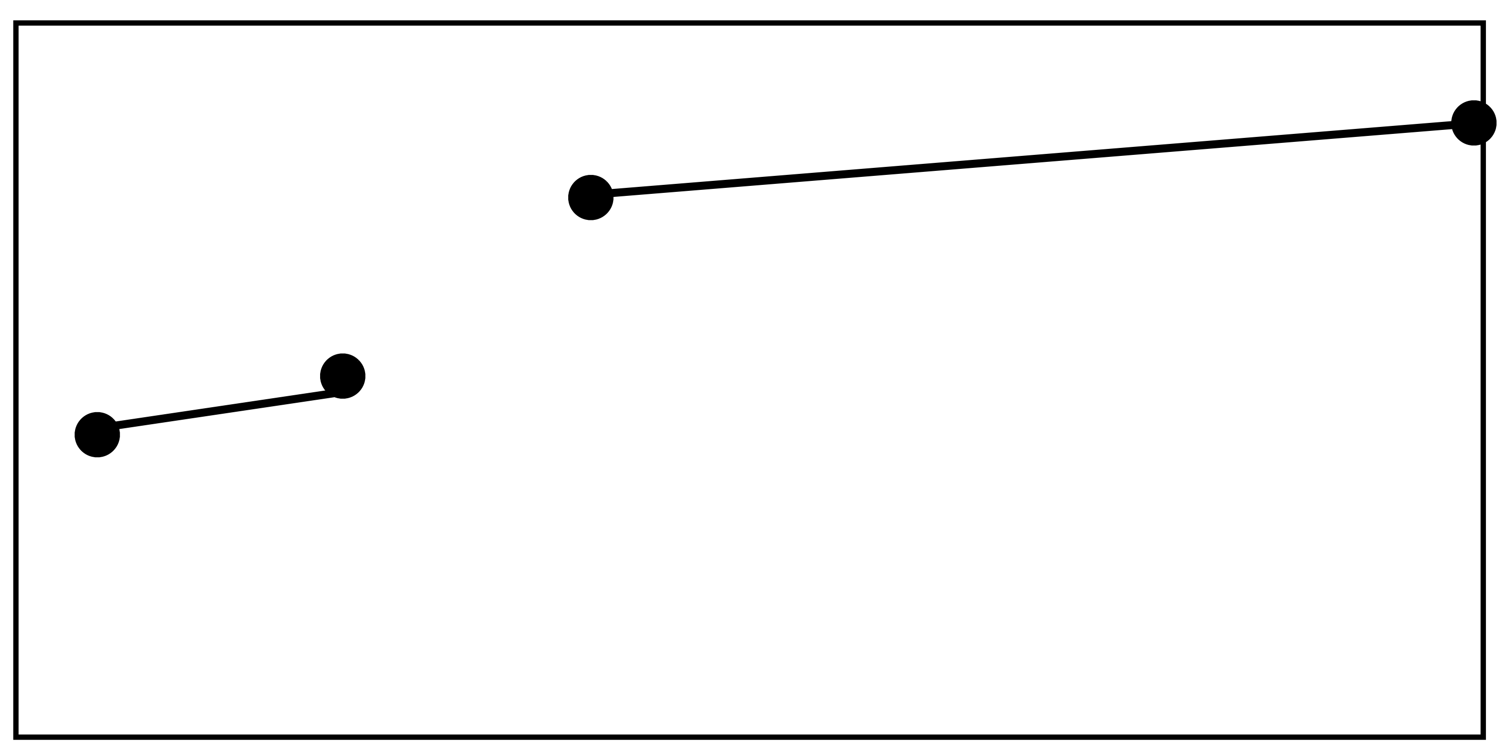
$$\psi : E(K) \to \mathbf{G}_a(K) \quad \text{(diff alg map)}$$

$$\ker(\psi) = E(K)_{tors}$$

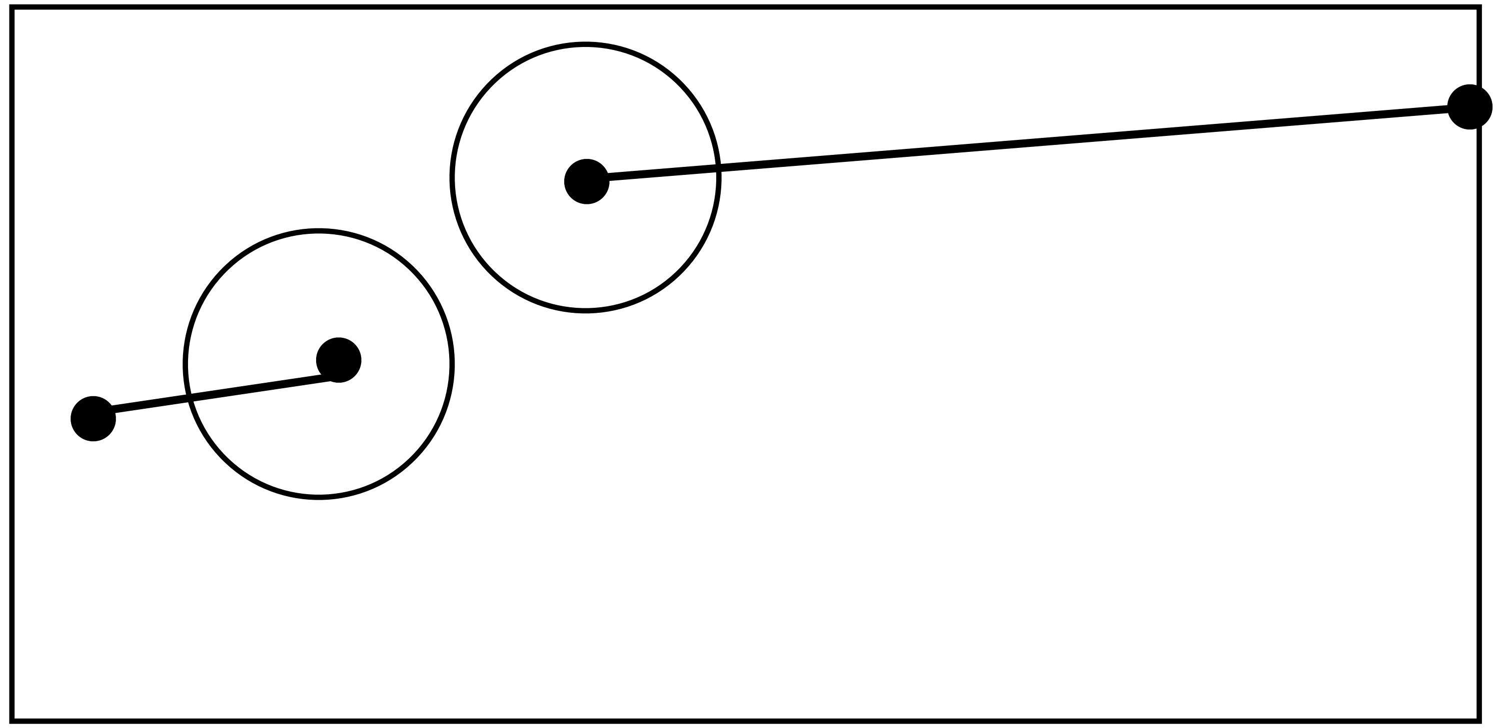
Setup: $\mathcal{E} : y^2 = x(x-1)(x-t)$

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathbf{A}^1 \end{array} \quad \text{universal family}$$

$\mathcal{E}_t(\mathbf{C}) =$



$\sqrt{x(x-1)(x-t)}$



$$H_1(\mathcal{E}_t(\mathbf{C}), \mathbf{Z}) = \mathrm{Span}\{\gamma_1, \gamma_2\}$$

$$\omega = dx/y$$

$$\int_{\gamma_1} \omega = \eta_1 \qquad \int_{\gamma_2} \omega = \eta_2$$

$\eta_1(t), \eta_2(t)$ periods

Picard-Fuchs Equations:

$$\Lambda = 4t(1-t)\frac{d^2}{dt^2} + 4(1-2t)\frac{d}{dt} - 1$$

kills periods

Manin-Map:

$$\psi(X,Y) = \Lambda \int^{(X,Y)} dx/y$$

Buium’s Picard-Fuchs

$$\Lambda = \phi^2 + \alpha_1 \phi + p\alpha_0$$

Buium’s Manin Map

$$\psi(P) = \Lambda\left(\frac{1}{p^r}\log_\omega(P)\right)$$