

AUTOMORPHISMS OF THE AFFINE LINE OVER NON-REDUCED RINGS

TAYLOR ANDREW DUPUY

ABSTRACT. The affine space \mathbb{A}_B^1 only has automorphisms of the form $aT + b$ when B is a domain. In this paper we study univariate polynomial automorphisms over non-reduced rings which have an extremely rich structure. Geometrically these groups appear naturally in transition maps of affine bundles in arithmetic geometry.

CONTENTS

1. Introduction	1
1.1. Motivation	2
1.2. Plan of the Paper	2
1.3. Acknowledgements	3
2. Automorphisms of the Affine Line over Non-reduced rings	3
2.1. Subgroups of Bounded Degree	3
3. The “Greenberg transform” of Univariate Polynomial Automorphisms	6
3.1. Witt Vectors	6
3.2. The Greenberg Transform	7
3.3. Automorphisms of the Affine Line as Algebraic Groups	8
4. Solvability	10
4.1. Solvable Groups	10
4.2. Abelian Normal Subgroups	11
5. Adjoint Representations	12
5.1. An Algorithm for Computing Inverses	12
5.2. The Adjoint Representation	13
5.3. Examples of Explicit Representations	14
References	15

1. INTRODUCTION

Throughout this paper $p \in \mathbb{N}$ will always denote a prime and all rings will be commutative with a unit. For a ring B we use the notation $\text{Aut}(\mathbb{A}_B^1) = \text{Aut}_B(B[T])^{op}$ and identify this group with collections of polynomials under composition.

For R be a q -torsion free ring where $q \in R$ and $qR \in \text{Spec}(R)$ we will let $R_d = R/q^{d+1}$. In this paper we prove three theorems about univariate polynomial automorphisms over such rings R . Two important examples to keep in mind are $R_n = K[t]/(t^{n+1})$ for K a field and $R_n = \mathbb{Z}/p^{n+1}$ which are the “geometric” and “arithmetic” cases in what follows.

Theorem 1. *For R a q -torsion free ring where $q \in R$ and $qR \in \text{Spec}(R)$ the collection*

$$\tilde{A}_d(R, q) := \{\psi \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1) : \forall m \geq 2, \deg(\psi \bmod q^m) \leq d2^{m-2}\}$$

forms a subgroup under composition.

This implies for example that for every R a q -torsion free ring with $qR \in \text{Spec}(R)$ and every $\psi \in \text{Aut}(\mathbb{A}_{R_n}^1)$ that every iterate of ψ has bounded degree. In particular it implies that every $\psi \in \text{Aut}(\mathbb{A}_{\mathbb{Z}/p^{n+1}}^1)$ has finite order.

We define the set of polynomials $A_d(R, q)$ which are of the form

$$(1.1) \quad f(T) \equiv a_0 + a_1T + qa_2T^2 + q^2a_3T^4 + \cdots + q^{d-1}a_dT^d \in R_{d-1}[T]$$

which are invertible under composition. It was proved in [Dup12] that for any q -torsion free ring R the set $A_d(R) \subset \text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ under composition is a group. We reproduce this proof in Example 5.

When $R = \mathbb{Z}$ and $q = p$ we have this implies for example that the polynomial $f(T) = 1 + T + pT^2 + p^2T^3 + p^3T^4$ has finite order under composition mod p^4 and that every iterate has degree less or equal to four mod p^4 .

Theorem 2.

- *There exists a finite dimensional group scheme over \mathbb{F}_p whose group of \mathbb{F}_p -points is isomorphic to $A_n(\mathbb{Z}, p)$.*
- *There exists (an infinite dimensional) group scheme over \mathbb{F}_p whose group of \mathbb{F}_p -points is isomorphic to $\text{Aut}(\mathbb{A}_{\mathbb{Z}/p^n}^1)$.*

These groups can in some sense be considered as “the Greenberg transform” of $A_n(\mathbb{Z}, p)$ and $\text{Aut}(\mathbb{A}_{\mathbb{Z}/p^n}^1)$ respectively.

Theorem 3. *For R a q -torsion free ring where $q \in R$ and $qR \in \text{Spec}(R)$ the groups $A_n(R, q)$ and $\text{Aut}(\mathbb{A}_{R_n}^1)$ are solvable.*

In particular with implies that for every $m \in \mathbb{Z}$ the groups $\text{Aut}(\mathbb{A}_{\mathbb{Z}/m}^1)$ are solvable.

1.1. Motivation. We will take a moment to motivate these groups. Let B be a ring. Recall that an \mathbb{A}^1 -**bundle** over a scheme X/B is a scheme E/B together with a morphisms $\pi : E \rightarrow X$ with the property that for every point $x \in X$ there exists an affine open subset U containing x and an isomorphisms $\psi : \pi^{-1}(U) \cong U \times_B \mathbb{A}_B^1$ with the property that $\pi|_{\pi^{-1}(U)} = p_1 \circ \psi$ where $p_1 : U \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the first projection onto U . Given two such isomorphisms $\psi, \psi' : \pi^{-1}(U) \cong U \times_B \mathbb{A}_B^1$ we can consider the

$$\psi' \circ \psi^{-1} : U \times_B \mathbb{A}_B^1 \rightarrow U \times_B \mathbb{A}_B^1.$$

Maps of this forms essentially the subject of this paper when B is a non-reduced ring. These appear in the author’s paper [Dup12] when E is the first p -jet spaces of a curve modulo p^n .

1.2. Plan of the Paper. In section 2 we introduce the groups and notation that we are going to use. In particular subsection 2.1 proves theorem 1.

In section 3 we prove theorem 2 which shows that univariate polynomials automorphisms over \mathbb{Z}/p^n are really algebraic groups. This section starts by proving this in a simple example and moves to the more general case.

In section 4 we prove solvability of univariate polynomial automorphisms by proving certain normal subgroups are abelian. This section starts by proving the theorem in a simple case then proves the more general case.

We apply the theory developed in section 4 in section 5. In subsection 5.1 we given an algorithm for computing . The remainder of the subsection gives module structures to the abelian subgroups introduced in section 4 and gives examples of explicit representations for the adjoint actions of on these abelian subgroups.

1.3. Acknowledgements. We would like to thank Alexandru Buium and Danny Krashen for their useful comments and suggestions.

2. AUTOMORPHISMS OF THE AFFINE LINE OVER NON-REDUCED RINGS

We introduced the groups $\tilde{A}_d(2, R, q)$ (denoted \tilde{A}_d in that paper) of polynomial automorphisms modulo p^2 with degree bounded by d and proved that they form a subgroup.

This section aims to generalize the result groups $\tilde{A}_d(2, R, q)$.

2.1. Subgroups of Bounded Degree. The main aim of this section is to prove the following result:

Theorem 4. *For R a q -torsion free ring with $q \in R$ and $qR \in \text{Spec}(R)$, the collection*

$$(2.1) \quad \tilde{A}_d(n, R, q) = \{\psi \in \text{Aut}(\mathbb{A}_{R, n-1}^1) : \deg(\psi \bmod p^m) \leq 2^{m-2}d, \quad 2 \leq m \leq n\}$$

is a subgroup.

Before proving this result in general we give several examples of the proof in special cases.

Example 5. We will show that for each $d \geq 1$, $\tilde{A}_d(2, R, q)$ is a group.

Let $\psi(T) = a_0 + a_1T + qf(T)$ and $\tilde{\psi}(T) = \tilde{a}_0 + \tilde{a}_1T + q\tilde{f}(T)$ with $\text{ord}_T(f), \text{ord}_T(\tilde{f}) \geq 2$ we get

$$(2.2) \quad \begin{aligned} \psi(\tilde{\psi}(T)) &\equiv a_0 + a_1[\tilde{a}_0 + \tilde{a}_1T + q\tilde{f}(T)] + qf(\tilde{a} + \tilde{a}T) \\ &= a_0 + a_1\tilde{a}_0 + (a_1\tilde{a}_1)T \\ &\quad + q(a_1\tilde{f}(T) + f(\tilde{a}_0 + \tilde{a}_1T)) \end{aligned}$$

Note that $a_1\tilde{f}(T) + f(\tilde{a}_0 + \tilde{a}_1T)$ has degree no larger than $\max\{\deg f, \deg \tilde{f}\}$. Also, since a_1 and \tilde{a}_1 are units, the degree of $\psi \circ \tilde{\psi}$ is exactly $\max\{\deg \psi, \deg \tilde{\psi}\}$ in the case that $\deg \psi \neq \deg \tilde{\psi}$. This means that if ψ and $\tilde{\psi}$ are inverse to each other then $\deg \psi = \deg \tilde{\psi}$ and hence $\tilde{A}_d(2, R, q)$ is closed under inverses. This is enough information to show that $\psi \circ \tilde{\psi}^{-1} \in \tilde{A}_d(2, R, q)$ which shows $\tilde{A}_d(2, R, q)$ is a subgroup. \square

Example 6. We will show that the set of $\psi \in \text{Aut}(\mathbb{A}_{R_2}^1)$ satisfying

$$\begin{aligned} \deg(\psi \bmod q^2) &\leq d \\ \deg(\psi \bmod q^3) &\leq 2d \end{aligned}$$

form a subgroup. If we write

$$\psi(T) = a_0 + a_1T + qf(T) + q^2g(T) \bmod q^3$$

for $f(T) \in R_1[T], g(T) \in R_0[T]$ with $\text{ord}_T(f) \geq 2$ and $\text{ord}_T(g) \geq 3$. Note that we have

$$\begin{aligned} \deg(\psi \bmod q^2) &\geq \deg(f \bmod q), \\ \deg(\psi \bmod q^3) &\geq \deg(g \bmod q), \deg(f \bmod q^2), \end{aligned}$$

Composing ψ with $\tilde{\psi}$ gives

$$\begin{aligned} \psi(\tilde{\psi}(T)) &= a_0 + a_1\tilde{\psi}(T) \\ &\quad + q[f(\tilde{a}_0 + \tilde{a}_1T) + qf'(\tilde{a}_0 + \tilde{a}_1T)\tilde{f}(T)] \\ &\quad + q^2g(\tilde{a}_0 + \tilde{a}_1T) \end{aligned}$$

Since the invertible polynomials of degree less than d are a group modulo q^2 we only need to check that $\deg(\psi(\tilde{\psi}(T))) \leq 2d$. We can just check each term is bounded by $2d$,

- The degree of $f(\tilde{a}_0 + \tilde{a}_1T) \bmod q^2$ is bounded by $2d$,
- The degree of $f'(\tilde{a}_0 + \tilde{a}_1T)\tilde{f}(T) \bmod q$ is bounded by $(d-1) + d$,
- The degree of $g(\tilde{a}_0 + \tilde{a}_1T) \bmod q$ is bounded by $2d$,

which completes the proof. \square

Example 7. We will show that the set of $\psi \in \text{Aut}(\mathbb{A}_{R_3}^1)$ satisfying

$$\begin{aligned} \deg(\psi \bmod q^2) &\leq d \\ \deg(\psi \bmod q^3) &\leq 2d \\ \deg(\psi \bmod q^4) &\leq 4d \end{aligned}$$

form a subgroup. The proof relies on the previous two cases. We will write such a ψ as

$$\psi(T) = a_0 + a_1T + qf(T) + q^2g(T) + q^3h(T) \bmod q^4$$

for $f(T) \in R_2[T], g(T) \in R_1[T], h(T) \in R_0[T]$ where $\text{ord}_T f \geq 2$, $\text{ord}_T g \geq 3$ and $\text{ord}_T h \geq 4$. Note that our conditions on degree imply that

$$\begin{aligned} \deg(\psi \bmod q^2) &\geq \deg(f \bmod q), \\ \deg(\psi \bmod q^3) &\geq \deg(g \bmod q), \deg(f \bmod q^2), \\ \deg(\psi \bmod q^4) &\geq \deg(h \bmod q), \deg(g \bmod q^2), \deg(f \bmod q^3). \end{aligned}$$

We will make use of these bounds in the subsequent computations. Composing ψ with $\tilde{\psi}$ gives

$$\begin{aligned} \psi(\tilde{\psi}(T)) &= a_0 + a_1\tilde{\psi}(T) \\ &\quad + q[f(\tilde{a}_0 + \tilde{a}_1T) + qf'(\tilde{a}_0 + \tilde{a}_1T)(\tilde{f}(T) + q\tilde{g}(T)) + q^2\frac{f''(\tilde{a}_0 + \tilde{a}_1T)}{2}\tilde{f}(T)^2] \\ &\quad + q^2[g(\tilde{a}_0 + \tilde{a}_1T) + qg'(\tilde{a}_0 + \tilde{a}_1T)\tilde{f}(T)] \\ &\quad + q^3h(\tilde{a}_0 + \tilde{a}_1T). \end{aligned}$$

From example 6 we just need to show that each term in this polynomial is bounded by $4d$.

- The degree of $f(\tilde{a}_0 + \tilde{a}_1T) \bmod q^3$ is bounded by $4d$,
- The degree of $f'(\tilde{a}_0 + \tilde{a}_1T)\tilde{f}(T) \bmod q^2$ is bounded by $(2d-1) + (2d)$
- The degree of $f''(\tilde{a}_0 + \tilde{a}_1T)\tilde{g}(T) \bmod q$ is bounded by $(d-1) + (2d)$
- The degree of $f''(\tilde{a}_0 + \tilde{a}_1T)\tilde{f}(T)^2 \bmod q$ is bounded by $(d-2) + 2d$

- The degree of $g(\tilde{a}_0 + \tilde{a}_1 T) \bmod q^2$ is bounded by $4d$,
- The degree of $g'(\tilde{a}_0 + \tilde{a}_1 T)\tilde{f}(T) \bmod q$ is bounded by $(2d - 1) + d$,
- The degree of $h(\tilde{a}_0 + \tilde{a}_1 T)$ is bounded by $4d$.

This shows that $\psi(\tilde{\psi}(T))$ has degree less than $4d$. \square

We will now give a remarkable theorem which generalizes the examples above.

Proof of Theorem 4. We prove this by induction on n where the statement is that the set $\tilde{A}_d(n, R, q)$ is a group. The base case is proved since $\tilde{A}_d(2, R, q)$ is a subgroup. We will suppose that $\tilde{A}_d(n-1, R, q)$ is a subgroup. Let $\psi(T), \tilde{\psi}(T) \in \tilde{A}_d(n, R, q)$. We just need to show that $\psi \circ \tilde{\psi} \in \tilde{A}_d(n, R, q)$, i.e. that the degree of $\psi \circ \tilde{\psi}$ is bounded by $2^{n-2}d$.

We will write ψ as

$$\psi(T) = a_0 + a_1 T + qf_1(T) + q^2 f_2(T) + \cdots + q^{n-1} f_{n-1}(T) \bmod q^n$$

where f_i is defined q^{n-i} and $\text{ord}_T f_i \geq i + 1$. Note that we have

$$d2^{m-2} \geq \deg(\psi(T) \bmod q^m) \geq \deg(f_i(T) \bmod q^{m-i})$$

For $2 \leq m \leq n-1$ and $2 \leq i \leq m-1$. We will now examine each of the terms of $\psi \circ \tilde{\psi}$. Here we have terms

$$\psi(\tilde{\psi}) = f_0(\psi) + qf_1(\psi) + \cdots + q^{n-1} f_{n-1}(\tilde{\psi}) \bmod q^n.$$

The general term is

$$\begin{aligned} q^i f_i(\tilde{\psi}) &= q^i f_i(\tilde{f}_0 + q\tilde{f}_1 + q^2 \tilde{f}_2 + \cdots + q^{n-i-1} \tilde{f}_{n-i-1}) \bmod q^n \\ (2.3) \quad &= q^i [f_i(\tilde{f}_0) + f'_i(\tilde{f}_0)A + \cdots + \frac{f_i^{(n-i-1)}(\tilde{f}_0)}{(n-i-1)!} A^{n-i-1}] \end{aligned}$$

where

$$A = \tilde{f}_1 + q\tilde{f}_2 + q^2 \tilde{f}_3 + \cdots + q^{n-i-2} \tilde{f}_{n-i-1}.$$

Lemma 8. For $i = 1, \dots, n$ the j th term in 2.3 is $q^{i+j} \frac{f_i^{(j)}(\tilde{f}_0)}{j!} A^j$ and $0 \leq j \leq n-i-1$. The degree of this term is bounded by

$$d_{n-j-1} - j + jd_{n-i-j}$$

where $\deg(\psi \bmod q^n), \deg(\tilde{\psi} \bmod q^n) \leq d_{n-1}$ for each $n \leq m$. (In our application $d_j = 2^{j-2}d$.)

Proof. First,

$$q^{i+j} A = q^{i+j} (\tilde{f}_1 + q\tilde{f}_2 + \cdots + q^{n-i-2} \tilde{f}_{n-i-1})$$

Since this is a term in $q^{i+j-1} \tilde{\psi} \bmod q^n$ and corresponds to a term in $\tilde{\psi} \bmod q^{n-i-j+1}$ which has degree bounded by d_{n-i-j} . Since A appears in the with multiplicity j in $q^{i+j} \frac{f_i^{(j)}(\tilde{f}_0)}{j!} A^j$ it contributes $j \cdot d_{n-i-j}$ to the degree bound of $q^{i+j} \frac{f_i^{(j)}(\tilde{f}_0)}{j!} A^j$.

Now we look at $q^{i+j} f_i^{(j)}(\tilde{f}_0)$. The expression $q^{i+j} f_i$ is a term in $q^j \psi \bmod q^n$ which corresponds to a term of $\psi \bmod q^{n-j}$ and hence has a degree bounded by d_{n-j-1} . Since we are taking j derivatives in our expression it has a contribution of $d_{n-j-1} - j$.

Putting the information from the two factors of $q^{i+j} \frac{f_i^{(j)}(\tilde{f}_0)}{j!} A^j$ together we have an overall degree bound

$$d_{n-j-1} - j + jd_{n-i-j}$$

as advertised. \square

All the terms in expression 2.3 are of the form $q^{i+j} \frac{f_i^{(j)}(\tilde{f}_0)}{j!} A^j$ where i varies from 0 to $n-1$ and j varies from 0 to $n-i-1$. We will now use the estimates in Lemma 8 to finish our proof. Again, we suppose that $d_j = 2^{j-1}d$. Plugging this into our expression we have

$$\begin{aligned} 2^{n-j-2}d - j + j2^{n-i-j-1}d &\leq 2^{n-j-2}d + j2^{n-i-j-1}d \\ &\leq 2^{n-j-2}d + j2^{n-j-1}d \\ &= 2^{n-2}d(2^{-j} + j2^{-j-2}) \end{aligned}$$

and since $2^{-j} + j2^{-j-2} \leq 1$ for all $j \geq 0$ we have our desired bound. \square

Example 9. The polynomial

$$\psi(T) = T + qT^d + q^2T^{2d} + q^3T^{4d} + q^4T^{8d} + q^5T^{16d} \pmod{q^6} \in \tilde{A}_d(6, \mathbb{Z}, p).$$

In particular has finite order under composition.

Corollary 10. Let $\psi \in \text{Aut}(\mathbb{A}_{\mathbb{Z}/q^n}^1)$ and let $\psi^r(T) = \psi(\psi(\dots(\psi(T))\dots))$ where the composition occurs r times. For every $r \geq 1$ we have

$$\deg(\psi^r(T)) \leq 2^{n-2}(\deg(\psi \pmod{q^2})).$$

Proof. Take ψ of degree d . It certainly has degree $d \pmod{q^2}$ satisfies $\deg(\psi \pmod{q^n}) \leq d2^{n-2}$ for each $n \geq 2$ so ψ is in the subgroup $\tilde{A}(d, R, q)$. \square

3. THE ‘‘GREENBURG TRANSFORM’’ OF UNIVARIATE POLYNOMIAL AUTOMORPHISMS

Let R be a q -torsion free ring where $q \in R$ and $qR \in \text{Spec}(R)$. In [Dup12] (section 4.1) we introduced the groups $A_d(R, q) \subset \text{Aut}(\mathbb{A}_{R_{d-1}}^1)$ consisting of polynomial automorphisms of the form

$$\psi(T) = a_0 + a_1T + qa_1T^2 + \dots + q^{d-1}a_{d-1}T^d \pmod{q^d}$$

and proved that they were a subgroup.

The aim of this section is to prove the following theorem:

Theorem 11. The groups $A_n(\mathbb{Z}, p)$ and $\text{Aut}(\mathbb{A}_{\mathbb{Z}/p^n}^1)$ are isomorphic to group of \mathbb{F}_p -points of an algebraic group.

This follows from a Greenberg-like transform.

3.1. Witt Vectors. An excellent reference for Witt vectors is chapter one of [Haz09]. Let R be a p -torsion free ring. The ring of p -typical **Witt vectors** of R , $W(R)$ is the set $R^{\mathbb{N}}$ together with a **Witt addition** and **Witt multiplication** which define a ring structure:

$$\begin{aligned} [x_0, x_1, x_2, \dots] +_W [y_0, y_1, y_2, \dots] &= [s_0, s_1, s_2, \dots], \\ [x_0, x_1, x_2, \dots] *_W [y_0, y_1, y_2, \dots] &= [m_0, m_1, m_2, \dots]. \end{aligned}$$

Here $s_i, m_i \in \mathbb{Z}[x_0, x_1, \dots, x_i, y_0, y_1, \dots, y_i]$ the **Witt addition** and **Witt multiplication** polynomials. They are the unique polynomials so that for every ring A the **Ghost Map**

$$\begin{aligned} w : {}_W(A) &\rightarrow A^{\mathbb{N}} \\ [x_0, x_1, x_2, \dots] &\mapsto [w_0, w_1, w_2, \dots] \end{aligned}$$

is a ring homomorphism. Here $w_j(x) = \sum_{n=0}^j p^j x_j^{p^{n-j}}$ are the **Witt polynomials**. Also, in the map w above, we give $A^{\mathbb{N}}$ its usual componentwise addition and multiplication.

Example 12. To compute the first two Witt addition and multiplication polynomials one needs to solve the “universal” system of equations¹

$$\begin{aligned} w_0(x +_W y) &= w_0(x) + w_0(y) \\ w_1(x +_W y) &= w_1(x) + w_1(y) \\ w_0(x *_W y) &= w_0(x)w_0(y) \\ w_1(x *_W y) &= w_1(x)w_1(y) \end{aligned}$$

which amounts to the system

$$\begin{aligned} s_0 &= x_0 + y_0 \\ s_0^p + ps_1 &= (x_0^p + px_1) + (y_0^p + py_1) \\ m_0 &= x_0y_0 \\ m_0^p + pm_1 &= (x_0^p + px_1)(y_0^p + py_1) \end{aligned}$$

which has the solution

$$\begin{aligned} s_0 &= x_0 + y_0 \\ s_1 &= x_1 + y_1 + \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} x^j y^{p-j} \\ m_0 &= x_0y_0 \\ m_1 &= x_1y_0^p + y_0^p y_1 + px_1y_1 \end{aligned}$$

□

It is a remarkable theorem of Witt’s

If we make a ring out of the set R^n by using only the first n Witt addition and multiplication polynomials we get the **ring of truncated p -typical Witt vectors** $W_{n-1}(R)$. We will also make use of the following important property of Witt vectors

Theorem 13 (Witt). $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^{n+1}$

3.2. The Greenberg Transform. The Greenberg transform first appeared in Lang’s thesis and can be found in [Lan52] and [Lan54]. It is essentially a way of converting polynomials over \mathbb{Z}/p^n to polynomials in more indeterminates over \mathbb{F}_p using Witt vectors.

Example 14. To compute the second Greenberg transform of $f(x, y) = x^2 + y \in \mathbb{Z}[x, y]$ we compute $x^2 + y$ using Witt additions and Witt multiplications

$$\begin{aligned} f([x_0, x_1], [y_0, y_1]) &= [x_0, x_1]^2 + [y_0, y_1] \\ &= [x_0^2, 2x_0^p x_1 + px_1^2] + [y_0, y_1] \\ &= [x_0^2 y_0, 2x_0^p x_1 + px_1^2 + y_1 + \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} x_0^{2j} y_0^{p-j}] \end{aligned}$$

¹ Universal meaning that we view everything just as symbols and don’t worry about what ring or characteristic they are in.

and get the polynomials

$$x_0^2 y_0, \quad 2x_0^p x_1 + px_1^2 + y_1 + \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} x_0^{2j} y_0^{p-j}.$$

□

In general, the n th Greenberg Transform of a polynomial $f(x_0, x_1, \dots, x_m) \in \mathbb{Z}[x_0, x_1, \dots, x_m]$ is the set of polynomials

$$g_0, g_1, \dots, g_n \in \mathbb{Z}[x_{0,1}, \dots, x_{0,n}; x_{1,0}, \dots, x_{1,n}; \dots; x_{m,0}, \dots, x_{m,n}]$$

where the g_i are defined by the equation

$$[g_0, \dots, g_n] = f([x_{0,0}, \dots, x_{0,n}], \dots, [x_{n,0}, \dots, x_{n,n}]).$$

The n th Greenberg transform of an ideal $I \triangleleft \mathbb{Z}[x_0, x_1, \dots, x_m]$ is the ideal $I' \triangleleft \mathbb{Z}[x_{0,1}, \dots, x_{0,n}; x_{1,0}, \dots, x_{1,n}; \dots; x_{m,0}, \dots, x_{m,n}]$ generated by the n th Greenberg transforms of the polynomials in I . The n th Greenberg transform of a subscheme of affine space $X = V(I) \subset \mathbb{A}_{\mathbb{Z}}^m$ is the scheme $\text{Gr}_n X \subset \mathbb{A}_{\mathbb{Z}}^{nm}$ defined by $\text{Gr}_n X = V(I')$ where I' is the n th Greenberg transform of I .

In general one can define the n th Greenberg transform as a functor $\text{Gr}_n : \text{Sch}_{\mathbb{Z}} \rightarrow \text{Sch}_{\mathbb{F}_p}$, which is characterized by $(\text{Gr}_n X)(\mathbb{F}_p) = X(\mathbb{Z}/p^n)$.

3.3. Automorphisms of the Affine Line as Algebraic Groups. We begin with an instructive example.

Example 15. We will explain how to apply the Greenberg Transform to $A_2(\mathbb{Z}, p)$ of univariate polynomial automorphisms of degree two modulo p^2 . We identify $f(T) = a + bT + pcT^2$ and $g(T) = a' + b'T + pc'T^2$ with $[a_0, a_1] + [b_0, b_1]T + [0, c_1]T^2$ and $[a'_0, a'_1] + [b'_0, b'_1]T + [0, c'_1]T^2$ then we multiply out the vectors as we normally would giving Witt multiplications

$$\begin{aligned} & [a_0, a_1] + [b_0, b_1]T + [0, c_1]T^2 \circ [a'_0, a'_1] + [b'_0, b'_1]T + [0, c'_1]T^2 \\ = & [a_0, a_1] + [b_0, b_1][a'_0, a'_1] + [0, c_1][a'_0, a'_1]^2 \\ & + ([b_0, b_1][b'_0, b'_1] + 2[0, c_1][b'_0, b'_1][a'_0, a'_1])T \\ & + ([b_0, b_1][0, c'_1] + [0, c_1][b'_0, b'_1]^2) \end{aligned}$$

which gives

$$\begin{aligned} [a''_0, a''_1] &= [a_0, a_1] + [b_0, b_1][a'_0, a'_1] + [0, c_1][a'_0, a'_1]^2 \\ [b''_0, b''_1] &= [b_0, b_1][b'_0, b'_1] + 2[0, c_1][b'_0, b'_1][a'_0, a'_1] \\ [c''_0, c''_1] &= [b_0, b_1][0, c'_1] + [0, c_1][b'_0, b'_1]^2 \end{aligned}$$

which tells us how to transform the coordinates. One can multiply these out to get explicit polynomials using the rules for Witt addition and Witt multiplication:

$$\begin{aligned} a''_0 &= a_0 + b_0 a'_0 \\ a''_1 &= a_1 + b_1 (a'_0)^p + p a'_1 b_1 + c_1 (a'_0)^{2p} + p c_1 (2 a'_1 (a'_0)^p + p (a'_1)^2) \\ b''_0 &= b_0 b'_0 \\ b''_1 &= b_0^p b'_1 + (b'_0)^p b_1 + p b_1 b'_1 + 2((b'_0 a'_0)^p c_1 + p c_1((b'_0)^p a'_1 + b'_1 (b'_0)^p + p b'_1 a'_1)) \\ c''_0 &= 0 \\ c''_1 &= b_0^p c'_1 + p b_1 c'_1 + (b'_0)^{2p} c_1 + p c_1 (2 (b'_0)^p b'_1 + p (b'_1)^2) \end{aligned}$$

If we assume that $a_0, a_1, b_0, b_1, c_1, a'_0, a'_1, b'_0, b'_1, c'_1$ are in \mathbb{F}_p these simplify to

$$\begin{aligned} a''_0 &= a_0 + b_0 a'_0 \\ a''_1 &= a_1 + b_1 a'_0 c_1 (a'_0)^2 \\ b''_0 &= b_0 b'_0 \\ b''_1 &= b_0 b'_1 + b'_0 b_1 + 2(b'_0 a'_0 c_1 + b'_1 b'_0) \\ c''_0 &= 0 \\ c''_1 &= b_0 c'_1 + (b'_0)^2 c_1. \end{aligned}$$

These relations define an algebraic group G/\mathbb{F}_p where $G \cong \mathbb{A}_{\mathbb{F}_p}^5$ as varieties and the group multiplication is given by

$$\mu((a_0, a_1, b_0, b_1, c_1), (a'_0, a'_1, b'_0, b'_1, c'_1)) = (a''_0, a''_1, b''_0, b''_1, c''_1).$$

From Theorem 13 it is clear that $G(\mathbb{F}_p) \cong A_2(\mathbb{Z})$. \square

The following theorem will define what we mean by ‘‘The Greenberg Transform’’ of the group $A_d(\mathbb{Z}, p)$.

Theorem 16. *There exists an algebraic group G/\mathbb{F}_p which is a subscheme of \mathbb{A}^N where $N = n + \frac{n(n+1)}{2}$ with the the property that*

$$G(\mathbb{F}_p) \cong A_d(\mathbb{Z}, p).$$

Proof. We proceed as in example 15 and identify

$$a_0 + a_1 T + p a_2 T^2 + p^2 a_3 T^3 + \cdots + p^{n-1} a_n T^n \pmod{p^n}$$

with

$$[a_{0,0}, \dots, a_{0,n-1}] + [a_{1,0}, \dots, a_{1,n-1}]T + [0, a_{2,1}, \dots, a_{2,n-1}]T^2 + \cdots + [0, 0, \dots, 0, a_{n,n-1}]T^n.$$

We then multiply out two such vectors using Witt addition and Witt multiplication to get some algebraic relations.

The formula for N comes from the adding $n + n + (n-1) + \cdots + 1$. \square

Remark 17. ‘‘The Greenberg Transform’’ of $A_d(\mathbb{Z}, p)$ isn’t a genuine Greenberg Transform since $A_d(\mathbb{Z}, p)$ isn’t a scheme.

The proof for the full group of automorphisms is quite similar.

Theorem 18. *The group $\text{Aut}(\mathbb{A}_{\mathbb{Z}/p^{n+1}}^1)$ is isomorphic to the \mathbb{F}_p points of an algebraic variety.*

Proof. The proof is similar. We replace coordinate a_0, a_1, \dots appearing in polynomials

$$a_0 + a_1 T + a_2 T^2 + \cdots + a_d T^d \pmod{p^{n+1}} \in \text{Aut}(\mathbb{A}_{\mathbb{Z}/p^{n+1}}^1)$$

with Witt coordinates $[a_{i0}, \dots, a_{in}]$ where $0 \leq i$ and impose the additions and multiplications as usual. The only difference from the proof for $A_n(\mathbb{Z}, n, p)$ is that that we have an infinite number of indeterminates and that we need to stipulate that the polynomial are affine linear modulo p . To do this we adjoin an extra symbol y and the additional equation $a_{10}y - 1$ as usual in algebraic geometry. \square

4. SOLVABILITY

4.1. Solvable Groups. Recall that a group G is **solvable** if and only if it admits a composition series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_l = G$ such that for every $0 \leq i \leq l$ the factor groups $G_{i+1}/G_i =: A_{i+1}$ are abelian.

We construct the class of groups which are **built by abelians** inductively: A group G is built by abelians if one of the follow holds

base case: G is abelian

- inductive step:**
- G is an extension of a group built by abelians by an abelian group.
 - G is an extension of an abelian group by a group built by abelians.

Lemma 19. *A group G is solvable if and only if it is built by abelians.*

Proof. Suppose that G is solvable and let $l(G)$ denote the length of the minimal composition series. We will show that it is built by abelians by induction on the length $l(G)$ of a minimal composition series for G .

Suppose that $l(G) = 1$. Then $G_1 = G_1/G_0 = A_1$ and G_1 is abelian.

Now suppose the proposition is true for $l(G) = n - 1$. Given a composition series for $G = G_n$ we have a composition series for G_{n-1} which shows that G_{n-1} is solvable. By inductive hypothesis G_{n-1} is built by abelians. The exact sequence

$$1 \rightarrow G_{n-1} \rightarrow G_n \rightarrow A_n \rightarrow 1,$$

which shows that G_n is built by abelians. In particular every solvable group is built from extending abelian groups by solvable groups.

We will now prove the converse. Let G be a group built by abelians. Let $c(G)$ be the minimal number of admissible extensions required to build G . Our proof will be by induction on $c(G)$. If $c(G) = 1$ then G is solvable since every abelian group is solvable.

Suppose $c(G) = n$ and that $G = G_n$ is an extension of an abelian group A_n by a group built by abelians G_{n-1} . This means we have an exact sequence

$$1 \rightarrow G_{n-1} \rightarrow G_n \rightarrow A_n \rightarrow 1$$

which implies that G_n is solvable since G_{n-1} is by inductive hypothesis.

Suppose now that G_n is an extension of G_{n-1} by an abelian group A :

$$1 \rightarrow A \rightarrow G_n \rightarrow G_{n-1}.$$

Let $p_n : G_n \rightarrow G_{n-1}$ with $\ker(p_n) = A$ as above. Since G_{n-1} is built from abelians it is solvable by inductive hypothesis. In particular there exists a sequence of groups subgroups

$$1 = \Gamma_0 < \dots < \Gamma_m = G_{n-1}$$

such that $\Gamma_j/\Gamma_{j-1} = B_j$ where A is abelian. Define $G'_j = \pi^{-1}(\Gamma_j)$. We have $G'_{j-1} \triangleleft G'_j$ and $A \subset G'_j$. We also have $G'_j/A \cong \Gamma_j$ so $G'_j/G'_{j-1} \cong (G'_j/A)/(G'_{j-1}/A) \cong \Gamma_j/\Gamma_{j-1} = B_j$ which is abelian. In addition $G'_0 = A$ so we have constructed a composition series for G_n and hence G_n is abelian. □

This lemma just says that solvable groups are built from abelian groups.

4.2. Abelian Normal Subgroups. The following lemma will allow us to build the groups $A_d(R, q)$ out of abelian ones.

Lemma 20. *Let R be a q -torsion free ring with $qR \in \text{Spec}(R)$. The kernel the natural map $\pi_d : A_d(R, q) \rightarrow A_{d-1}(R, q)$ is isomorphic to R_0^{d+1} .*

Proof. The group $N_d(R, q) := \ker(\pi_d : A_d(R, q) \rightarrow A_{d-1}(R, q))$ consists of elements of the form

$$(4.1) \quad \psi(T) = q^{d-1}a_0 + (1 + a_1q^{d-1})T + q^{d-1}a_2T^2 + \cdots + q^{d-1}a_dT^d \pmod{q^d}$$

whose reduction mod q^{d-1} is the identity. This is clearly a normal subgroup. We now show that it is closed under composition; every ψ can be written as $\psi(T) = T + q^{d-1}\psi'(T)$. Suppose that $\varphi(T) = T + q^{d-1}\varphi'(T)$ then we have

$$\varphi(\psi(T)) = \psi(T) + q^{d-1}\varphi'(T + q^{d-1}\psi'(T)) = \psi(T) + \varphi'(T) = T + \psi'(T) + \varphi'(T) \pmod{q^d}.$$

From this expression it is now clear that $N_d(R, q) \cong R_0^d$ where the isomorphism is given by

$$q^{d-1}a_0 + (1 + a_1q^{d-1})T + q^{d-1}a_2T^2 + \cdots + q^{d-1}a_dT^d \pmod{q^d} \mapsto (a_0, a_1, a_2, \dots, a_d).$$

□

Theorem 21. *If R is a q -torsion free ring with $qR \in \text{Spec}(R)$ then the groups $A_d(R, q)$ are solvable.*

Proof. The proof is by induction on d . For $d = 1$, $A_1(R, q) \cong R_0 \otimes R_0^\times$ which is clearly solvable. We will now assume the proposition is true for $d - 1$ and prove it for d . For each d we have the exact sequence

$$N_d(R, q) \rightarrow A_d(R, q) \rightarrow A_{d-1}(R, q) \rightarrow 1$$

Which shows that $A_d(R, q)$ is an extension of $A_{d-1}(R, q)$ by the abelian group $N_d(R, q) \cong R_0^{d+1}$ and since $A_{d-1}(R, q)$ is solvable by hypothesis we are done. □

We spend the rest of this section generalizing the above results for the groups $\text{Aut}(\mathbb{A}_{R/q^n}^1)$ and $\text{Aut}(\mathbb{A}_{\mathbb{Z}/m}^1)$. We will show that both of these groups are solvable.

We start with the following simple lemma:

Lemma 22. *Let R be a q -torsion free ring and suppose $f_1, f_2 \in \text{Aut}(\mathbb{A}_{R/q^n}^1)$. If $r + s \geq n$ and*

$$\begin{aligned} f_1(T) &= T \pmod{q^r} \\ f_2(T) &= T \pmod{q^s} \end{aligned}$$

then

$$(4.2) \quad f_1 \circ f_2 = f_2 \circ f_1 \pmod{q^n}$$

Proof. Write $f_1(T) = T + q^r g_1(T)$ and $f_2(T) = T + q^s g_2(T)$ then we have

$$\begin{aligned} f_1(f_2(T)) &= f_2(T) + q^r g_1(f_2(T)) \\ &= T + q^s g_2(T) + q^r g_1(T + q^s g_2(T)) \\ &= T + q^s g_2(T) + q^r g_1(T) \pmod{q^n} \end{aligned}$$

where the last line follows from the fact that if $a \equiv b \pmod{q^r}$ then $aq^s \equiv bq^s \pmod{q^n}$. □

Corollary 23. *Let R be a q -torsion free ring. Let*

$$\begin{aligned} N_{n,r}(R, q) &:= \{f \in A_n(R) : f(T) \equiv T \pmod{q^r}\} \\ &= \ker(\pi_{n,r} : A_n \rightarrow A_r), \\ K_{n,r}(R, q) &:= \{f \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1) : f(T) \equiv T \pmod{q^r}\} \\ &= \ker(\pi_{n,r} : \text{Aut}(\mathbb{A}_{R_{n-1}}^1) \rightarrow \text{Aut}(\mathbb{A}_{R_{r-1}}^1)), \end{aligned}$$

If $r > n/2$ then both of these groups are abelian.

Proof. This follows from Lemma 22 since every pair of polynomial in $K_{n,r}(R, q)$ for $r > n/2$ commutes. Since $N_{n,r}(R, q) \subset K_{n,r}(R, q)$ we are done. \square

The arguement in the proof of Lemma 22 can actually be used to prove something slightly more general.

Lemma 24. *Let R be a ring and $I, J \triangleleft R$ with $I^2 = 0$. The group*

$$\ker(\text{Aut}(\mathbb{A}_R^1) \rightarrow \text{Aut}(\mathbb{A}_{R/I}^1))$$

is abelian.

This means for $R = \mathbb{Z}/m$ where $m = p_1^{n_1} \cdots p_s^{n_s}$ and $I = (m') \subset \mathbb{Z}/m$ where $m' = p_1^{r_1} \cdots p_s^{r_s}$ and $r_i > n_i/2$ for $i = 1, \dots, s$ we can apply our technique of solvability. We summarize our discussion in the following theorem.

Theorem 25. *The following groups are solvable*

- $\text{Aut}(\mathbb{A}_{R/q^n}^1)$, where R q -torsion free
- $\text{Aut}(\mathbb{A}_{\mathbb{Z}/m}^1)$

Proof. These groups are built by Abelian. \square

Remark 26. The author recognizes that he could have simply proved that $\text{Aut}(\mathbb{A}_{R/q^n}^1)$ was solvable first and then used the fact that $A_d(R, q)$ was a solvable group to prove solvability here but decided to present it this way as this was the way he proved it first.

5. ADJOINT REPRESENTATIONS

5.1. An Algorithm for Computing Inverses. Let R be a q -torsion free ring where $q \in R$ and $qR \in \text{Spec}(R)$. We now move to the question of computing inverses in the group $\text{Aut}(\mathbb{A}_{R/q^n}^1)$ efficiently. Note that if $\psi(T) = T + q^r f(T) \in K_{n,r}(R, q)$ then its inverse is easily computable since the group is abelian and isomorphic $R/q^{n-r}[T]$.

Also note that if $\psi(T) = a_0 + a_1 T \in A_1(R, 1) = \text{Aut}(\mathbb{A}_{R/q}^1)$ its inverse is also easily computable. Suppose that $\psi(T) \in \text{Aut}(\mathbb{A}_{R/q^n}^1)$ and let $\phi(T)$ be a lift of the inverse of $\pi_{n,r}(\psi)$ where $r > d/2$. Then $\psi \circ \phi \in K_{n,r}(R, q)$ and its inverse is readily computable. This gives a recursive algorithm for computing inverses.

This gives us the following recursive algorithm for computing inverses

Algorithm 27. *For $\psi \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ we can compute ψ^{-1} using*

$$(5.1) \quad \psi^{-1}(T) = \begin{cases} T - q^r f(T) & , \psi \in K_{n,r}(R, q), r > n/2 \\ -a_1^{-1} a_0 + a_1^{-1} T, & \psi \in A_1(R, q), \\ \text{lift}(\pi_{n, \lceil n/2 \rceil}(\psi)^{-1}) \circ (\psi \circ \text{lift}(\pi_{n, \lceil n/2 \rceil}(\psi)^{-1}))^{-1}, & \psi \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1) \setminus N_{n, \lceil n/2 \rceil} \text{ and } d \neq 1 \end{cases}$$

Where $\text{lift} : \text{Aut}(\mathbb{A}_{R_{n-1}}^1) \rightarrow \text{Aut}(\mathbb{A}_{R_{d-1}}^1)$ is just a map of sets such that $\pi_{n,r} \circ \text{lift} = \text{id}$.

5.2. The Adjoint Representation. Let R be a q -torsion free ring with $qR \in \text{Spec}(R)$. In the previous section we defined the groups

$$K_{n,r}(R, q) := \ker(\pi_{n,r} : \text{Aut}(\mathbb{A}_{R_{n-1}}^1) \rightarrow \text{Aut}(\mathbb{A}_{R_{r-1}}^1))$$

which had the property that they were Abelian when $r > n/2$. There are the elements of $\text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ which can be thought of as q -adically close to the identity and hence should be viewed as the ‘‘Lie algebra’’ of $\text{Aut}(\mathbb{A}_{R_{n-1}}^1)$. It is natural then to ask if $K_{n,r}(R, q)$ is an R -module and if there exists an R -linear adjoint action. The answer to both these questions is yes which we will now show.

In what follows we define the **Adjoint Action** of $\text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ on $K_{n,r}(R, q)$ by

$$(5.2) \quad \text{Ad}_f(g) = f \circ g \circ f^{-1}$$

for $f \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ and $g \in K_{n,r}(R, q)$.

Define the R -multiplication on $K_{n,r}(R)$ by

$$(5.3) \quad c\dot{g}(T) = c \cdot (T + q^r h(T)) := T + q^r ch(T).$$

Where $g(T) = T + q^r h(T) \in N_{n,r}(R, q)$ and $c \in \mathcal{O}$. We have the following theorem

Theorem 28. *The adjoint action of $\text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ on $K_{n,r}(R)$ is R -linear for $r > n/2$. That is, for all $f \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1)$, all $g \in K_{n,r}(R)$ and all $c \in R$ we have*

$$(5.4) \quad \text{Ad}_f(c \cdot g) = c \cdot \text{Ad}_f(g).$$

Proof. Take $f \in \text{Aut}(\mathbb{A}_{R_{n-1}}^1)$ and $g \in K_{n,r}(R, q)$ and write it as $g(T) = T + q^s h(T)$.

$$\begin{aligned} f \circ (c \cdot g) \circ f^{-1}(T) &= f(T + cq^s h(T)) \circ f^{-1}(T) \\ &= f(f^{-1}(T) + cq^s h(f^{-1}(T))) \\ &= f(f^{-1}(T) + cq^s h(f^{-1}(T))) \\ &= a_0 + a_1(f^{-1}(T) + cq^s h(f^{-1}(T))) \\ &\quad + \sum_{j=1}^{n-1} q^{j-1} a_j (f^{-1}(T) + cq^s h(f^{-1}(T)))^j \\ &= a_0 + a_1(f^{-1}(T)) + \sum_{j=1}^n q^{j-1} a_j f^{-1}(T)^j + a_1 cq^s h(f^{-1}(T)) \\ &\quad + \sum_{j=1}^n q^{j-1} a_j \left[\sum_{l=1}^j \binom{j}{l} f^{-1}(T)^{j-l} (cq^s h(f^{-1}(T)))^l \right] \\ &= T + a_1 cq^s h(f^{-1}(T)) + \sum_{j=1}^n q^{j-1} a_j [j f^{-1}(T)^{j-1} (cq^s h(f^{-1}(T)))] \\ &= T + cq^s h(f^{-1}(T)) \left(a_1 + \sum_{j=1}^n q^{j-1} j a_j f^{-1}(T)^{j-1} \right) \\ &= T + cq^s h(f^{-1}(T)) f'(f^{-1}(T)) \end{aligned}$$

Where we reduced the sum in the binomial expansion using the fact that $(q^s)^l = 0 \pmod{q^n}$ for $l > 1$.

If we can show that

$$(5.5) \quad f \circ g \circ f^{-1}(T) = T + q^s h(f^{-1}(T)) f'(f^{-1}(T))$$

we are done. This is indeed the case if we take the above computation with $c = 1$. \square

This result is surprising since it says that we can study composition of polynomials over \mathbb{Z}/p^n using representations.

Corollary 29. *We have*

$$(5.6) \quad N_{2m,m}(R, q) \cong (R/q^m)^{\oplus m+1} \oplus R/q^m \oplus R/q^{m-1} \oplus \cdots R/q$$

as R -modules.

5.3. Examples of Explicit Representations. In this section we work over the “universal ring”

$$R = \mathbb{Z}[a, b, c, d, 1/b][q].$$

Example 30. The group $N_{4,2}(R, q)$ consist of a subgroup of degree four polynomials mod q^4 and we have $N_{4,2}(R, q) \cong (R/q^2)^{\oplus 4} \oplus R/q$.

The group action of $A_d(R, q)$ on the group $N_d(R, q)$ by conjugation gives a linear map

$$\text{Ad} : A_d(R, q) \rightarrow \text{GL}_{d+1}(R_0)$$

The kernel of this map contains $N_d(R, q)$ which means that $\text{Ad}_{d-1}(R, q)$ is well defined on the quotient $A_d(R, q)/N_d(R, q) \cong A_{d-1}(R, q)$.

We can compute several representations of A_n acting on certain subgroups

Example 31. The action of $a+b+qcT^2+q^2dT^3 \in A_3(R, q)$ on the normal subgroup $N_{3,1}(R, q)$ yields

$$\begin{bmatrix} b & -a & a^2/b & -a^3/b^2 \\ 0 & 1 & -2a/b & 3a^2/b^2 \\ 0 & 0 & 1/b & -3a/b^2 \\ 0 & 0 & 0 & b^{-2} \end{bmatrix}$$

Example 32. The action of $a+b+qcT^2+q^2dT^3+q^3eT^4 \in A_4(R, q)$ on the normal subgroup $N_{4,1}(R, q)$ yields

$$(5.7) \quad \begin{bmatrix} b & -a & a^2/b & -a^3/b^2 & a^4/b^3 \\ 0 & 1 & -2a/b & 3a^2/b^2 & -4a^3/b^3 \\ 0 & 0 & 1/b & -3a/b^2 & 6a^2/b^3 \\ 0 & 0 & 0 & b^{-2} & -4a/b^3 \\ 0 & 0 & 0 & 0 & b^{-3} \end{bmatrix}$$

Example 33. The following matrix describes the action of $a+b+qcT^2+q^2dT^3 \in A_4(R, q)$ on $N_{2,2}(R, q)$.

$$(5.8) \quad \begin{bmatrix} -\frac{2acq}{b} + b & \frac{a^2cq}{b^2} - a & \frac{a^2}{b} & -\frac{a^4cq}{b^4} - \frac{a^3}{b^2} & \frac{a^4q}{b^3} \\ \frac{2cq}{b} & -\frac{2acq}{b^2} + 1 & -\frac{2a}{b} & \frac{4a^3cq}{b^4} + \frac{3a^2}{b^2} & -\frac{4a^3q}{b^3} \\ 0 & \frac{cq}{b^2} & \frac{1}{b} & -\frac{6a^2cq}{b^4} - \frac{3a}{b^2} & \frac{6a^2q}{b^3} \\ 0 & 0 & 0 & \frac{4acq}{b^4} + \frac{1}{b^2} & -\frac{4aq}{b^3} \\ 0 & 0 & 0 & -\frac{cq}{b^4} & \frac{q}{b^3} \end{bmatrix}$$

REFERENCES

- [Dup12] T. Dupuy. Arithmetic deformation classes associated to curves. (*preprint*), 2012.
- [Haz09] M. Hazewinkel. Witt vectors. part 1. *Handbook of algebra*, 6:319–472, 2009.
- [Lan52] S. Lang. On quasi-algebraic closure. *Ann. of Math*, 55(2):373–390, 1952.
- [Lan54] S. Lang. Some applications of the local uniformization theorem. *American Journal of Mathematics*, pages 362–374, 1954.