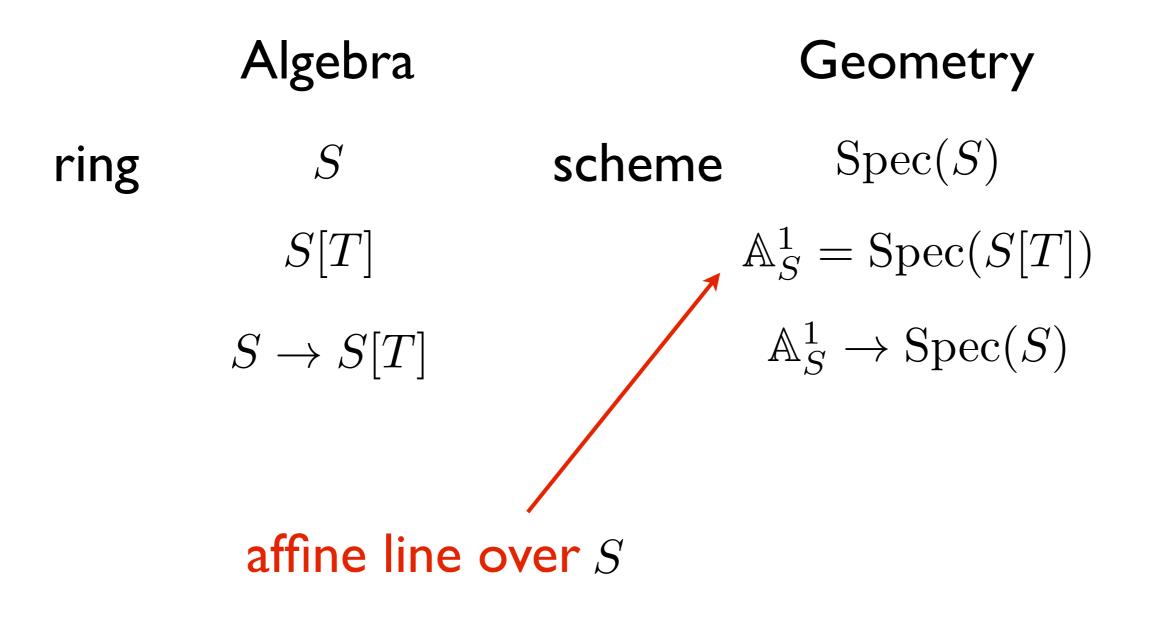
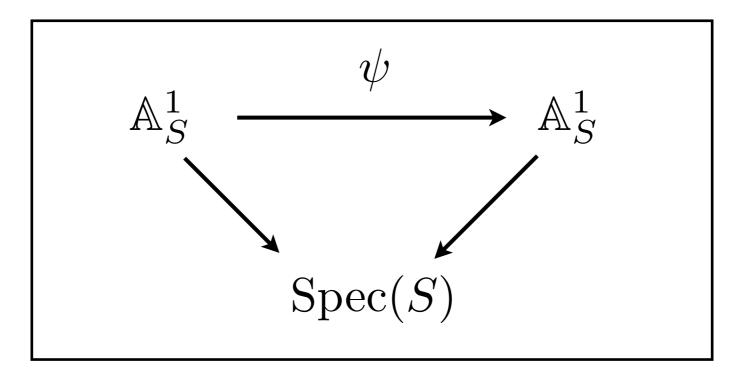
Automorphisms of the Affine Line over Nonreduced Rings

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The Affine Line



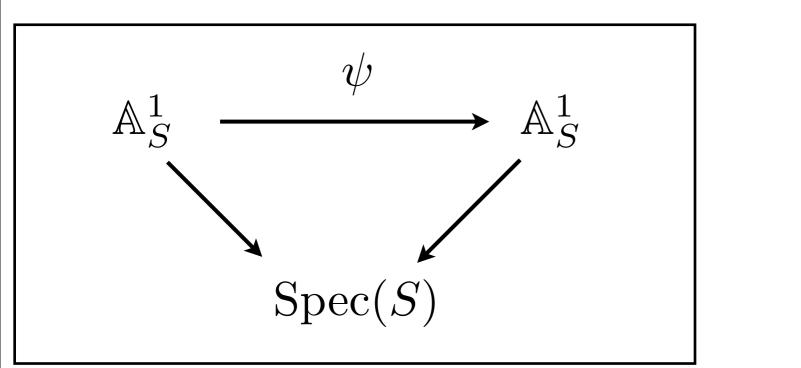
Automorphisms of Affine Line



SchemesRings $Aut(\mathbb{A}^1_S)$ $Aut_S(S[T])^{op}$

group of polynomials invertible under composition

Affine Linear Subgroup



 $AL_1(S) \subset Aut(\mathbb{A}^1_S)$ $\psi(T) = a + bT$ $a \in S$ $b \in S^{\times}$

Group Law $\psi_1(T) = a_1 + b_1 T$, $\psi_2(T) = a_2 + b_2 T$ $(\psi_1 \circ \psi_2)(T) = a_1 + b_1 a_2 + b_1 b_2 T$

Automorphisms over Domains

Theorem.

 $S \text{ a domain} \\ \Longrightarrow \operatorname{Aut}(\mathbb{A}^1_S) = \operatorname{AL}_1(S)$

proof.

$$S[T] = S[\psi(T)] \implies \deg(\psi(T)) \le 1$$

Automorphisms of the affine line over domains are really really really boring.

Non-Boring Automorphisms

 $S = \mathbb{Z}/p^2 \text{ (Ring with nilpotents!)}$ $\psi(T) = T + pT^{100} \mod p^2$ $\psi^{-1}(T) = T - pT^{100} \mod p^2$ $\psi(T) \in \operatorname{Aut}(\mathbb{A}^1_{\mathbb{Z}/p^2})$

Iterates have bounded degree: $deg(\psi^n(T)) \le 100$ Has finite order: $\psi(T)$ has order p

MAIN POINTS

- Univariate polynomials under composition have finite order (over \mathbb{Z}/p^n)
- Iterates of a univariate polynomial under composition have bounded degree.
- Univariate polynomials under composition are really algebraic groups! (over \mathbb{Z}/p^n)
- Univariate polynomials automorphism groups are solvable!

Examples

polynomial	order	coefficient ring
1+T	2	$\mathbb{Z}/2^3$
1+T	16	$\mathbb{Z}/2^4$
$T + 2^2 T^4$	4	$\mathbb{Z}/2^4$
$T + 2^{3}T^{4}$	2	$\mathbb{Z}/2^4$
$T + 2^2 T^{10} + 2^3 T^5$	8	$\mathbb{Z}/2^4$
$T^2 + 2^2 T^{10}$	8	$\mathbb{Z}/2^4$

More Examples

This is the typical case

Even More Examples

$\operatorname{Aut}(\mathbb{A}^1_{\mathbb{Z}/5^4})$

- order25 $80T^{10} + 350T^9 + 620T^8 + 300T^7 + 180T^6 + 145T^5 + 560T^4 + 525T^3 + 265T^2 + 571T + 191,$
- order125 $555T^{10} + 400T^9 + 605T^8 + 305T^7 + 435T^6 + 470T^5 + 250T^4 + 490T^3 + 515T^2 + 346T + 356,$
- order500 $230T^{10} + 405T^9 + 335T^8 + 410T^7 + 205T^6 + 325T^5 + 620T^4 + 195T^3 + 10T^2 + 62T + 160,$

order625 $370T^{10} + 70T^9 + 75T^8 + 65T^7 + 385T^6 + 450T^5 + 200T^4 + 560T^3 + 395T^2 + 606T + 487,$

order125 $390T^{15} + 330T^{14} + 300T^{13} + 290T^{12} + 220T^{11} + 230T^{10} + 580T^9 + 220T^8 + 575T^7 + 430T^6 + 600T^5 + 365T^4 + 230T^3 + 395 * T^2 + T + 285$

Our Setup

We Study: $\operatorname{Aut}(\mathbb{A}^1_S) = \operatorname{Aut}_S(S[T])^{op}$

Where: $S = R/q^n$ (non reduced!) $qR = \langle q \rangle$ prime

Rings we are thinking about:

 $R = \mathbb{Z} \qquad q = p \qquad \text{(wittfinitesimal)}$ $R = F[t] \qquad q = t \qquad \text{(infinitesimals)}$ $R = \frac{\text{coord ring of affine}}{\text{scheme}} \qquad q = p \qquad \left(\begin{array}{c} \text{corresp. to} \\ \mathcal{O}(U \times \mathbb{A}^1_{\mathbb{Z}}) \end{array} \right)$

Subgroups: Abelian ones! important feature

 $\mathfrak{g}_{r,s} \subset \operatorname{Aut}(\mathbb{A}^1_{R/q^r}), s \ge r/2 \longleftarrow q^s \cdot q^s \equiv 0 \mod q^r$

$$\begin{aligned} & \operatorname{reduction\,map}_{\mathfrak{g}_{r,s}} := \ker(\operatorname{Aut}(\mathbb{A}^1_{R/q^r}) \to \operatorname{Aut}(\mathbb{A}^1_{R/q^s})) \\ & \psi(T) = T + q^s f(T) \in \mathfrak{g}_{s,r}, \qquad f(T) \in R/q^{r-s}[T] \\ & \bigwedge_{\text{Should be viewed as q-adically close to} \\ & \text{the identity! Like a Lie algebra!} \end{aligned}$$

Group Law:

$$(T + p^{s}f(T)) \circ (T + p^{s}g(T)) = T + p^{s}(f(T) + g(T))$$

Subgroups: Bounded Degree

Defn/Proposition

$$\widetilde{A}_d(n, R, q) \subset \operatorname{Aut}(\mathbb{A}^1_{R/q^n})$$
$$\forall m \ge 2, \ \operatorname{deg}(\psi \mod q^m) \le d2^{m-2}$$

Corollaries

- I) Every iterate of $\psi \in Aut(\mathbb{A}^1_{R/q^n})$ has bounded degree.
- 2) Every $\psi \in Aut(\mathbb{A}^1_{\mathbb{Z}/p^n})$ has finite order.

Corollaries

I) Every iterate of $\psi \in Aut(\mathbb{A}^1_{R/q^n})$ has bounded degree.

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The Point:

If $\psi \in \operatorname{Aut}(\mathbb{A}^1_{R/q^n})$ and $\deg(\psi) = d$ then $\psi \in \widetilde{A}_d(n, R, q)$

Remark:

The explicit bound on the degree using this method is super shitty.

$$\begin{split} \widetilde{A}_d(R,q) &\subset \operatorname{Aut}(\mathbb{A}^1_{R/q^n}) \\ \forall m \geq 2, \ \deg(\psi \mod q^m) \leq d2^{m-2} \end{split}$$

example:
$$n = 2$$

 $d = whatever$

 $\widetilde{A}_d(2, R, q)$ polynomials mod q^2 of degree less than d

$$\psi(T) = a_0 + a_1T + qf(T)$$

$$\widetilde{\psi}(T) = \widetilde{a}_0 + \widetilde{a}_1T + q\widetilde{f}(T)$$

$$\operatorname{ord}_T(f), \operatorname{ord}_T(\tilde{f}) \ge 2$$

composing these polynomials gives

$$\psi(\widetilde{\psi}(T)) \equiv a_0 + a_1[\widetilde{a}_0 + \widetilde{a}_1T + q\widetilde{f}(T)] + qf(\widetilde{a} + \widetilde{a}T)$$
$$= a_0 + a_1\widetilde{a}_0 + (a_1\widetilde{a}_1)T$$
$$+ q(a_1\widetilde{f}(T) + f(\widetilde{a}_0 + \widetilde{a}_1T))$$

which shows the set is closed under composition.

next case:
$$\widetilde{A}_d(3, R, q)$$
 ord_T(f) ≥ 2 ord_T(g) ≥ 3
 $\psi(T) = a_0 + a_1T + qf(T) + q^2g(T) \mod q^3$
 $\deg(\psi \mod q^2) \leq d$
 $\deg(\psi \mod q^3) \leq 2d$

Want to show when we compose two of these guys we get one back. Look at:

$$deg(\psi \mod q^2) \geq deg(f \mod q),$$

$$deg(\psi \mod q^3) \geq deg(g \mod q), deg(f \mod q^2),$$

composing gives

$$\psi(\widetilde{\psi}(T)) = a_0 + a_1 \widetilde{\psi}(T) + q[f(\widetilde{a}_0 + \widetilde{a}_1 T) + qf'(\widetilde{a}_0 + \widetilde{a}_1 T) \widetilde{f}(T)] + q^2 g(\widetilde{a}_0 + \widetilde{a}_1 T)$$

$$\frac{d}{\deg(\psi \mod q^2)} \ge \underbrace{\deg(f \mod q),}_{\deg(g \mod q) \in \underline{\mathsf{g}}(f \mod q),}_{\operatorname{deg}(\psi \mod q^3)} \ge \underbrace{\deg(g \mod q),}_{\operatorname{deg}(g \mod q),}_{\operatorname{deg}(f \mod q),}_{\psi(\widetilde{\psi}(T))} = \underbrace{a_0 + a_1\widetilde{\psi}(T)}_{\operatorname{deg}(\widetilde{a}_0 + \widetilde{a}_1T) \oplus f'(\widetilde{a}_0 + \widetilde{a}_1T)\widetilde{f}(T)}_{\operatorname{deg}(\widetilde{a}_0 + \widetilde{a}_1T)}_{\operatorname{deg}(\widetilde{a}_0 + \widetilde{a}_1T) \mod q)} \le 2d,$$

$$\frac{d}{deg(\psi(\widetilde{\psi}(T)))} \le 2d$$

Algebraic Groups

- Algebraic varieties where group laws are given by polynomial expressions.
- Example: matrix groups like the general linear group

As Algebraic Groups!

Theorem.

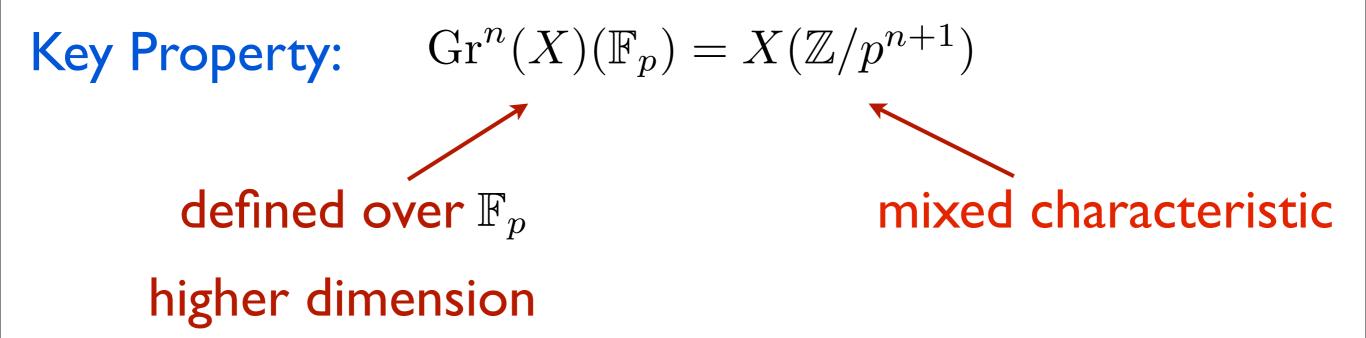
There exist G/\mathbb{F}_p finite dimensional such that $G(\mathbb{F}_p) \cong A_n(\mathbb{Z}, p).$

There exist G/\mathbb{F}_p infinite dimensional such that $G(\mathbb{F}_p) \cong \operatorname{Aut}(\mathbb{A}^1_{\mathbb{Z}/p^n}).$

Algebraicity Idea:

Apply (Greenberg Transform = p-Jet Functors)!

 $\operatorname{Gr}^n(X) = J_p^n(X) \mod p$



Solvability

Theorem. The groups $Aut(\mathbb{A}^1_{R_n})$ and $A_n(R,q)$ are solvable.

Solvable Groups

The collection of solvable groups is built inductively:

base defn: If a group is abelian then it is solvable.

inductive part: a group is solvable when one of the following is true

I) It is the the extension of an abelian group by a solvable group.

2) It is the extension of a solvable group by an abelian group.

extension of H by V $1 \rightarrow V \rightarrow E \rightarrow H \rightarrow 1$

Example

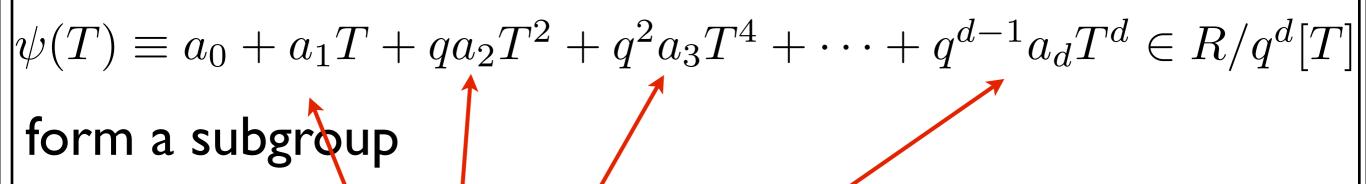
Claim: $AL_1(S)$ is solvable.

proof.

$$\begin{split} \mathrm{AL}_1(S) &\cong S \rtimes S^\times \\ & 1 \to S \to S \rtimes S^\times \to S^\times \to 1 \end{split}$$

Subgroups: Newton Polygonish condition

The collection of invertible polynomials of the form



 $\mathbf{A}_{d}(R,q) \subset \operatorname{Aut}(\mathbb{A}^{1}_{R/q^{d}})$

Coefficients are increasingly divisible by q

proof is similar to the computations we did before.

Idea Behind Solvability

Use the groups

$$g_{r,s} = \ker(\operatorname{Aut}(\mathbb{A}^1_{R/q^r}) \to \operatorname{Aut}(\mathbb{A}^1_{R/q^s}))$$

 $s \ge r/2$

or their variants to build up the groups we want.

Example. $A_2(R,q)$ is solvable. proof is by induction.

Expected Degree Bounds

The Theorem we stated earlier is far from optimal for automorphisms of quotients of the integers.

Here is the expected better bound (for a typical element):

$$\psi \in \operatorname{Aut}(\mathbb{A}^1_{\mathbb{Z}/p^r})$$

 $\operatorname{deg}(\psi^n \mod p^r) \leq (\operatorname{deg}(\psi) - 1)(r - 1) + 1$