Dimensional Analysis

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Abstract

Hannes Schenck explained this to me.

1 Units and Buckingham's Theorem

Given a physical system there are basic units such as mass M, time T and length L and then there are "derived units" like speed $S = LT^{-1}$, force $F = MLT^{-2}$ and energy $E = FL$. Really a unit is derived from a collection of others if there is an algebraic dependence relation. For a given physical system (an experiment or thought experiment) one can take all the units you can measure (or more that will vary while you are considering them) and form an abelian group. For example with the units above if we chose the bijection $M = (1, 0, 0)$, $T = (0, 1, 0)$ and $L = (0, 0, 1)$ then speed is just $LT^{-1} = (1, -1, 0)$, force is $(1, -2, 1)$ and energy is $(1, 2, -1)$. Really the choice of which units are "derived" is quite arbitrary. All we really know is that we have a collection of units (for example M, T, L, S, F, E) and that they satisfy certain relations. There is a physical map from these units to the physical world for which one has prescribed dimensions. A dimensionless unit is one which is in the kernel of this map. In our example, $\pi_1 = SL^{-1}T$ is such an element (there are acually two more $\pi_2 = F^{-1}MLT^{-2}$ and $\pi_3 = E^{-1}ML^2T^{-2}$). Note that the free abelian group on our parameters is generated by the fundamental units together with the dimensionless ones.

Remark 1. The following can be found in the award winning paper [Whi68] by Whitney (yes, the $2n + 1$ guy).

Ok, I can now state Buckingham's "Theorem": It says roughly that any physical system can be described strictly in terms of dimensionless parameters. More precisely it says that the system can be described by a simple relation $\Phi(\pi_1, \pi_2, \ldots, \pi_n) = 0$ where Φ is some real valued continuous function. I appears that this function is algebraic.

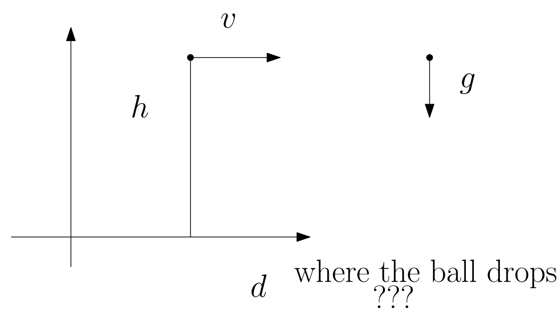
So, one can always choose dimensionless units for say your differential equation. Also, this allows one to systematically reduce the dimensions of your physical systems (say in a large metabolic system). This theorem is extremely useful for determining the form of simple relationships in physics after using some good (and simple) physical assumptions on your system.

2 Examples

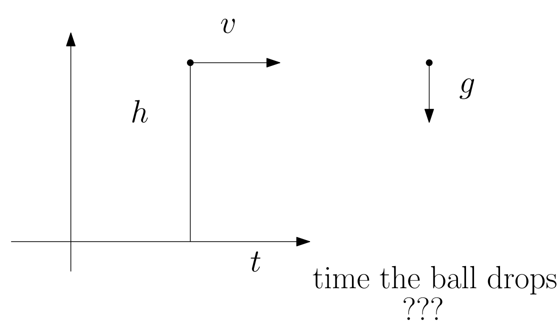
Example 1 (Free Falling Body)**.** Consider a body falling at from height h with sideways velocity v. There is the force of gravity acting on this body to consider.

Simple Falling Body

(A)



(B)



How far will the ball fall?

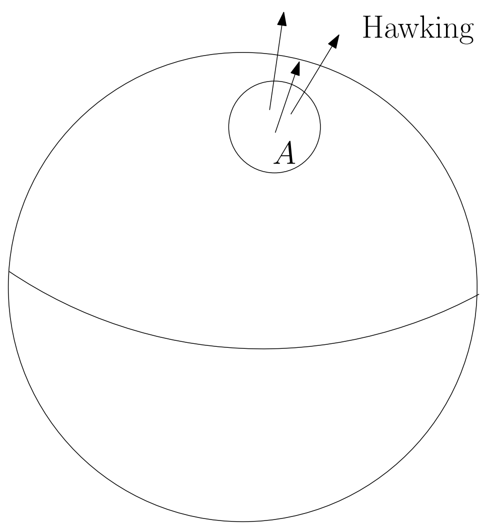
$[v] = L_y T^{-1}$, $[h] = L_x$, $[d] = [L]$, $[g] = L_x T^{-1}$. The gravitational constant is not so hard because $G = gm_1m_2/r$ is the gravitational potential (which has units of energy). One sees that $\pi_1 = (v/d)^2(h/g)$. Since Φ is now a one dimensional real relation we much have that π_1 is constant. This means that d is proportional to we have $d \propto \sqrt{g/h}v$ or that $d = C \cdot \sqrt{g/h}v$ for some scalar c.

Example 2 (Hawking Radiation)**.** Hawking Radiation is what makes a Black Hole decay. At the event horizon of a Schwarzschild blackhole (a blackhole that is assumed to be Non-rotating for simplicity) particles and antiparticles can be simultaneously created from the vaccum (for example a photon and a electron anti-election can be created). The idea is then that some of the particles fall into the black hole and some of them escape thus making the Blackhole emit energy (mass is really energy afterall).

The rate at which a blackhole emits energy $P = E/s$ is called the hawking radiation. There are a number of units involved are $[A] = L^2, [G] = EL/M^2$, $[M] = M$, $[P] = E/TL^2$, $[c] = LT^{-1}$, $[\hbar] = ET$, $[E] = ML^2T^{-2}$. We guess that

$$P \cdot A \propto \frac{\hbar c^6}{(GM)^2}.$$

Example 3 (Casimir Effect)**.** The Casimir effect is a Force felt between two metalic plates which are really close to each other. Physicists believe that the



"non-zero ground energy of the vacuum" is responsible for this effect. Here is their explanation: The second quantization of Maxwell's equations is just a bunch of quantum harmonic oscillators at every points. Recall that the energies for the quantum harmonic oscillator are $E_n = \hbar\omega(n + 1/2)$ so that the ground-state $E_0 = \hbar\omega/2$ which is non-zero. The total vacuum energy is just $E = \sum_k \hbar\omega_k$ where the ω_k are the frequencies associated to the allowed wave numbers. When we introduce two metal plates certain ω_k's are not allowed; since the plates are conducting, the electric $\vec{E}$ must satisfy $\vec{E} \cdot v = 0$ for any tangent vector in that plane. Electons moving in the plate will create only vector fields which are orthogonal to the plate. Since there are fewer frequencies in between the planes the ground state energy of the quantized electic field in between that plates $E' = \sum_{k'} \hbar\omega_{k'}$ smaller. (Note that both of these energies are mathematically crazy since we are summing over energies in an uncountable number of places.) Since change in energy is Force (or equivalently, force applied across a distance in energy) there should be a force accounting for this change in the energy. As crazy as physicists sound, this actually happens.

What is involved? $[\hbar] = ET$, $[c] = L/T$, $[F] = E/L_x$, $[A] = L^2$ the area of the plates and $[d] = L$ the distance between the plates. It seems reasonable to assume that the force is distributed evenly about the plates. We have $[\hbar c] = EL$, $[F] = E/L$ so $[F/A] = [\hbar c]/[d^4]$ and we guess that

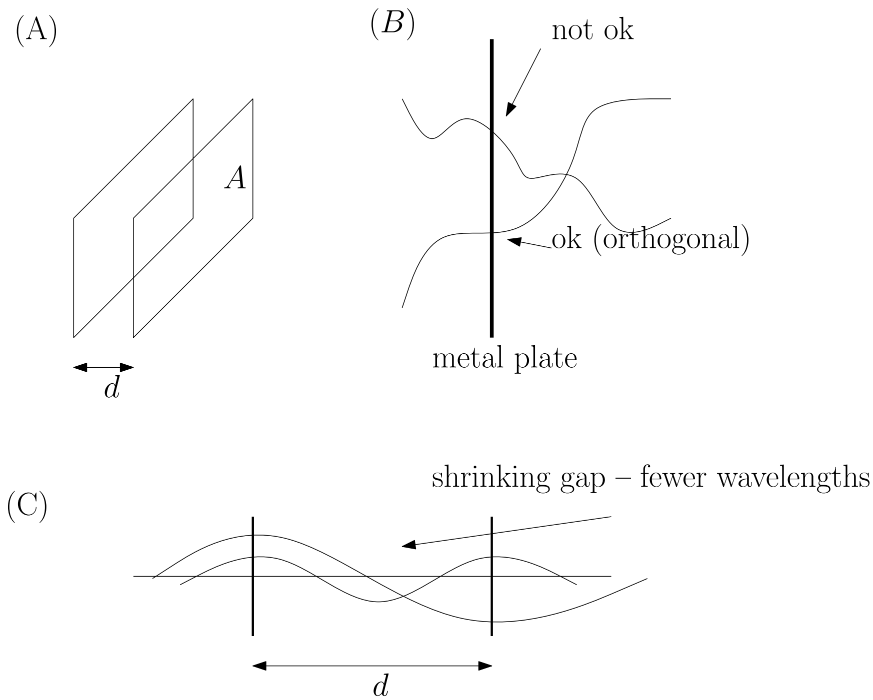
$$F/A \propto \frac{\hbar c}{d^4}.$$

In actuality on finds that $F/A = -\frac{\hbar c\pi^2}{240d^4}$.

References

[Whi68] Hassler Whitney. The mathematics of physical quantities: Part i: Mathematical models for measurement. *The American Mathematical Monthly*, 75(2):115–138, 1968.

Casimir Effect



—Similarly if one attempts to solve for time t rather than distance one gets $\pi = hg/t^2$ which is again constant. This shows that $t = \sqrt{h/g}$.

If one solves for both at the same time one will find that $\Phi(\pi_1, \pi_2) = 0$. Often one can guess the shape of these curves by looking at just a couple of data points. –Input actual value for Casimit Forms