Arithmetic Deformation Classes Associated to Curves

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Čech Cohomology

X scheme

G sheaf of groups

$\mathcal{U} = \{U_i\}_{i=1}^n$ open cover

$$\check{H}^1(X, G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)$$

Cocycles: (g_{ij})

$$\check{Z}^1(\mathcal{U}, G) \subset \prod_{i,j} G(U_{ij})$$

$$g_{ij} g_{jk} g_{ki} = 1$$

$$g_{ij}^{-1} = g_{ji}$$

Cohomology: $(g_{ij}) \sim (g'_{ij})$

$$\Longleftrightarrow \exists\ (h_i) \in \prod_i G(U_i)$$

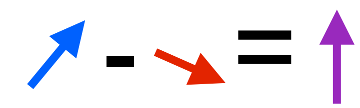
$$h_i g_{ij} h_j^{-1} = g'_{ij}$$

$$\check{H}^1(\mathcal{U}, G) = \check{Z}^1(\mathcal{U}, G)/\sim$$

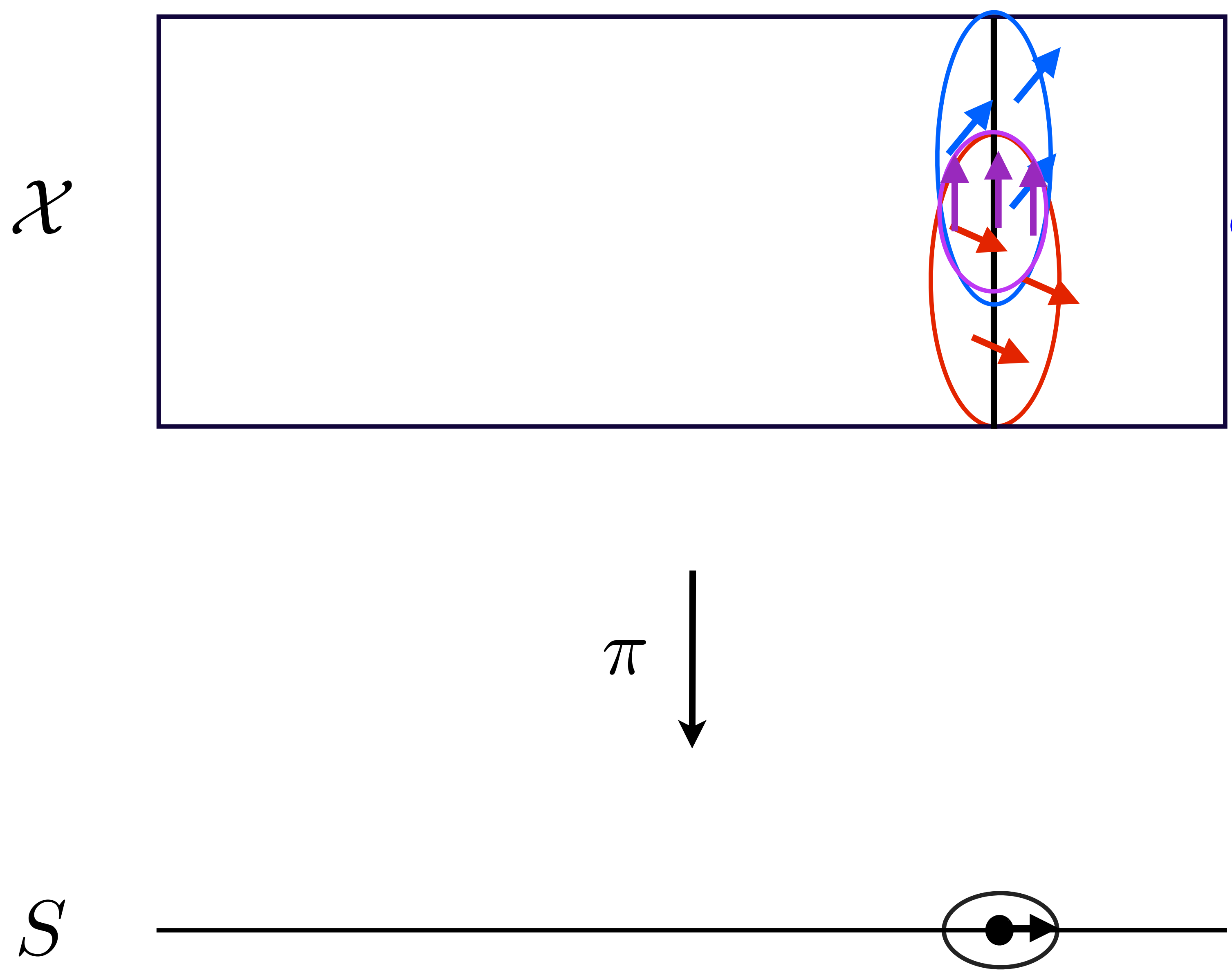
$$\mathrm{KS} : T_P S \to H^1(\mathcal{X}_P, T_{\mathcal{X}_P})$$

$$\delta_i \in T_{\mathcal{X}}(U_i)$$
$$\delta_j \in T_{\mathcal{X}}(U_j)$$

$$X_P = \pi^{-1}(P)$$



$$\delta_i - \delta_j = \delta_{ij} \in T_{\mathcal{X}_P}(U_{ij})$$



$$\delta \in T_{S,P}(U)$$

S = Moduli Space
Well-Defined
LES SES

Kodaira-Spencer Map

$$\mathrm{KS} : \{ \text{ derivations on } K \ \} \to H^1(X, T_X)$$

$\delta : K \to K$ $\quad$ $K =$ field with a derivation

Cover

$$X = \bigcup_i U_i$$

Get Local Lifts

$$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \qquad \delta_i|_K = \delta$$

Take Differences

$$\delta \mapsto [\delta_i - \delta_j] \in H^1(X, T_{X/K})$$

Deligne-Illusie Map

$\mathrm{DI}_0 : \{\ p\text{-derivations on } R\ \} \to H^1(X_0, F^*T$

$\delta : R \to R$

Cover

$X = \bigcup_i U_i$

Get Local Lifts

$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i)$

$\delta_i|_{F}$

Take Differences

$\delta \mapsto [\delta_i - \delta_j] \in H^1(X_0, F^*T_{X_0})$
$\mod p$

need to explain this doodad

Does this relate to Serre-Tate for Ab?
What is history for E?
Is this f_1?

not surj (missing zero)

Higher Order classes

NOTATION!
F, delta, X_0

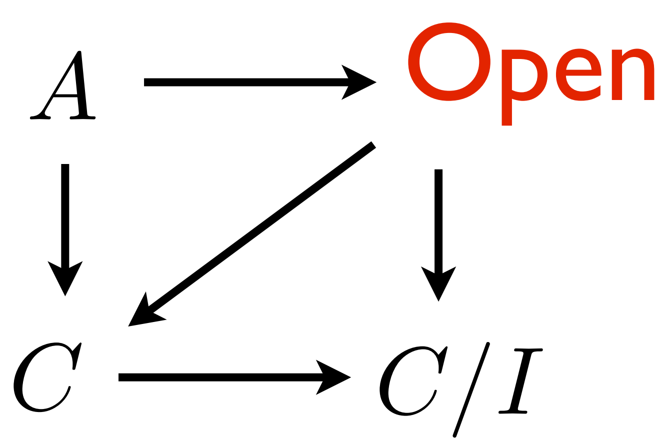
RETAIN INFORMATION MOD P^2 not MOD P^3

Is it injective?
What does it do?
What doesn't it do?
Is it surjective?
How doesn't it do this?

Get Local Lifts

Infinitesimal Lifting Property

$I^2 = 0$



Is this smoothness?

Is this Hensel?

Geometric Setting

$$C = D_1(B) := B[\varepsilon]/\langle \varepsilon^2 \rangle$$

$$I = \langle \varepsilon \rangle$$

Arithmetic Setting

$C =$

$I =$

How can I think about this?

The Frobenius Tangent Sheaf

$$D \in F^* T_{X_0}$$

Derivations of the Frobenius

$$\begin{aligned} D(a+b) &= D(a) + D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

Derivations of the Frobenius

$$\begin{aligned} D(a+b) &= D(a) + D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

$$\delta_1, \delta_2 : A \to B \qquad B \in \mathsf{CRing}_A$$

$$D(a) := \delta_1(a) - \delta_2(a) \quad \text{mod p}$$

$$\mathrm{CRAP} = \frac{a^p + b^p - (a+b)^p}{p}$$

Additivity:

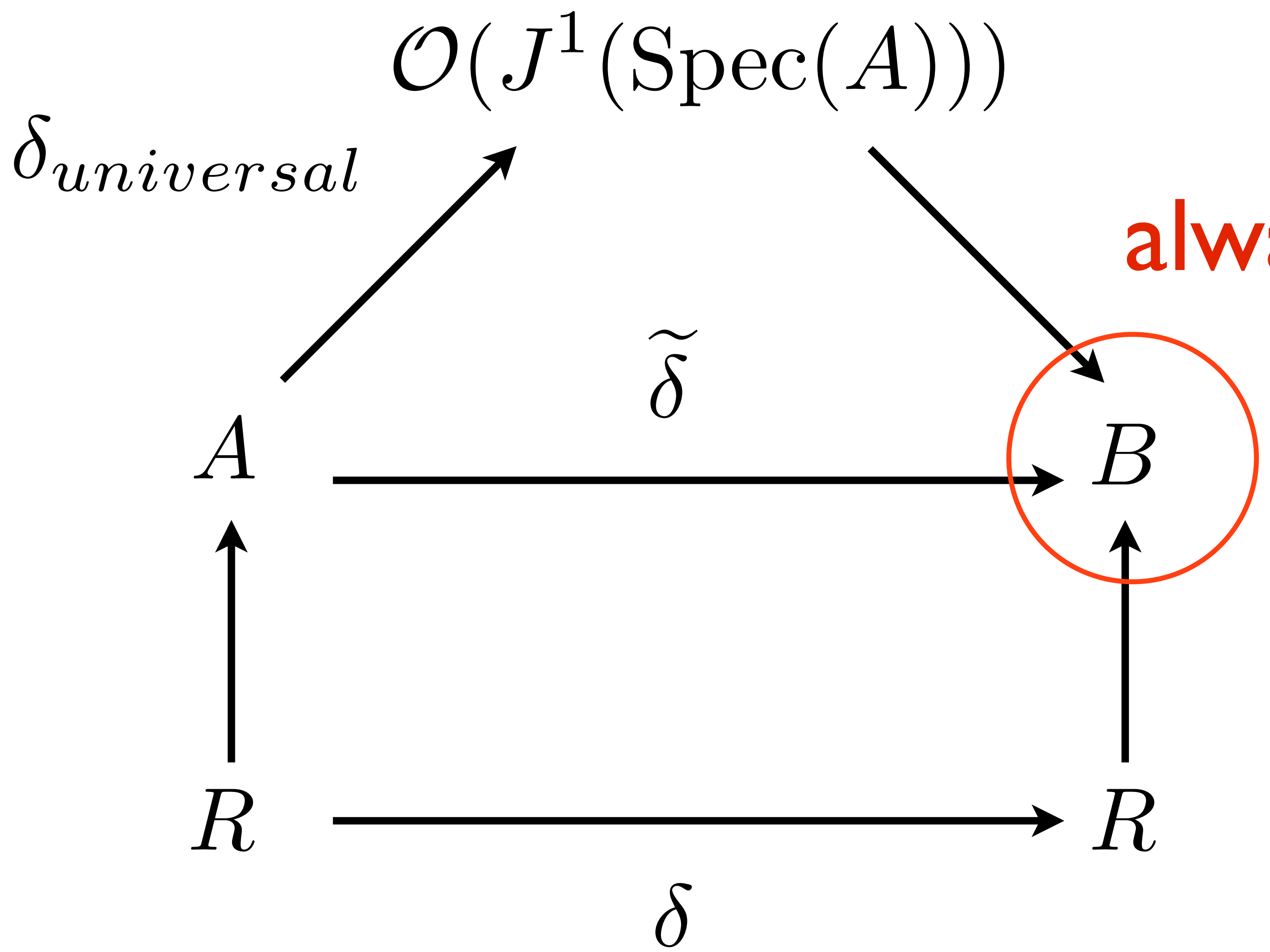
$$\begin{aligned} D(a+b) &= \delta_1(a+b) - \delta_2(a+b) \\ &= \delta_1(a) + \delta_1(b) + \mathrm{CRAP} - (\delta_2(a) + \delta_2(b) + \mathrm{CRAP}) \\ &= D(a) + D(b) \end{aligned}$$

Product Rule:

$$\delta_i(ab) = \delta_i(a)b^p + a^p\delta_i(b) + \cancel{p\delta_i(a)\delta_i(b)}$$

Arithmetic Jet Spaces

$R = \widehat{\mathbb{Z}}_p^{ur}$



always an A-algebra

Example $\mathbb{A}^1_R = \mathrm{Spec}\ R[x]$ $\quad {}^1_R \hat{\times} \widehat{\mathbb{A}}^1_R$

$\mathcal{O}(J^1(\mathbb{A}^1_R)) = R[x][\dot{x}]\widehat{\ } = R[x]\{\dot{x}\}$ Rest r Series

EXAMPLE USED IN THE FUTURE!!!

FREE OBJECT THROUGH WHICH EXTRA FACTORS

Geometric Descent

X/K smooth projective

$K = \overline{K} \qquad \mathrm{char}(K) = 0$

$\delta : K \to K$

$$K^\delta = \{r \in K : \delta(r) = 0\}$$

What is X/K?

Theorem

T.F.A.E.

1. $\mathrm{KS}(\delta) = 0$
2. $J^1(X) \cong TX$ as schemes over X
3. $\exists X'/K^\delta$ such that $X' \otimes_{K^\delta} K \cong X$

Descent to the constants

Arithmetic Descent

$X/\widehat{\mathbb{Z}}_p^{ur}$ smooth

$\delta : \widehat{\mathbb{Z}}_p^{ur} \to \widehat{\mathbb{Z}}_p^{ur}$

How can we get an equation that describes when we have a lift?

Theorem

T.F.A.E.

1. $\mathrm{DI}_0(\delta) = 0$
2. $J^1(X)_0 \cong F^* T_{X_0}$ as schemes over X_0
3. X_1 admits a lift of the p-Frobenius

Descent to the field with one element

X descends to Borger-Buium $\mathbb{F}_1 \implies \mathrm{DI}_0(\delta_p) = 0$

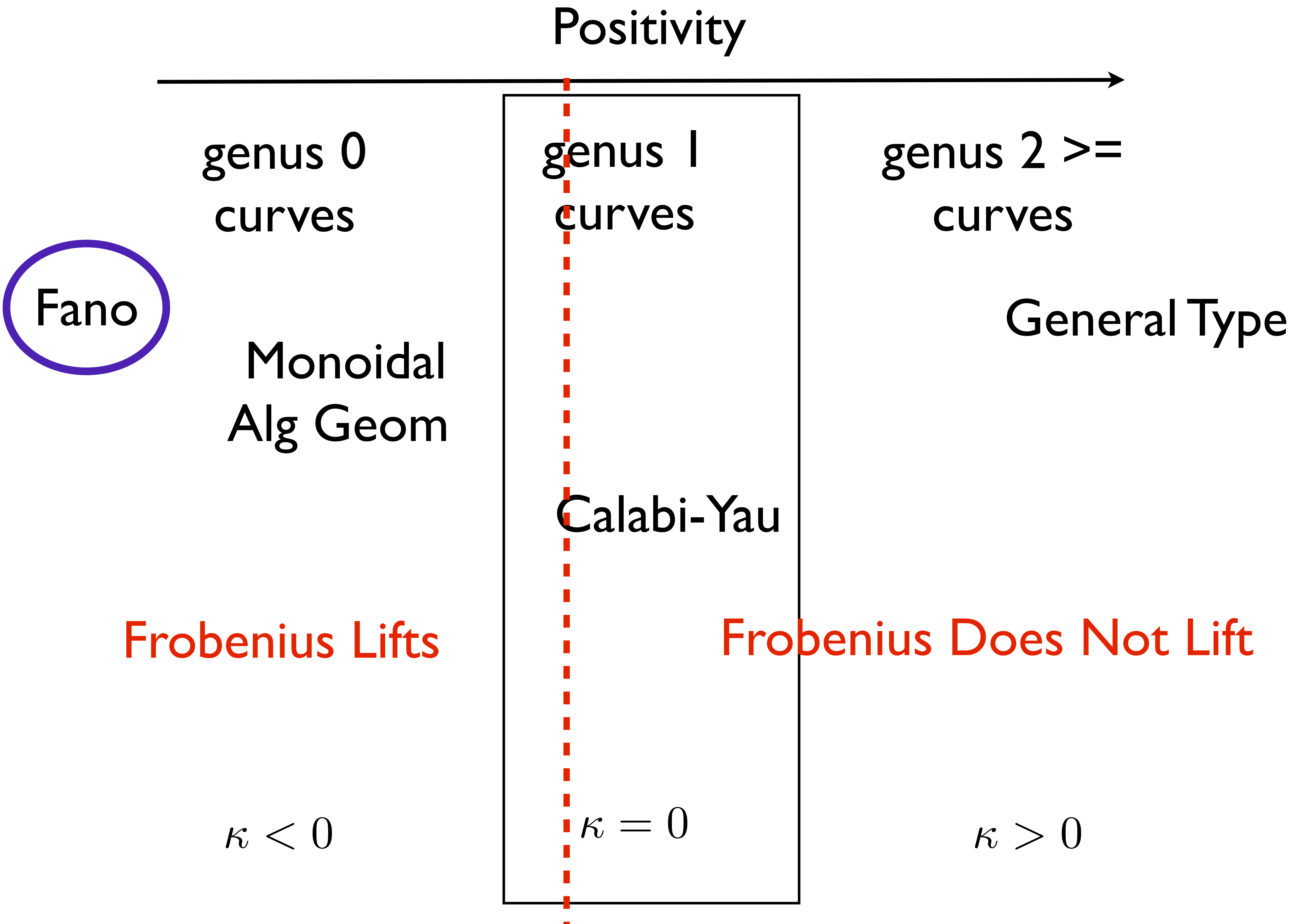
$$\widehat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta : \zeta^n = 1, p \nmid n]\widehat{\ }$$

$$(\widehat{\mathbb{Z}}_p^{ur})^\delta = \{r : \delta(r) = 0\} \qquad \delta(r) = \frac{\phi(r) - r^p}{p}$$

$$= \text{ Monoid of roots of unity}$$

$$:= M$$

$$\mathrm{DI}_0(\delta) = 0 \implies \exists X_1'/M_1 \text{ such that } X_1' \otimes_M \widehat{\mathbb{Z}}_p^{ur}/p^2 \cong X$$


Positivity
genus 0 curves
genus 1 curves
genus 2 >= curves
Fano
General Type
Monoidal Alg Geom
Calabi-Yau
Frobenius Lifts
Frobenius Does Not Lift
$\kappa < 0$
$\kappa = 0$
$\kappa > 0$

What is the principle of computations?
What care about explicit comp?
What makes the computation difficult?

Simple Problem:

Describe the set of λ such that

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - \lambda X_0 X_1 X_2 X_3 X_4 X_5 = 0$$

admits a lift of the Frobenius

Relation of Deformation classes to Jet Sp

PHS ~> torsor

Setup	Torsor	Group Scheme	Cohom Class
X/K	$J^1(X)$	$T_{X/K}$	$\mathrm{KS}(\delta)$
$X/\widehat{\mathbb{Z}}_p^{ur}$	$J^1(X)_0$	$F^*T_{X_0}$	$\mathrm{DI}_0(\delta)$

Main Theorem of Talk

	Setup	Torsor	Group Scheme	Cohom Class
	X/K	$J^1(X)$	$T_{X/K}$	$\mathrm{KS}(\delta)$
	$X/\widehat{\mathbb{Z}}_p^{ur}$	$J^1(X)_0$	$F^*T_{X_0}$	$\mathrm{DI}_0(\delta)$
	$X/\widehat{\mathbb{Z}}_p^{ur}$ curves	$J^1(X)_n$	L_n	$\mathrm{DI}_n(\delta)$
$g \geq 2$ (new)	$\widehat{X}/\widehat{\mathbb{Z}}_p^{ur}$ curves	$J^1(X)$	$\widehat{L}$	$\widehat{\mathrm{DI}}(\delta)$

$- \otimes_{R_n} R_0$

How do you get this lift?

What happens in elliptic curve case?
Higher Dimensions?
Procedure?
-"Structures" OK
-Cocycle

What does th
mean?

need p-formal completion

- Arithmetic Jet Spaces are affine bundles
- Affine bundles have associated cohomology classes
- the cohomology class associated to the jet space controls the D

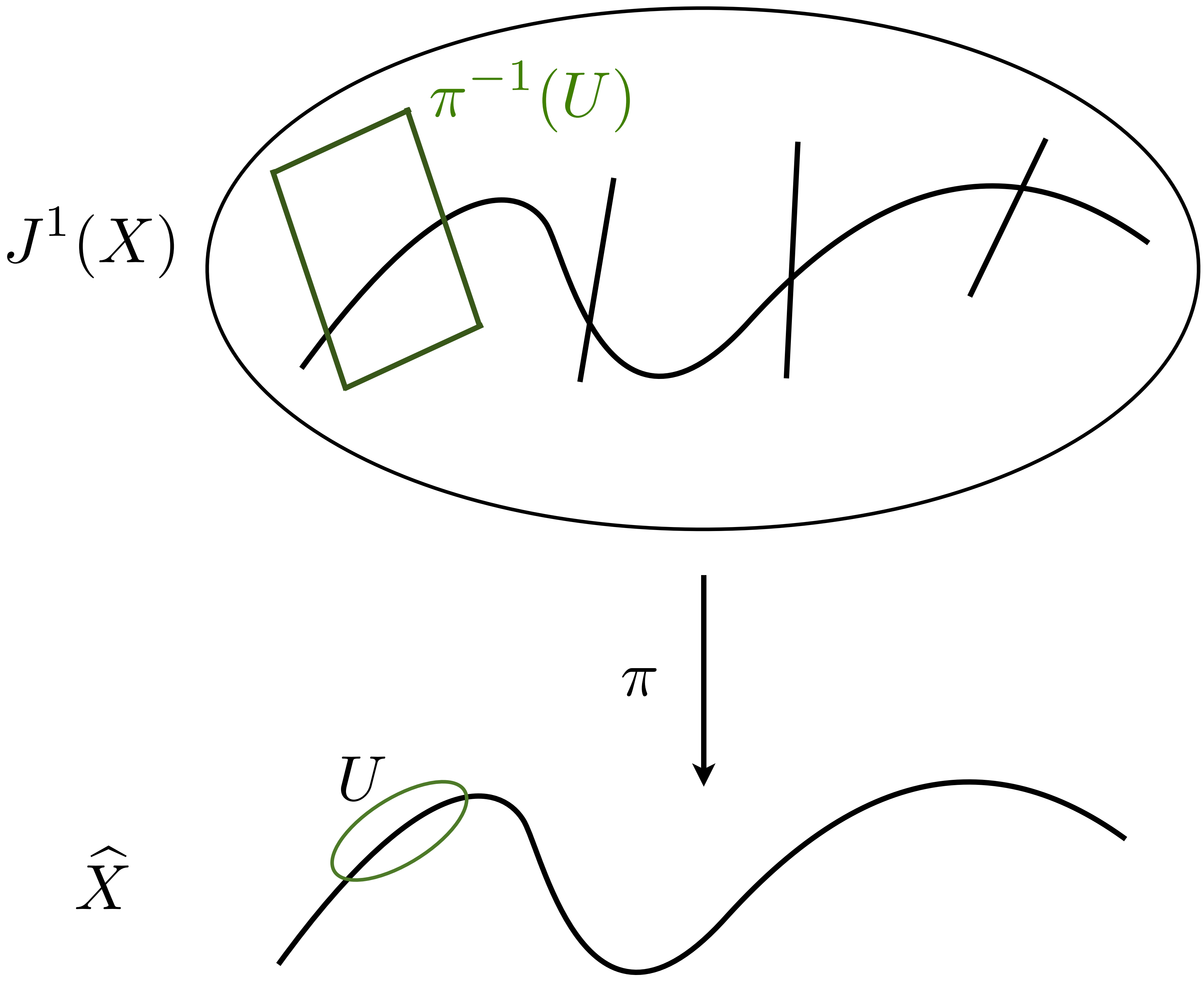
Natural To Investigate

- Arithmetic Jet Spaces admit reductions of the structure group.
- Elliptic Curves may admit multiple reductions of the structure group!!!

WARNING

Lemma

$$U \to \mathbb{A}^n_R \text{ étale } \implies J^1(U) \simeq \widehat{U} \hat{\times} \widehat{\mathbb{A}}^n$$

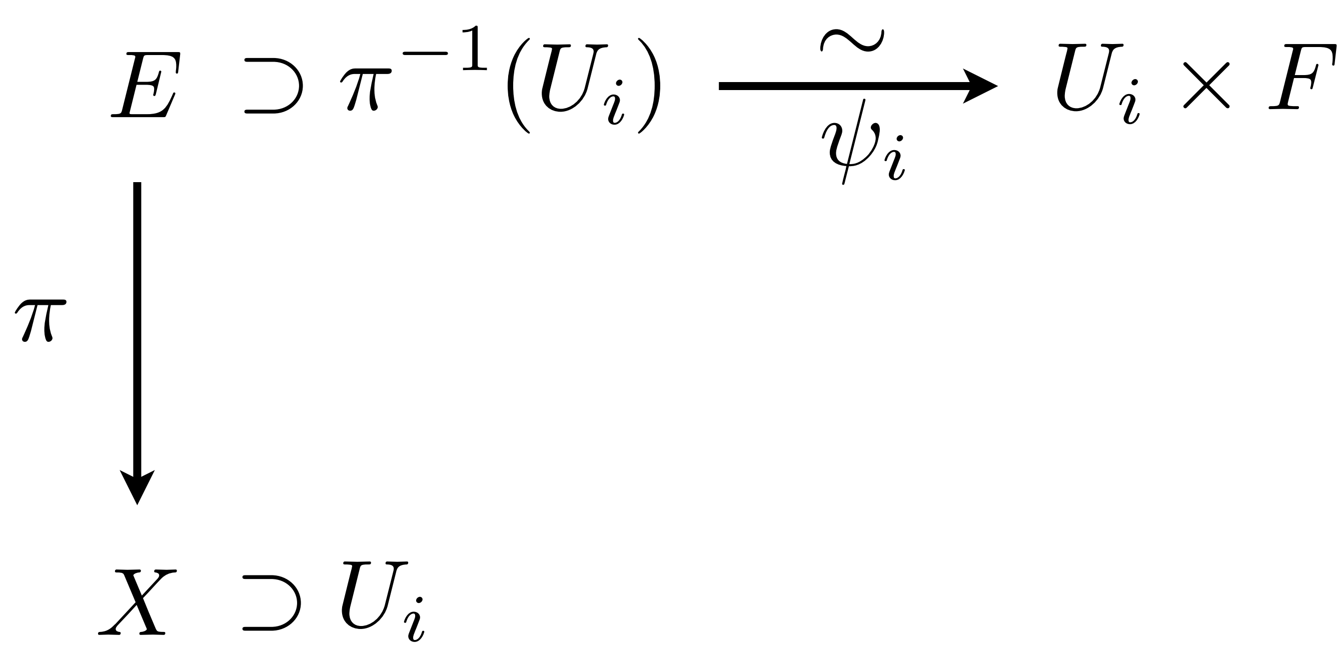

$\pi^{-1}(U)$
$J^1(X)$
π
U
$\widehat{X}$

Local Trivialization of F-bundles

$$
\begin{array}{ccc}
E \supset \pi^{-1}(U) & \xrightarrow[\psi]{\sim} & U \times \\
\big\downarrow \pi & & \\
X \supset U & &
\end{array}
$$

What is E and F?

$$
\begin{array}{ccc}
E \supset \pi^{-1}(U_i) & \xrightarrow[\psi_i]{\sim} & U_i \times F \\
\pi \big\downarrow & & \\
X \supset U_i & &
\end{array}
$$

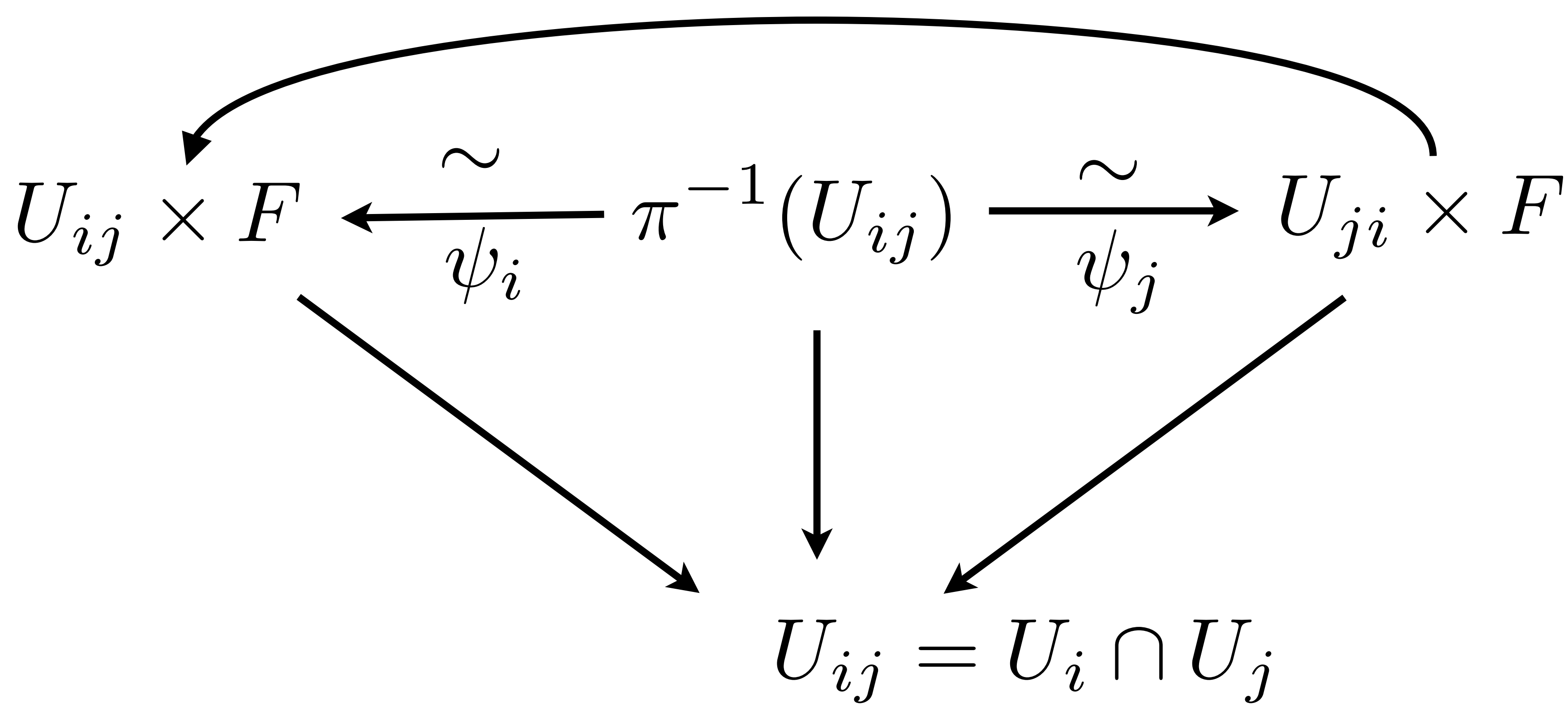


Trivializing Cover

$$X = \bigcup_i U_i$$

Transition Map

$$\psi_{ij} := \psi_i \circ \psi_j^{-1} \in \underline{\mathrm{Aut}}(F)(U_{ij})$$



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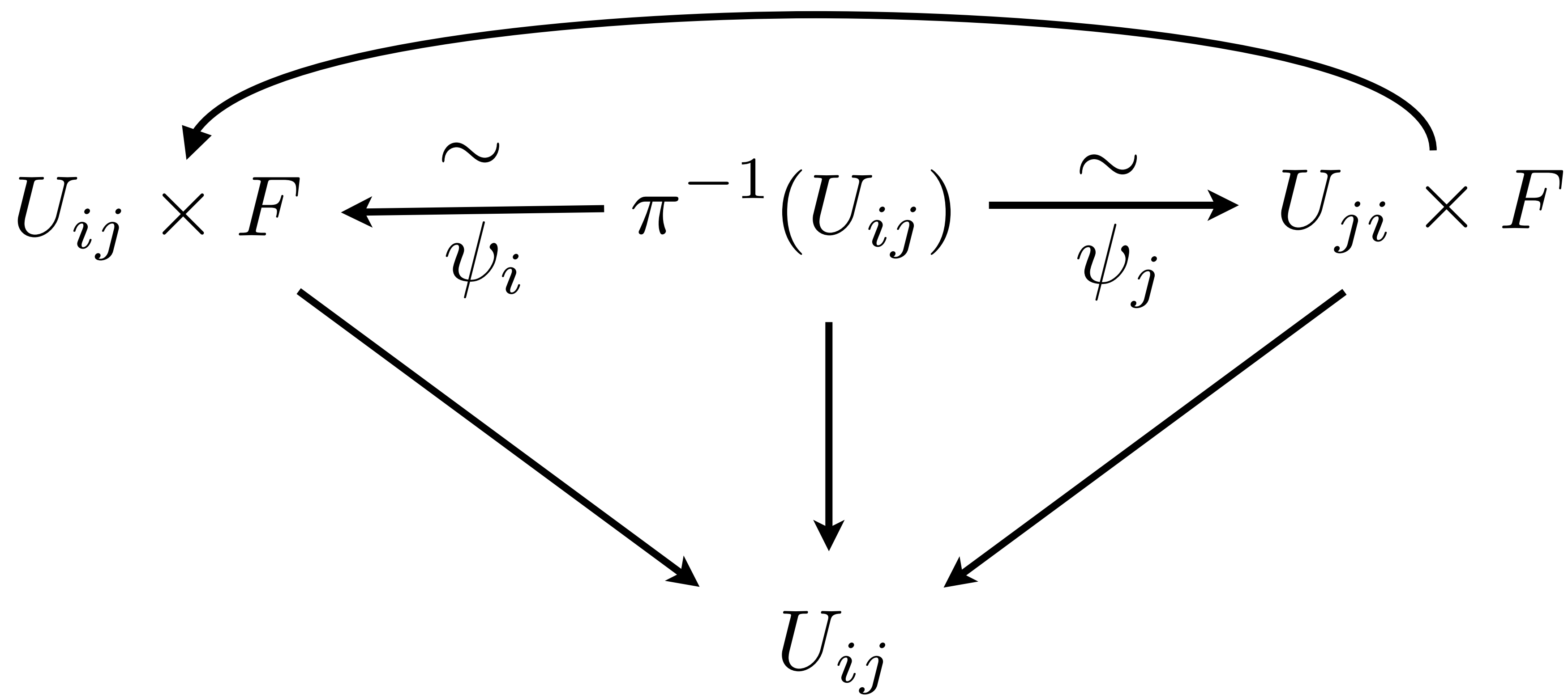

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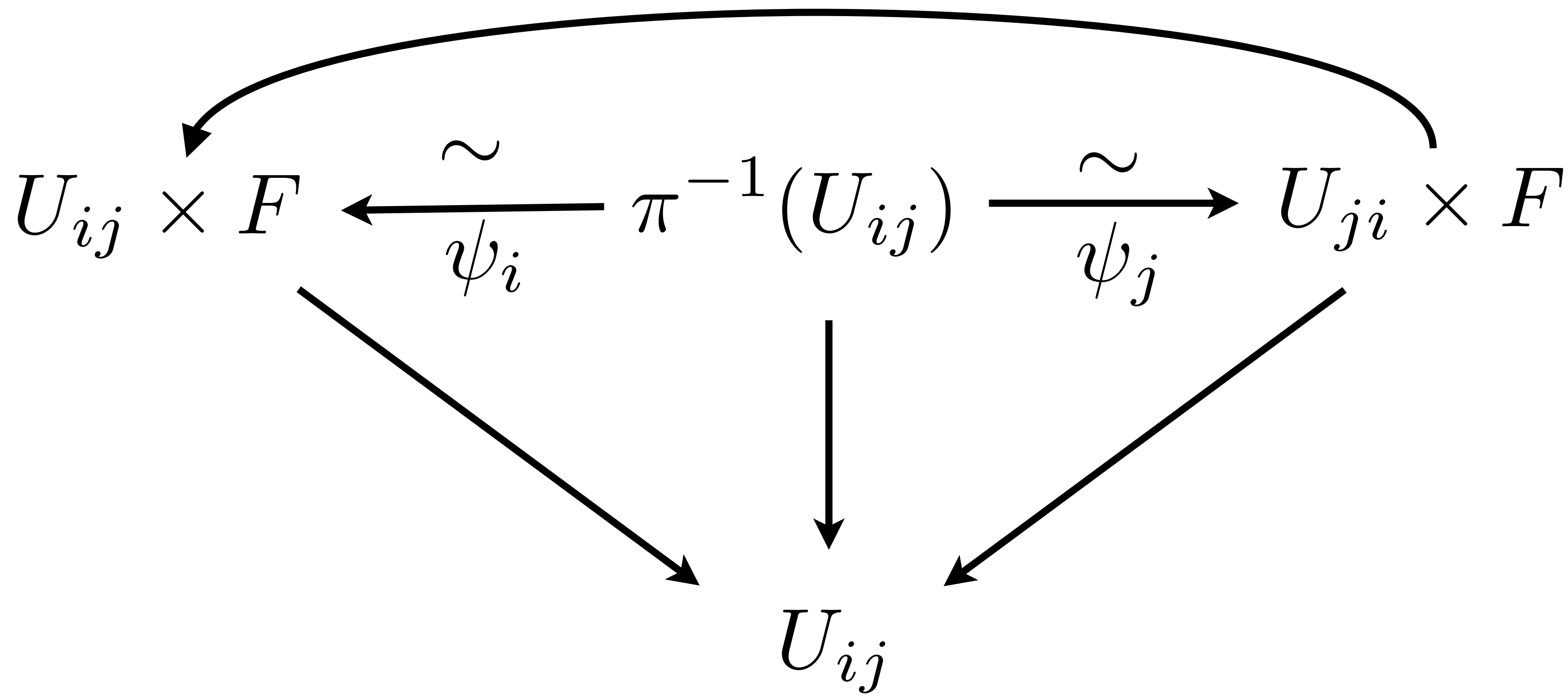

Class associated to Bundle

$$\rightsquigarrow [\psi_{ij}] \in \check{H}^1(X, \underline{\mathrm{Aut}}(F))$$

$$J^1(X) \supset \pi^{-1}(\widehat{U}_i) \xrightarrow[\psi_i]{\sim} \widehat{U}_i \hat{\times} \widehat{\mathbb{A}}^m$$

$$\Big\downarrow \pi$$

$$\widehat{X} \supset \widehat{U}_i$$

$$m = \dim(X)$$

Who is the Big Class?

What is the data actually given by?

What is the data given by mod p and mod p^2?

$$\rightsquigarrow \quad \beta := [\psi_{ij}] \in H^1(\widehat{X}, \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^m))$$

Controls Deligne-Illusie

Fiber Bundle

$$E \supset \pi^{-1}(U_i) \xrightarrow[\psi_i]{\sim} U_i \times F$$

$$\pi \downarrow$$

$$X \supset U_i$$

$$H \leq \underline{\mathrm{Aut}}(F)$$

Extra Condition

$$\psi_{ij} \in H(U_{ij})$$

Definition

$\{(U_i, \psi_i)\}$ = H-atlas for E

Definition

Σ = H-structure = Maximal H-atlas

EXAMPLE

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{aligned} U_1 &= \text{Spec } R[x] \\ U_2 &= \text{Spec } R[y] \end{aligned}$$

Do we need to know about the A^1 example?

$$U_1 \cap U_2 = \text{Spec } R[x, y]/\langle xy - 1 \rangle$$

$$J^1(\mathbb{P}^1_R) = ???$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{aligned} U_1 &= \mathrm{Spec}\ R[x] \\ U_2 &= \mathrm{Spec}\ R[y] \end{aligned}$$

$$\mathcal{O}(J^1(U_1)) = \mathcal{O}(U_1)[\dot{x}]^{\frown} \xrightarrow{\sim} \mathcal{O}(U_1)[T]^{\frown} \qquad \dot{x} \mapsto T$$

$$\mathcal{O}(J^1(U_2)) \xrightarrow{\sim} \mathcal{O}(U_2)[T]^{\frown} \qquad \dot{y} \mapsto T$$

$$\mathcal{O}(U_{12})[\dot{x}]^{\frown} = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]^{\frown}$$

$$\dot{x} \mapsto T \quad \sim \swarrow \qquad \qquad \searrow \sim \quad \dot{y} \mapsto T$$

$$\mathcal{O}(U_{12})[T]^{\frown} \qquad \qquad \mathcal{O}(U_{12})[T]^{\frown}$$

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$$\mathcal{O}(U_{12})[\dot{x}]^{\widehat{}} = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]^{\widehat{}}$$

$$\dot{x} \mapsto T \quad \sim \swarrow \qquad\qquad \searrow \sim \quad \dot{y} \mapsto T$$

$$\mathcal{O}(U_{12})[T]^{\widehat{}} \qquad\qquad \mathcal{O}(U_{12})[T]^{\widehat{}}$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad U_1 = \mathrm{Spec}\ R[x], \quad U_2 = \mathrm{Spec}\ R[y]$$

$$\mathcal{O}(U_{12})[\dot{x}]^{\wedge} = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]^{\wedge}$$

$$\dot{x} \mapsto T \ \sim \swarrow \qquad \searrow \sim \ \dot{y} \mapsto T$$

$$\mathcal{O}(U_{12})[T]^{\wedge} \qquad \mathcal{O}(U_{12})[T]^{\wedge}$$

$$T \mapsto \dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})} \mapsto \frac{-T}{y^p(y^p + pT)}$$

$$x = 1/y \Longrightarrow \dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})}$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad U_1 = \text{Spec } R[x] \qquad U_2 = \text{Spec } R[y]$$

$$\mathcal{O}(U_{12})[\dot{x}]\widehat{\ } = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]\widehat{\ }$$

$$\dot{x} \mapsto T \;\sim\; \swarrow \qquad \searrow \;\sim\; \dot{y} \mapsto T$$

$$\mathcal{O}(U_{12})[T]\widehat{\ } \qquad\qquad \mathcal{O}(U_{12})[T]\widehat{\ }$$

$$T \overset{\psi_1^*}{\mapsto} \dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})} \overset{(\psi_2^*)^{-1}}{\mapsto} \frac{-T}{y^p(y^p + pT)} = \psi_2^{-1*}\psi_1^*(T)$$

$$J^1(U_{12})$$

$$\psi_1 \swarrow \qquad \searrow \psi_2$$

$$\widehat{U}_{12}\hat{\times}\widehat{\mathbb{A}}^1 \qquad\qquad \widehat{U}_{12}\hat{\times}\widehat{\mathbb{A}}^1$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad U_1 = \text{Spec } R[x] \qquad U_2 = \text{Spec } R[y]$$

$$\mathcal{O}(U_{12})[\dot{x}]\widehat{\ } = \mathcal{O}(J^1(U_{12})) = \mathcal{O}(U_{12})[\dot{y}]\widehat{\ }$$

$$\dot{x} \mapsto T \ \sim \swarrow \qquad \searrow \sim \ \dot{y} \mapsto T$$

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$$J^1(U_{12})$$

$$\psi_1 \swarrow \qquad \searrow \psi_2$$

$$\widehat{U}_{12} \hat{\times} \widehat{\mathbb{A}}^1 \qquad \qquad \widehat{U}_{12} \hat{\times} \widehat{\mathbb{A}}^1$$

$$J^1(U_{12})$$

$$\widehat{U}_{12} \,\hat{\times}\, \widehat{\mathbb{A}}^1 \xleftarrow{\psi_1} J^1(U_{12}) \xrightarrow{\psi_2} \widehat{U}_{12} \,\hat{\times}\, \widehat{\mathbb{A}}^1$$

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad U_1 = \text{Spec } R[x], \quad U_2 = \text{Spec } R[y]$$

$$\frac{-T}{y^p(y^p + pT)} = \psi_2^{-1*}\psi_1^*(T) = (\psi_1\psi_2^{-1})^*(T) = \psi_{12}^*(T)$$

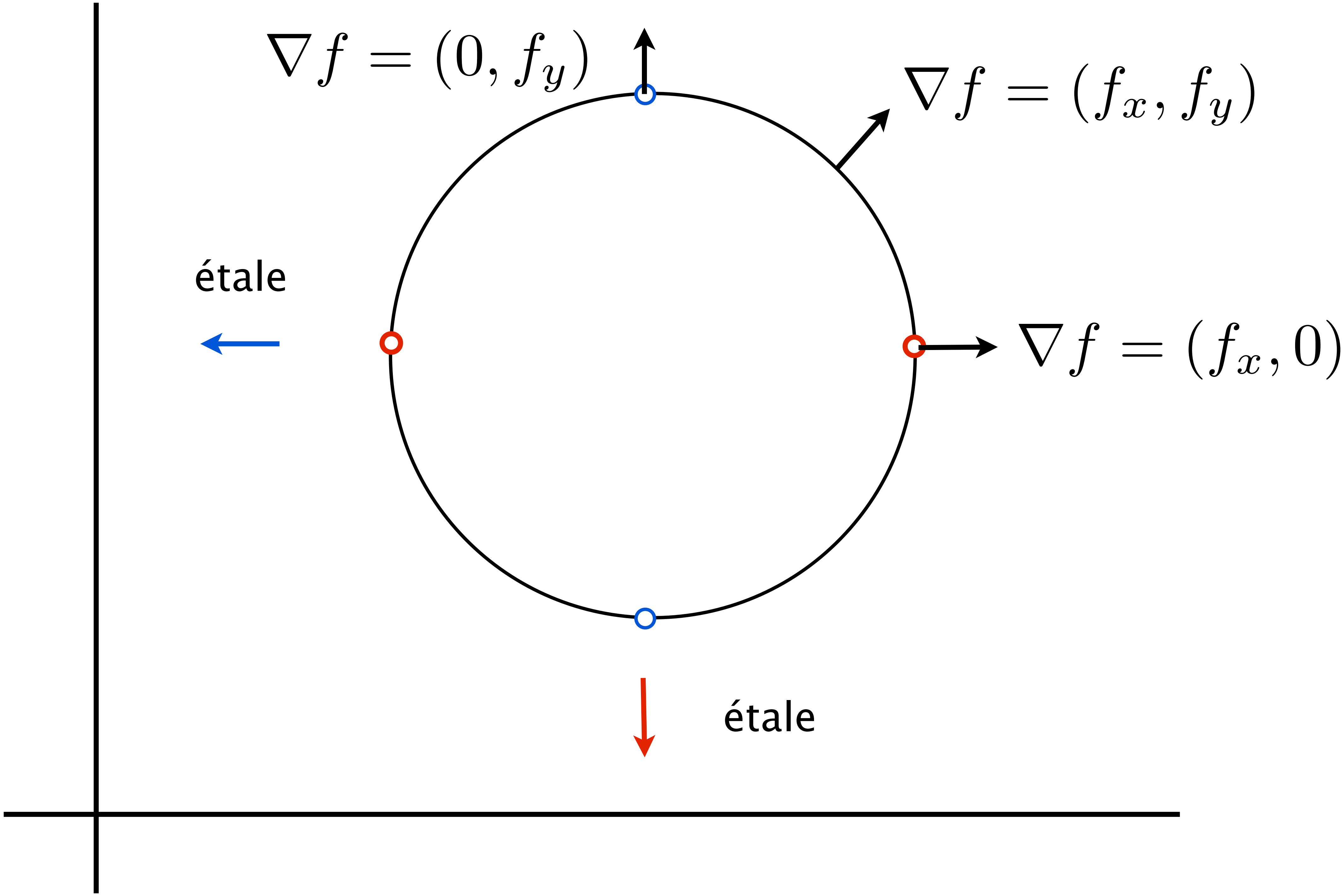
$$\psi_{12}(T) = -\frac{T}{y^p(y^p + pT)}$$

$$[\psi_{12}] \in H^1(\mathbb{P}^1, \underline{\text{Aut}}(\widehat{\mathbb{A}}^1))$$

what is the point?

can you tell that PP^1 has a lift of the frobenius mod p^2 from this class already?

EXAMPLE $X : f(x,y) = 0$

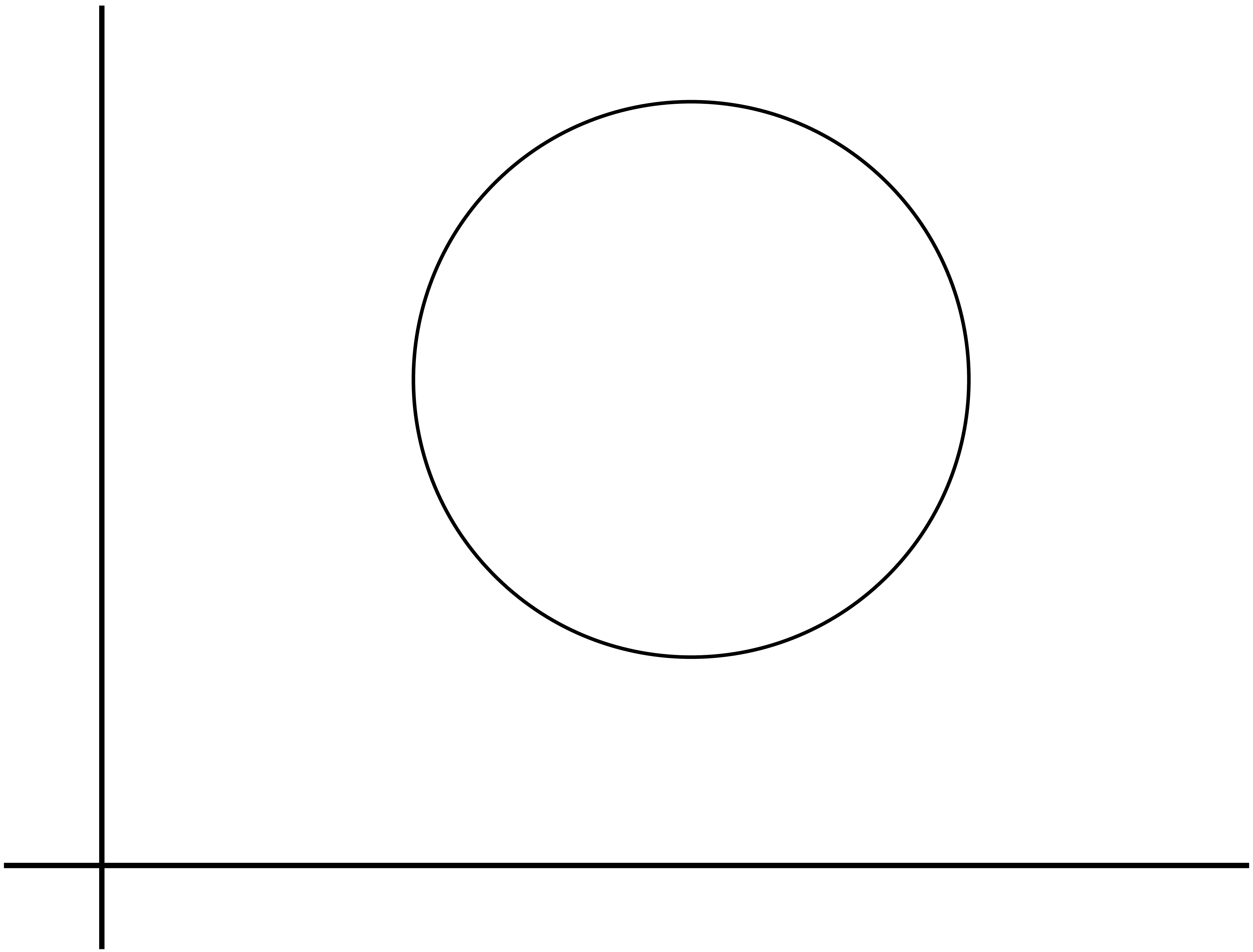


$U_1 = X \setminus V(f_x)$

$U_2 = X \setminus V(f_y)$

$X = \text{Spec } R[x,y]/\langle f(x,y) \rangle$

$= U_1 \cup U_2$



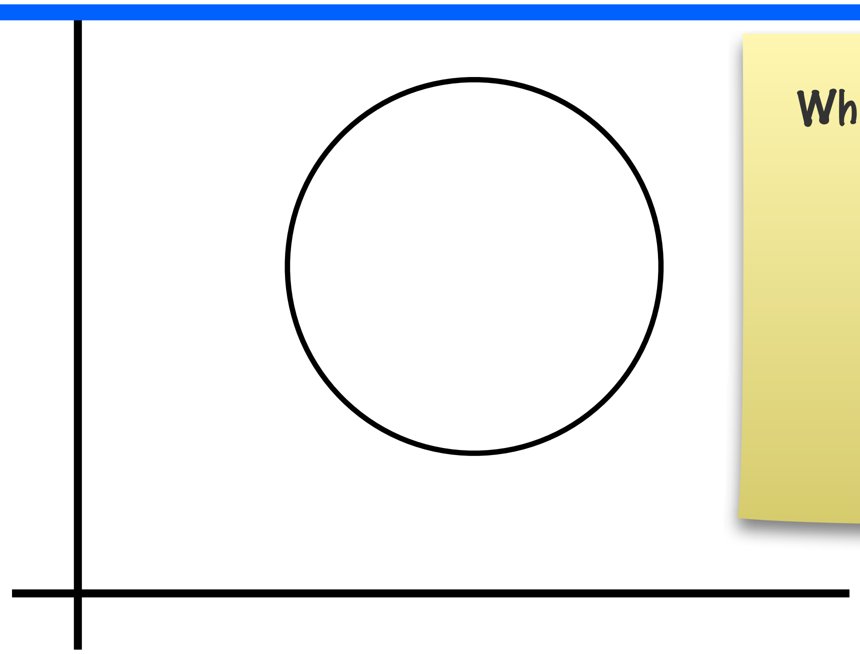
Why do you
show this c

$U_1 = X \setminus V(f_x)$

$U_2 = X \setminus V(f_y)$

$X = \text{Spec } R[x,y]/\langle f(x,y) \rangle$

$$X = \operatorname{Spec} R[x,y]/\langle f(x,y)\rangle$$



What is f^{\phi}?

$$f(x,y) = 0$$

$$\delta f = 0$$

$$\frac{\phi(f) - f^p}{p} = 0$$

$$U_1 = X \setminus V(f_x)$$

$$U_2 = X \setminus V(f_y)$$

$$\frac{f^{\phi}(x^p + p\dot{x}, y^p + p\dot{y}) - f(x,y)^p}{p} = 0$$

$$f^{\phi}(x^p + p\dot{x}, y^p + p\dot{y}) = ???$$

$$g(T) = a_0 + a_1 T + \cdots \text{ then } g^{\phi}(T) = \phi(a_0) + \phi(a_1)T + \cdots$$

$$X = \text{Spec } R[x,y]/\langle f(x,y)\rangle$$

$$f^\phi(x^p + p\dot{x}, y^p + p\dot{y}) \;=\; f^\phi(x^p, y^p)$$

0th order

$$+ \;\; p\left[\frac{\partial f^\phi}{\partial x}(x^p, y^p)\dot{x} + \frac{\partial f^\phi}{\partial y}(x^p, y^p)\dot{y}\right]$$

1st order

$$+\;\frac{p^2}{2}\left[\frac{\partial^2 f^\phi}{\partial x^2}(x^p, y^p)\dot{x}^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x^p, y^p)\dot{x}\dot{y} + \frac{\partial^2 f^\phi}{\partial y^2}(x^p, y^p)\dot{y}^2\right]$$

2nd order

$$+\cdots$$

$$= \sum_{d \geq 0} \frac{p^d}{d!} h_d$$

various orders

$$X = \operatorname{Spec} R[x,y]/\langle f(x,y)\rangle$$

$$\sum_{d\geq 0} \frac{p^d}{d!} h_d$$

$$0 = \frac{f^{\phi}(x^p + p\dot{x}, y^p + p\dot{y}) - f(x,y)^p}{p} = \frac{1}{p}\left[\sum_{d\geq 0} \frac{p^d}{d!} h_d - f(x,y)^p\right]$$

$$= \frac{f^{\phi}(x^p, y^p) - f(x,y)^p}{p} + \sum_{d\geq 1} \frac{p^{d-1}}{d!} h_d$$

$$= r + \frac{\partial f^{\phi}}{\partial x}(x^p, y^p)\dot{x} + \frac{\partial f^{\phi}}{\partial y}(x^p, y^p)\dot{y} + O(p)$$

$$X = \text{Spec } R[x,y]/\langle f(x,y)\rangle$$

$$0 = r + \frac{\partial f^{\phi}}{\partial x}(x^p, y^p)\dot{x} + \frac{\partial f^{\phi}}{\partial y}(x^p, y^p)\dot{y} + O(p)$$

$$0 \equiv r + f_x^p \dot{x} + f_y^p \dot{y} \mod p$$

$$\dot{y} \equiv -\frac{r + f_x^p \dot{x}}{f_y^p} \mod p$$

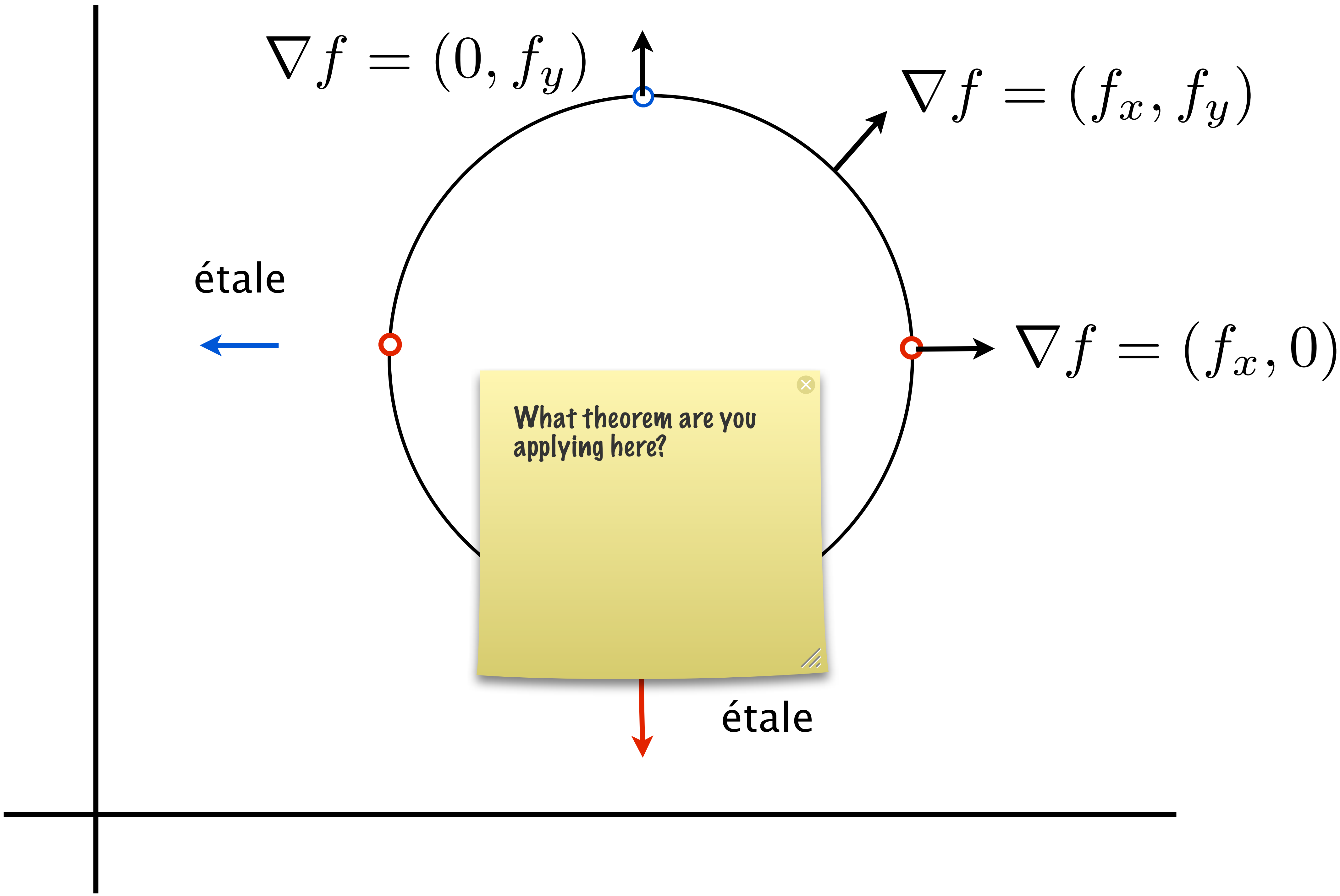
$$\dot{y} \equiv A + B\dot{x} + pC\dot{x}^2 \mod p^2$$

Transition maps take on a very particular form!

$$\dot{y} \equiv A + B\dot{x} + pC\dot{x}^2 + p^2 D\dot{x}^3 \mod p^3$$

$$\vdots$$

EXAMPLE $X : f(x,y) = 0$



$U_1 = X \setminus V(f_x)$

$U_2 = X \setminus V(f_y)$

$$X = \text{Spec } R[x,y]/\langle f(x,y)\rangle$$
$$= U_1 \cup U_2$$

$$X = \text{Spec } R[x,y]/\langle f(x,y)\rangle$$

$$U_1 = X \setminus V(f_x)$$

$$U_2 = X \setminus V(f$$

Convinced you tha
theorem is plausibl

$$\widehat{U}_{12} \times \widehat{\mathbb{A}}^1 \xleftarrow{\psi_1} J^1(U_{12}) \xrightarrow{\psi_2} \widehat{U}_{12} \times \widehat{\mathbb{A}}^1$$

$$\dot{y} = A + B\dot{x} + pC\dot{x}^2 + O(p^2)$$

$$\Longrightarrow \psi_{12}(T) = A + BT + pCT^2 + O(p^2)$$

What was the goal?

Does this say that
transition maps lie i
wacky subgroups?

$$\mathbb{P}^1 = U_1 \cup U_2 \qquad \begin{array}{l} U_1 = \text{Spec } R[x] \\ U_2 = \text{Spec } R[y] \end{array}$$

$$\dot{x} = \frac{-\dot{y}}{y^p(y^p + p\dot{y})} \quad \Longrightarrow \quad \psi_{12}(T) = -\frac{T}{y^p(y^p + pT)}$$

Transition maps for Jet Spaces lie in wacky subgroups!*

$$A_n \leq \underline{\mathrm{Aut}}(\mathbb{A}^1_{R_{n-1}})$$

$$\psi(T) = c_0 + c_1 T + p c_2 T^2 + \cdots + p^{n-1} c_{n-1} T^n \mod p^n$$

What is a Degree Structure?

Prop. These are groups

Prop.
$p > 6g-5 \implies \exists$ tri-canonical A_{n+1}-structure on $J^1(X)_n$

A_2

(transition maps for $J^1(X) \mod p^2$)

What is the point of this slide?

What does the coeycle that determines reducitons look like?

Example 1

$$\psi(T) = T + pT^2$$
$$\psi^{-1}(T) = T - pT^2$$

Example 2

$$(a_0 + a_1 T + p a_2 T^2) \circ (b_0 + b_1 T + p b_2 T^2)$$
$$= (a_0 + a_1 b_0 + p a_2 b_0^2) + (a_1 b_1 + 2p a_2 b_0 b_1) T + p(a_1 b_2 + a_2 b_1^2) T^2$$

A_2 (transition maps for $J^1(X) \mod p^2$)

Example 1

$$\psi(T) = T + pT^2$$

$$\psi^{-1}(T) = T - pT^2$$

What is the point of the cocycle you are going to cook up?

Example 2

$$(a_0 + a_1 T + p a_2 T^2) \circ (b_0 + b_1 T + p b_2 T^2)$$

$$= (a_0 + a_1 b_0 + p a_2 b_0^2) + (a_1 b_1 + 2 p a_2 b_0 b_1) T + p(a_1 b_2 + a_2 b_1^2) T^2$$

$$(a_0 + a_1 T + p a_2 T^2) \circ (b_0 + b_1 T + p b_2 T^2)$$

$$= (a_0 + a_1 b_0 + p a_2 b_0^2) + (a_1 b_1 + 2 p a_2 b_0 b_1) T + p(a_1 b_2 + a_2 b_1^2) T^2$$

Prop/Def

$$\tau_2(c_0 + c_1 T + p c_2 T^2) := \frac{c_2}{c_1}$$

$$\tau_2(f \circ g) = \tau_2(f) \cdot m(g) + \tau_2(g)$$

$$\frac{a_1 b_2 + a_2 b_1^2}{a_1 b_1} = \frac{b_2}{b_1} + \frac{a_2}{a_1} b_1$$

Consider the whole situation mod p

Theorem

$$\beta_0 = [a_{ij} + b_{ij}T] \in H^1(X_0, \mathrm{AL}_1)$$

$\mathrm{DI}_0(\delta)$ —B→ a_{ij} A: $[F^*T_{X_0}]$ → $b_{ij}T$

How to prove B? (General Nonsense)

idea used in A

$$\mathrm{AL}_1 = \underline{\mathrm{Aut}}(\mathbb{A}^1_{\mathbb{F}_p}) \; [illegible] \to \mathcal{O}_{X_0} \rtimes \mathcal{O}^{\times}_{X_0}$$

$$a + bT \circ c + dT = a + bc \; [illegible] \; b) \cdot (c, d) = (a + bc, bd)$$

$$\mathcal{O}_{X_0} \rtimes \mathcal{O}^{\times}_{X_0} \xrightarrow{\pi} \mathcal{O}^{\times}_{X_0}$$

$$H^1(X, \mathcal{O}_X \rtimes \mathcal{O}^{\times}_X) \xrightarrow{\pi} H^1(X, \mathcal{O}^{\times}_X)$$

$$\pi(\beta_0) = [b_{ij}] = [F^*T_{X_0}]$$

■

Conventions

$$\varphi_i : \mathcal{O}(U_i) \to L(U_i)$$

$$\varphi_i(1) = v_i$$

$$b_{ij} v_i = v_j$$

$$[L] = [b_{ij}]$$

idea of B

Conventions

$$\varphi_i : \mathcal{O}(U_i) \to L(U_i)$$

$$\varphi_i(1) = v_i$$

$$b_{ij} v_i = v_j$$

$$[L] = [b_{ij}]$$

$$(0, 1) = (a_{ij}, b_{ij}) \cdot (a_{jk}, b_{jk}) \cdot (a_{ki}, b_{ki})$$
$$= (a_{ij} + b_{ij}a_{jk} + b_{ij}b_{jk}a_{ki}, b_{ij}b_{jk}b_{ki})$$

$$(a_{ij}, b_{ij}) \mapsto \quad s_{ij} = \varphi_i(a_{ij}) \in L(U_{ij})$$

Conventions

$$\varphi_i : \mathcal{O}(U_i) \to L(U_i)$$
$$\varphi_i(1) = v_i$$
$$b_{ij}v_i = v_j$$
$$[L] = [b_{ij}]$$

$$s_{ij} + s_{jk} + s_{ki} = a_{ij}v_i + a_{jk}v_j + a_{ki}v_k$$
$$= a_{ij}v_i + a_{jk}b_{ij}v_i + a_{ki}b_{ij}b_{jk}v_i$$
$$= 0$$

Strategy: Find "images" of β in semi-direct products

use GROUP COCYCLES

GROUP COCYCLES

Defn. $$\Phi : G \to A$$

$$\Phi(g_1 g_2) = \Phi(g_1) + g_1 \cdot \Phi(g_2)$$

$$\Phi \quad \rightsquigarrow \quad \begin{array}{c} G \to A \rtimes \mathrm{Aut}(A) \\ g \mapsto (\Phi(g), \rho(g)) \end{array}$$

GROUP COCYCLES

Left Cocycle

$$\Phi : G \to A$$

$$\Phi(g_1 g_2) = \Phi(g_1) + g_1 \cdot \Phi(g_2)$$

$$\rightsquigarrow \quad \begin{matrix} G \to A \rtimes \mathrm{Aut}(A) \\ g \mapsto (\Phi(g), \rho(g)) \end{matrix}$$

Right Cocycle

$$\Phi : G \to A$$

$$\Phi(g_1 g_2) = \Phi(g_1)^{g_2} + \cdot \Phi(g_2)$$

$$\rightsquigarrow \quad \begin{matrix} G \to \mathrm{Aut}(A) \ltimes A \\ g \mapsto (\rho(g), \Phi(g)) \end{matrix}$$

cook up right cocycle + map:

$$f_{\tau_2} : H^1(X_1, A_2) \to H^1(X_0, \mathcal{O}^\times \rtimes \mathcal{O})$$

What does the 2 in tau_2 stand for?

What was iota_{right}?

apply to big class to get twisted cocycle:

$$f_{\tau_2}(\beta_1) \in H^1(X_0, \mathcal{O}^\times \ltimes \mathcal{O})$$

$$f_{\tau_1}(\beta_1) = [\alpha_i \alpha_j^{-1}]$$

image class:

genus>1

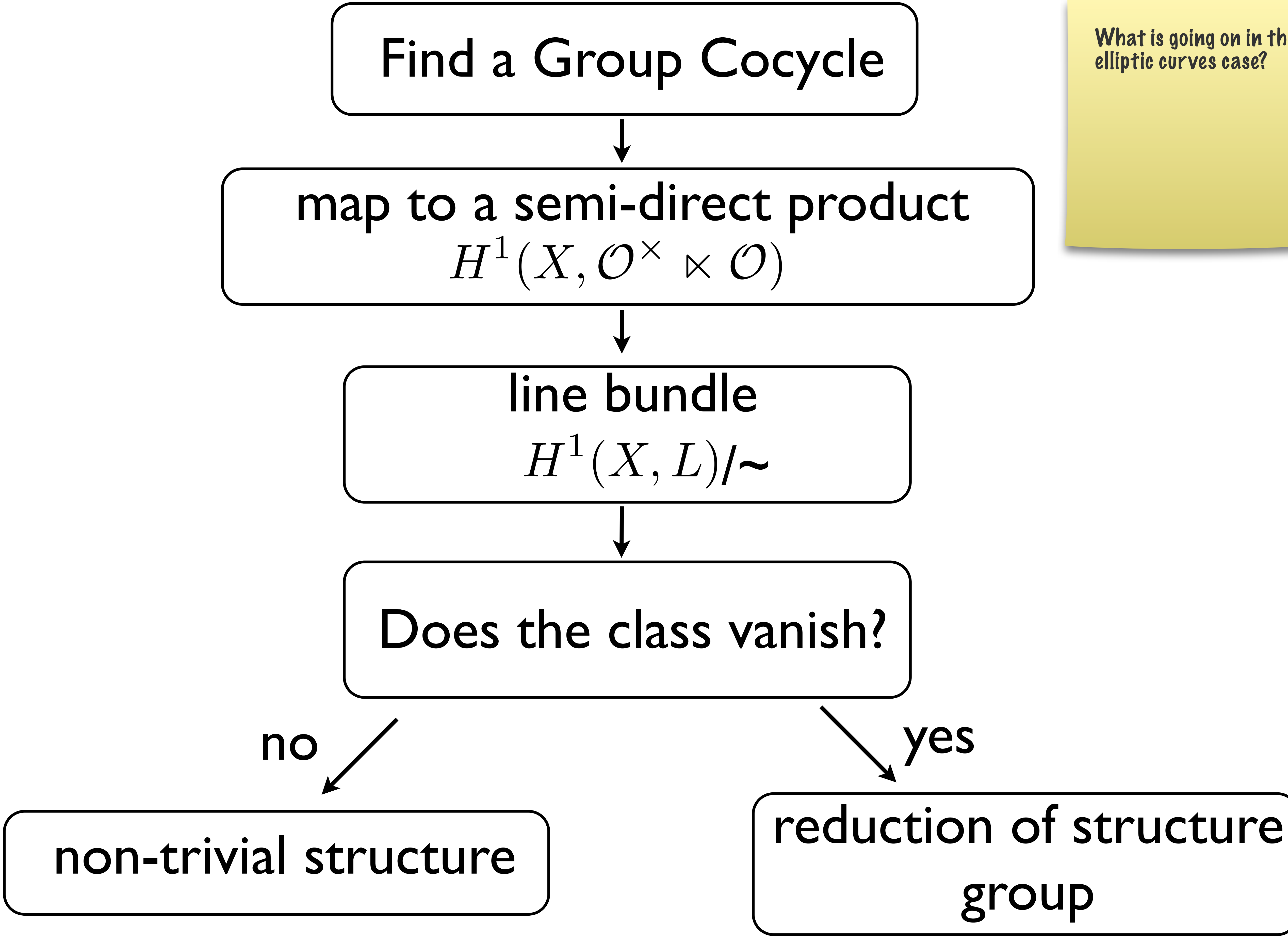
$$\iota_{right}(f_{\tau_2}(\beta_1)) \in \underline{H^1(X_0, \Omega^p_{X_0})} = 0$$

$\sim$

get affine linear structure on first jet space:

$$\beta_1 = [\psi_i \circ \psi_j^{-1}] \qquad f_{\tau_2}(\psi_{ij}) = f_{\tau_2}(\psi_{\alpha_i}) f_{\tau_2}(\psi_{\alpha_j}^{-1})$$

$$\psi_{\alpha_i}^{-1} \psi_{ij} \psi_{\alpha_j} \in \mathrm{Ker}(f_{\tau_2}) = \mathrm{AL}_1(\mathcal{O}_{X_1})$$



What is going on in the elliptic curves case?

Prop.

You can NOT have the same physical affine bundle with two different GL_1 structures

how line bundles are built

$$L = \frac{\coprod_i U_i \times \mathbb{A}^1}{\sim}$$

$$\downarrow \pi$$

$$X$$

morphisms = collection of maps

$$U_i \times \mathbb{A}^1 \xrightarrow{f_i} U_i \times \mathbb{A}^1$$

Multiple Structures?

+ compatibility

$$\begin{array}{ccc} U_{ij} \times \mathbb{A}^1 & \xrightarrow{f_i} & U_{ij} \times \mathbb{A}^1 \\ \psi_{ij} \downarrow & & \downarrow \psi'_{ij} \\ U_{ji} \times \mathbb{A}^1 & \xrightarrow{f_j} & U_{ji} \times \mathbb{A}^1 \end{array}$$

$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

$$f_j \circ \psi_{ij} = \psi'_{ij} \circ f_i$$

$$b_{ij} T = f_j(a_{ij} f_i^{-1}(T))$$

$$b_{ij} = f'_j(a_{ij} f_i^{-1}(T)) \cdot a_{ij} (f_i^{-1})'(T)$$

$$= f'_j(a_{ij} f_i^{-1}(T)) \cdot a_{ij} / f'_i(f_i^{-1}(T))$$

$$T = f_i(S)$$

$$= f'_j(a_{ij} S)) \cdot a_{ij} / f'_i(S)$$

$$S = 0$$

$$= f'_j(0) \cdot a_{ij} f'_i(0)^{-1}$$

THE END