An Introduction to p-derivations and the Torsor of Lifts of the Frobenius

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Texas Tech Algebra Seminar
November 12th, 2012

PART I:
Arithmetic Differential Algebra

PART II:
Deformation Theory

PART I

Arithmetic Differential Algebra

What is a p-derivation?

Fermat's Little Theorem

$\forall n \in \mathbb{Z}, \forall p$ prime

$$n \equiv n^p \mod p$$

$$n - n^p = p \cdot \text{CRAP}$$

This is a p-derivation

$$\text{CRAP} = \frac{n-n^p}{p}$$

$$\delta_p(n) = \frac{n-n^p}{p}$$

The Frobenius

$$F : A/p \mapsto A/p$$
$$a \mapsto a^p$$

Properties of p-derivations

$$\delta_p(n) = \frac{n - n^p}{p}$$

Zero mod p

Zero mod p

EXERCISE

Product Rule

$$\delta_p(ab) = \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b)$$

Sum Rule

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j} b^j$$

non-linear

(Buium, Joyal ~1994)

Abstract Definition: $\delta_p : A \to B$ is a **p-derivation** provided that

Always an A algebra

Product Rule:

$$\delta_p(ab) = \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b)$$

Sum Rule:

$$\delta_p(a+b) = \delta_p(a) + \delta_p(b) - \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^{p-j}b^j$$

Lifts of the Frobenius

Definition: A **lift of the Frobenius** is a ring homomorphism $\phi : A \to B$ such that

$$\phi(a) \equiv a^p \mod p$$

Proposition: If $\delta_p : A \to B$ is a p-derivation then

$$\phi(a) := a^p + p\delta_p(a)$$

is a lift of the Frobenius.

Conversely, if B is p-torsion free ring with a lift of the Frobenius $\phi : A \to B$ then

$$\delta_p(a) := \frac{\phi(a) - a^p}{p}$$

defines a p-derivation.

EXERCISE:

For $\delta_p : \mathbb{Z} \to \mathbb{Z}$
defined by $\delta_p(n) = \frac{n - n^p}{p}$

show $\delta_p(p^m) = p^{m-1} \cdot \ (\text{ unit mod } p \ )$

Example:

$$\delta_p(p) = \frac{p - p^p}{p}$$
$$= 1 - p^{p-1}$$

Idea: order of vanishing is "bumped down"

$$\delta_t = \frac{d}{dt}$$
$$\delta_t(t^n) = n \cdot t^{n-1}$$

EXERCISE

derivations ⟷ ring homomorphisms

$\delta : A \to A$

$f : A \to A[\varepsilon]/\langle \varepsilon^2 \rangle$

"dual numbers"

"infinitesimals"

p-derivations ⟷ ring homomorphisms

$\delta_p : A \to A$

$f : A \to W_1(A)$

"Witt vectors"

"wittfinitesimals"

"Wittferentiation"

Big Analogies

Dual Numbers

$D_1(A) = A[t]/\langle t^2 \rangle$

Truncated Witt Vectors

$W_1(A)$

Power Series

$D(A) = A[[t]]$

Witt Vectors

$W(A)$

General Philosophy

(Borger-Weiland ~ 2004)

- Replace dual ring functor with other functors to get different forms of differential algebra!
- Another Example: Difference Rings

What are Witt Vectors???

$$W_1(A) \cong_{sets} A \times A$$

Funky Addition

$$(a_0, a_1) +_W (b_0, b_1) = (a_0 + b_0, a_1 + a_1 + p\sum_{j=1}^{p-1} \tfrac{1}{p}\tbinom{p}{j} a_0^{p-j} b_0^j)$$

Funky Multiplication

$$(a_0, a_1) *_W (b_0, b_1) = (a_0 + b_0, a_1 b_0^p + a_0^p a_1 + p a_1 b_1)$$

exercise: $W_1(\mathbb{F}_p) = \mathbb{Z}/p^2$

What are Witt vectors in general?

$$W(A) \cong_{sets} A^{\mathbb{N}}$$

sum polyn

Funky Addition

$$(a_0, a_1, a_2, \ldots) +_W (b_0, b_1, b_2, \ldots) = (s_0(a,b), s_1(a,b), s_2(a,b), \ldots)$$

Funky Multiplication

$$(a_0, a_1, a_2, \ldots) *_W (b_0, b_1, b_2, \ldots) = (m_0(a,b), m_1(a,b), m_2(a,b), \ldots$$

mult polyn

$W(A) \cong_{sets} A^{\mathbb{N}}$

Defining Property of Ring Structure:

The map $W(A) \to A^{\mathbb{N}}$ defined by

$$(a_0, a_1, a_2, \ldots) \mapsto (w_0(a), w_1(a), w_2(a), \ldots)$$

is a ring homomorphism.

usual componentwise addn and mult

WHAT ARE THE w's???

$$w_0(a) = a_0$$

$$w_1(a) = a_0^p + pa_1$$

$$w_2(a) = a_0^{p^2} + pa_1^p + p^2 a_2$$

$$w_3(a) = a_0^{p^3} + pa_1^{p^2} + p^2 a_2^p + p^3 a_3$$

$$\vdots$$

Witt Polynomials

EXERCISE:

Compute some of the addition polynomials

$$w_0(a) + w_0(b) = w_0(s)$$
$$w_1(a) + w_1(b) = w_1(s)$$
$$w_2(a) + w_2(b) = w_2(s)$$
$$\vdots$$

exercise: $W(\mathbb{F}_p) = \mathbb{Z}_p$

$$a_0 + b_0 = s_0$$
$$(a_0^p + pa_1) + (b_0^p + pb_1) = s_0^p + ps_1$$
$$(a_0^{p^2} + pa_1^p + p^2 a_2) + (b_0^{p^2} + pb_1^p + p^2 b_2) = s_0^{p^2} + ps_1 + p^2 s_2$$
$$\vdots$$

PART II

Kodaira Spencer Theory

Čech Cohomology

X scheme

G sheaf of groups

$\mathcal{U} = \{U_i\}_{i=1}^n$ open cover

$$\check{H}^1(X, G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)$$

Cocycles: (g_{ij})

$$\check{Z}^1(\mathcal{U}, G) \subset \prod_{i,j} G(U_{ij})$$

$$g_{ij} g_{jk} g_{ki} = 1$$

$$g_{ij}^{-1} = g_{ji}$$

Cohomology: $(g_{ij}) \sim (g'_{ij})$

$$\iff \exists \ (h_i) \in \prod_i G(U_i)$$

$$h_i g_{ij} h_j^{-1} = g'_{ij}$$

$$\check{H}^1(\mathcal{U}, G) = \check{Z}^1(\mathcal{U}, G) / \sim$$

Kodaira-Spencer

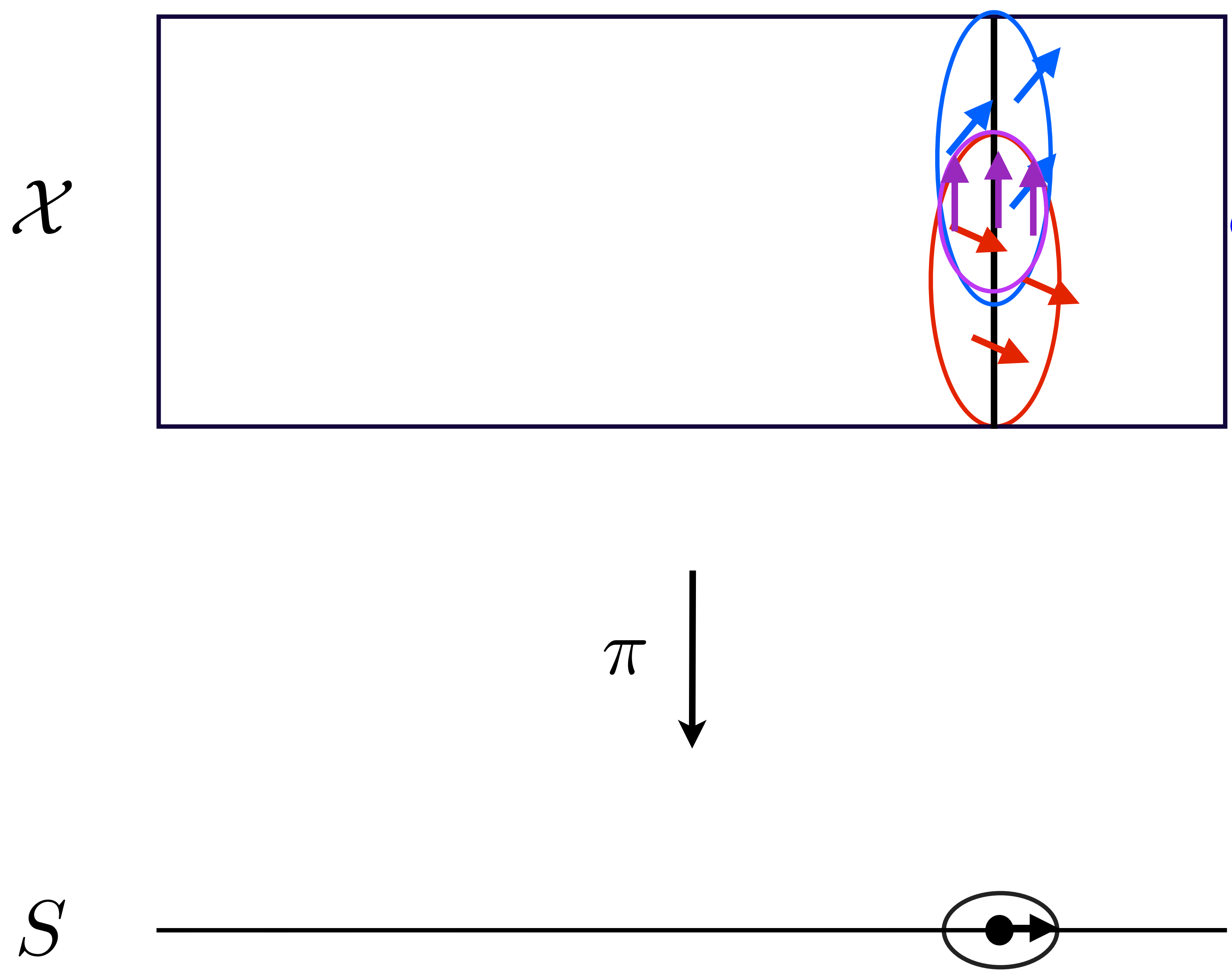
- What do Moduli Spaces look like?
- Moduli Space = Universal Space that parametrizes something (e.g. Curves)
- Kodaira Spencer <---> TANGENT SPACE

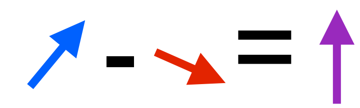
$$\mathrm{KS} : T_P S \to H^1(\mathcal{X}_P, T_{\mathcal{X}_P})$$

$\delta_i \in T_{\mathcal{X}}(U_i)$

$\delta_j \in T_{\mathcal{X}}(U_j)$

$$X_P = \pi^{-1}(P)$$





$\delta_i - \delta_j = \delta_{ij} \in T_{\mathcal{X}_P}(U_{ij})$

$\delta_P \in T_{S,P}$

S = Moduli Space
Well-Defined
LES SES

EXERCISE: Show the dimension of the moduli space of curves of genus g is 3g-3

compute this

$$\dim_K H^1(X, T_{X/K})$$

Need This

$$\dim_K H^0(X, L) - \dim_K H^1(X, L) = \deg(L) - \text{genus}(X) + 1$$

$$T_{X/K} = \omega_{X/K}^{\vee}$$

$$\deg(\omega_{X/K}) = 2g - 2$$

Kodaira-Spencer Map (Differential Alg Style)

$$\mathrm{KS} : \{ \text{ derivations on } K \text{ } \} \to H^1(X, T_X)$$

$\delta : K \to K$ $\quad K =$ field with a derivation

Cover

$$X = \bigcup_i U_i$$

Get Local Lifts

$$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \quad \delta_i|_K = \delta$$

Take Differences

$$\delta \mapsto [\delta_i - \delta_j] \in H^1(X, T_{X/K})$$

Deligne-Illusie Map

$$\mathrm{DI}_0 : \{\ p\text{-derivations on } R\ \} \to H^1(X_0, F^*T$$

$\delta : R \to R$ $R = \widehat{\mathbb{Z}}_p^{ur}$

Cover

$$X = \bigcup_i U_i$$

Get Local Lifts

$$\delta_i : \mathcal{O}(U_i) \to \mathcal{O}(U_i) \qquad \delta_i|_R$$

Take Differences

$$\delta \mapsto [\delta_i - \delta_j] \in H^1(X_0, F^*T_{X_0}) \mod p$$

need to explain this doodad

NOTATION!
F, delta, X_0

RETAIN INFORMATION
MOD P^2 not MOD P^3

Is it injective?
What does it do?
What doesn't it do?
Is it surjective?
How doesn't it do this?

Does this relate to Serre-Tate for Ab?
What is history for F?
Is this f_1?

not surj (missing zero)

Higher Order classes

Defn: a **derivation of the Frobenius** is a map $D : B \to A/p$ such that

$$\begin{aligned} D(a+b) &= D(a) + D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

THEOREM:
The pointwise difference of two p-derivations

$$\delta_1, \delta_2 : A/p^2 \to A/p$$

is a derivation of the Frobenius.

LEMMA: A derivation of the Frobenius

$$D : A/p^2 \to A/p$$

Gives a well-defined derivation of the Frobenius

$$D : A/p \to A/p$$

Derivations of the Frobenius

$$\begin{aligned} D(a+b) &= D(a)+D(b) \\ D(ab) &= D(a)b^p + a^p D(b) \end{aligned}$$

PROOF

$\delta_1, \delta_2 : A \to B \qquad B \in \mathsf{CRing}_A$

$D(a) := \delta_1(a) - \delta_2(a)$ mod p

$$\mathrm{CRAP} = \frac{a^p + b^p - (a+b)^p}{p}$$

Additivity:

$$\begin{aligned} D(a+b) &= \delta_1(a+b) - \delta_2(a+b) \\ &= \delta_1(a) + \delta_1(b) + \mathrm{CRAP} - (\delta_2(a) + \delta_2(b) + \mathrm{CRAP}) \\ &= D(a) + D(b) \end{aligned}$$

Product Rule:

$$\delta_i(ab) = \delta_i(a)b^p + a^p\delta_i(b) + p\delta_i(a)\delta_i(b)$$

Geometric Descent

X/K smooth projective

$K = \overline{K} \qquad \mathrm{char}(K) = 0$

$\delta : K \to K$

$$K^\delta = \{r \in K : \delta(r) = 0\}$$

Theorem

T.F.A.E.

1. $\mathrm{KS}(\delta) = 0$
2. $J^1(X) \cong TX$ as schemes over X
3. $\exists X'/K^\delta$ such that $X' \otimes_{K^\delta} K \cong X$

Descent to the constants

What is X/K?

Arithmetic Descent

$X/\widehat{\mathbb{Z}}_p^{ur}$ smooth

$\delta : \widehat{\mathbb{Z}}_p^{ur} \to \widehat{\mathbb{Z}}_p^{ur}$

How can we get an equation that describes when we have a lift?

Theorem

T.F.A.E.

1. $\mathrm{DI}_0(\delta) = 0$
2. $J^1(X)_0 \cong F^* T_{X_0}$ as schemes over X_0
3. X_1 admits a lift of the p-Frobenius

Descent to the field with one element

X descends to Borger-Buium $\mathbb{F}_1 \implies \mathrm{DI}_0(\delta_p) = 0$

THANK YOU!

How Does Delta Geometry Work?

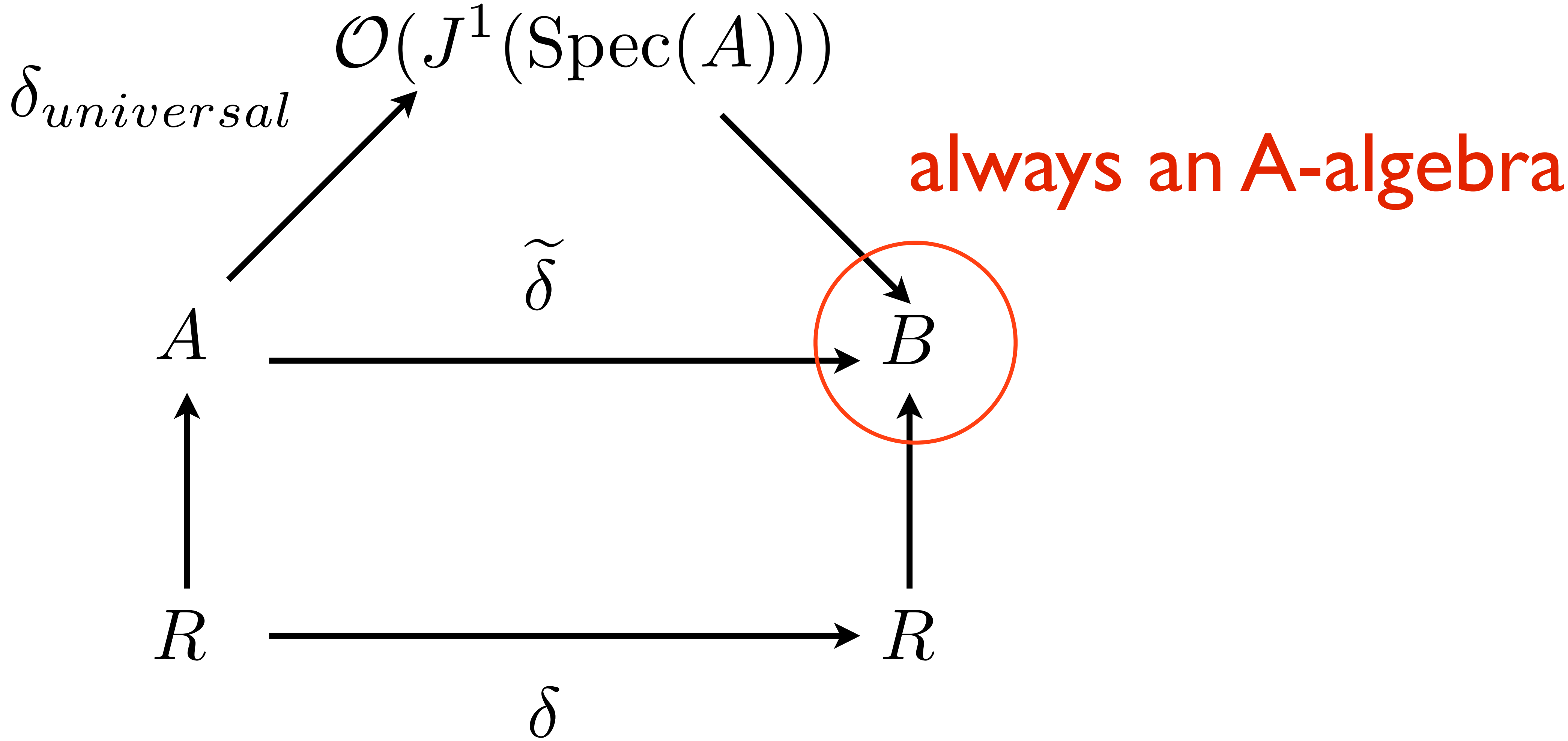
Step 1: Kolchin Style

Step 2: Jet Spaces reformulation

Step 3: Arithmetic Jet Spaces

Universal Property of p-Jet Spaces

$R = \widehat{\mathbb{Z}}_p^{ur}$



Example $\mathbb{A}^1_R = \mathrm{Spec}\ R[x]$ $J^1(\mathbb{A}^1_R) = \widehat{\mathbb{A}}^1_R \hat{\times} \widehat{\mathbb{A}}^1_R$

$\mathcal{O}(J^1(\mathbb{A}^1_R)) = R[x][\dot{x}]\widehat{\ } = R[x]\{\dot{x}\}$ Restricted Power Series

Many birds with one Stone

- Difference Ring
- Differential Ring
- Lambda Ring

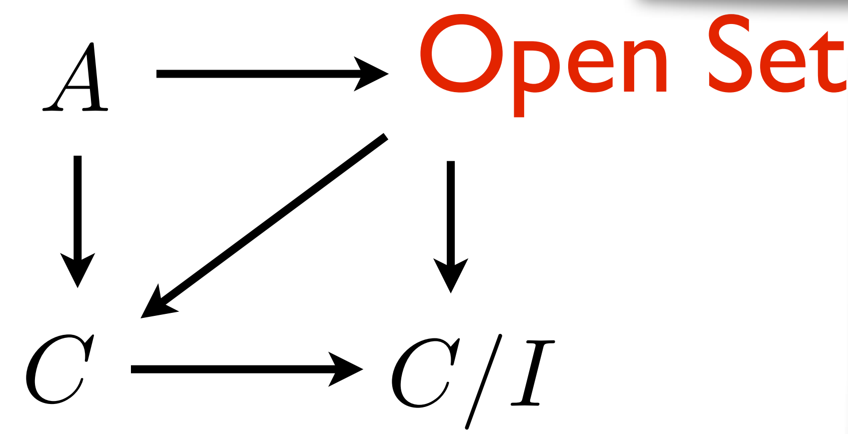
How do we get local lifts?

Get Local Lifts

How can I think about this?

Infinitesimal Lifting Property

$I^2 = 0$



Is this smoothness?

Is this Hensel?

derivation setting

$$C = D_1(B) := B[\varepsilon]/\langle \varepsilon^2 \rangle$$

$$I = \langle \varepsilon \rangle$$

p-derivation setting

$$C = W_1(B)$$

$$I = V(W_1(B))$$

Rigorous Formulation of Descent???

$$\widehat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta : \zeta^n = 1, p \nmid n]\widehat{\ }$$

$$(\widehat{\mathbb{Z}}_p^{ur})^\delta = \{r : \delta(r) = 0\} \qquad \delta(r) = \frac{\phi(r) - r^p}{p}$$

$$= \text{ Monoid of roots of unity}$$

$$:= M$$

$$\mathrm{DI}_0(\delta) = 0 \implies \exists X_1'/M_1 \text{ such that } X_1' \otimes_M \widehat{\mathbb{Z}}_p^{ur}/p^2 \cong X$$