Introduction to p-Jet Spaces

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Abstract

A quick introduction to p-jet spaces. I am going to add more to this and these notes are crude but I've made them available for people trying to learn about p-jets.

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1 p-derivations

1.1 p-derivations

Let A and B be rings, with B an A-algebra. A p-derivation $\delta_p : A \to B$ is a map of sets satisfying the following axioms

$$\begin{aligned} \delta_p(a+b) &= \delta_p(a) + \delta_p(b) + C_p(a,b) \\ \delta_p(ab) &= \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b) \\ \delta_p(1) &= 0 \\ C_p(x,y) &= \frac{x^p + y^p - (x+y)^p}{p} \in \mathbf{Z}[x,y] \end{aligned}$$

The category of rings with p-derivations is called the category of Λ_p-**rings**.

1.2 examples

Let A be a ring and $a \in A$. Recall that we have a well-defined morphism

$$\left[\frac{1}{a}\right] : aA \to A/\mathrm{ann}(a).$$

Example 1.1. $\delta : \mathbf{Z}/p^2 \to \mathbf{Z}/p$ given by $\delta(x) = (x - x^p)/p$ where we interpret $1/p$ as a map

$$\frac{1}{p} : p\mathbf{Z}/p^2 \to \mathbf{Z}/p.$$

Example 1.2. If $R = W_{p,\infty}(k)$ with k perfect of characterisic p then R has a unique lift of the Frobenius ϕ on it. It hence has a unique p-derivation $\delta(x) = (\phi(x) - x^p)/p$.

Theorem 1.3. *Let $R = W_{p,\infty}(k)$ where k is a perfect field of characteristic p.*

1. $\delta_p(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1} \cdot \mathrm{unit}$
2. $\delta_p(p^n \cdot \mathrm{unit}) = p^{n-1} \cdot \mathrm{unit}$
3. $(p^n, \delta_p(p^n), \delta_p^2(p^n), \ldots, \delta^r(p^n))_R = (p^{n-r})_R$

Proof. The first property is trivial. The second property follows from the computation

$$\begin{aligned} \delta_p(p^s \cdot u) &- \delta_p(p^s)u^p + p^{sp}\delta_p(u) + p\delta_p(p^s)\delta_p(u) \\ &= p^{s-1} \cdot u^p + p^{sp}\delta_p(u) + p^s \cdot \mathrm{unit} \cdot \delta_p(u) \\ &= p^{s-1}(\mathrm{unit} + p \cdot \mathrm{junk}). \end{aligned}$$

We prove the last property by induction on r. It is sufficient to show that $\delta_p^r(p^n) = p^{n-r} \cdot \mathrm{unit}$. We have

$$\delta_p(\delta_p^{r-1}(p^n)) = \delta_p(p^{n-r+1} \cdot \mathrm{unit}) = p^{n-r} \cdot \mathrm{unit},$$

where the first equality follows from inductive hypothesis and the second equality follows from the second proposition. □

1.3 First p-jet ring

Define $(-)_{p,1} : \mathsf{CRing} \to \mathsf{CRing}$ by

$$A_{p,1} = A[\dot{a} : a \in A]/(\mathrm{relations})$$

where (relations) are generated by

$$\begin{aligned} (\dot{ab + c}) &= \dot{a}b^p + a^p\dot{b} + p\dot{(a)}\dot{(b)} + \dot{(c)} + C_p(ab, c), \\ C_p(x, y) &= \frac{x^p + y^p - (x+y)^p}{p} \in \mathbf{Z}[x, y], \end{aligned}$$

For all $a, b, c \in A$.

Remark 1.4. If A is an R-algebra and R admits multiple p-derivations we may want to impose that the p-derivation on A extend the one on the base. Suppose $\delta_0 : R \to R$ is such a p-derivation on the base. The additional relation we impose is then $\dot{r} = \delta_0(r)$ where of course these are understood to be taken as an image in A.

In this relative setting we could write $(A/R, \delta_0)_{p,1}$ in place of $A_{p,1}$.

Example 1.5. A/R is finite type,

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r) = R[x]/(f)$$

where $x = (x_1, \ldots, x_n)$, $f = (f_1, \ldots, f_r)$ then

$$A_{p,1} = R[x, \dot{x}]/(f, \dot{f})$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$ and $\dot{f} = (\dot{f}_1, \ldots, \dot{f}_r)$. Here $\dot{(f_1)}, \ldots, \dot{(f_r)} \in R[x, \dot{x}]$ are computing using the rule for linear combinations above.

Theorem 1.6 (Universal Property)**.** *There is a universal p-derivation $\delta_{p,1} : A \to A_{p,1}$ mapping a to $\dot{a}$. It satisfies the following universal property:*

For every p-derivation $\delta : A \to B$ of the ring homomorphism $A \to B$ there exists a unique ring homomorphism $u_\delta : A_{p,1} \to B$ such that

$$\begin{array}{ccc} A & \xrightarrow{\delta} & B \\ & \searrow^{\delta_{p,1}} & \uparrow \\ & & A_{p,1} \end{array} \quad .$$

The ring homomorphism is the morphism of A-algebras defined by $u_\delta(\dot{a}) = \delta(a)$. [1]

Proof. It is clear the the morphism is well-defined from the definitions. ☐

1.4 Data of p-derivations

Lemma 1.7 (flatness over witt vectors= p-torsion free)**.** *Let A be an $R = W_{p,\infty}(k)$ algebra with k-perfect of characteristic p. The following are equivalent*

1. *A is flat over R*
2. *The multiplication by p morphism is injective.*
3. *A is p-torsion free*

Proof. It is clear the (2) and (3) are the same. We will show p-torsion free implies flat. Flatness is equivalent to $I \otimes_R A \to IA$ given by $i \otimes_R a \mapsto ia$ is injective. We have $I = m^n$ for some m where $m = (p)$ is the maximal ideal of R. A general element of $m^n \otimes A$ looks like $\sum_i p^{n'+n_i} \otimes a_i$ with n' be the the gcd of all of the $p^{n'+n_i}$ where we can assume wlog that a_i's are not divisible by any powers of p. Suppose $\sum_i p^{n'+n_i} \otimes a_i \mapsto p^{n'}(\sum_i p^{n_i} a_i) = 0$. Since multplication by p is injective we have $\sum_i p^{n_i} a_i = 0$. This is a contradiction since $\sum_i p^{n_i} a_i$ was cooked up to be a unit.

We will show that flatness implies p-torsion free. We prove the converse by contrapositive: If it is not p-torsion free it will not be flat. Suppose that multiplication by p is not injective on A. This means that the map $pR \otimes_R A \to pA$ is not an injection. This contradicts flatness. ☐

Theorem 1.8. *Let $B \in \mathsf{CRing}_A$, $A \in \mathsf{CRing}_R$ where $R = W_{p,\infty}(k)$ and k is a perfect field of characteristic p. Suppose that A and B are flat over R. The following data are equivalent.*

1. *A p-derivation $\delta : A \to B$ of the algebra map $A \to B$.*
2. *An action $\rho : A \to W_{p,1}(B)$ (meaning a morphism of rings such that $(\pi_{p,1})_B \circ g) = f : A \to B$ the algebra map.*
3. *A morphism of A-algebras $A_{p,1} \to B$.*

Proof. Follows from the definitions. ☐

Example 1.9. Let A and B be rings over $R = W_{p,\infty}(k)$ with k perfect of characteristic p. Suppose $p \neq 2$ and consider the diagram

$$\begin{array}{ccc} A_{p,1} & \xrightarrow{u} & B \\ \uparrow & & \\ A & & \end{array}$$

[1]Warning: The diagram is not a diagram in the categorical sense but it is an exercise to show that the universal property can be formulated in terms of diagrams

This induces $A \to B$.

If $f : A \to B$ is already given and $A = A/p^{n+1}$ then $(A)_{p,1} = (A)_{p,1}/p^n$. This follows from the fact that

$$\delta(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1}(1 - p^{n(p-1)})$$

when p is not a unit.

Hence we have a factorization

$$\begin{array}{ccc} A_{p,1} & \xrightarrow{u} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\pi^*_{p,1}} & A/p^n \end{array}$$

although $f : A \to B$ may not factor through a reduction modulo p^n in general.

Theorem 1.10. *Let B be a p-torsion free ring and ϕ a lift of the Frobenius on B inducing a lift of the Frobenius on $A_n = B/p^{n+1}$. This then induces a well-defined p-derivation*

$$\delta_p : A_n \to A_{n-1}.$$

Proof. In general, given any A and a lift of the Frobenius $\phi : A \to A$, one can try to define

$$\delta_p : A \to A/\mathrm{ann}(p)$$

via

$$\delta_p(a) = (\left[\frac{1}{p}\right] \circ g)(a)$$

where $g(a) = \phi(a) - a^p$, and $g : A \to pA$ at least.

The difficulty in defining δ_p comes from the equality

$$\left[\frac{1}{p}\right](g(a)g(b)) = p \cdot \left[\frac{1}{p}\right](g(a)) \cdot \left[\frac{1}{p}\right](g(b)) \text{ in } A/\mathrm{ann}(p).$$

We leave it to the reader to verify that this makes sense.

It is useful for the reader to note that if $A_n = B/p^{n+1}$ where B is p-torsion free then

$$\begin{aligned} \mathrm{ann}_A(p^j) &\cong p^{n-j}A \\ A/\mathrm{ann}_A(p^j) &\cong A/p^{n-j} \end{aligned}$$

These give maps $[1/p] : pA_n \to A_{n-1}$. ∎

Theorem 1.11. *Let A, B be flat over R. Suppose that A is of finite type over R. Let $f : A \to B$ be a morphism of rings inducing the morphism of rings $f_n : A_n \to B_n$. The following are equivalent*

1. *A lift of the Frobenius $\phi_n : A_n \to B_n$,*

$$\phi_n(a) \equiv f_0(a)^p \mod p$$

2. *A p-derivation $\delta_p : A_n \to B_{n-1}$*

3. *A morphism $(A_{p,1})_{n-1} \to B_{n-1}$ of A_{n-1}-algebras.*

Proof. To see that 2 implies 1 note that $\phi_n(a) := a^p + p\delta(a)$ defines a lift of the Frobenius. We will show that 3 and 2 are equivalent: Let $A = R[x]/(f)$ so that $(A_{p,1})_n = (R[x, \dot{x}]/(f, \dot{f}))/p^n = R_{n-1}[x, \dot{x}]/(f, \dot{f})$. The map clearly defines a p-derivation. (Note: $(\delta_{p,1})_n : A_n \to (A_{p,1})_{n-1}$ is universal).

We will not show 1 implies 2: The follows from what we said previously about p-derivations. ∎

2 p-Jets

2.1 p-jet spaces and p-arc spaces as functors

Let X/R be a scheme where $R = W_{p,\infty}(k)$, with k perfect of characteristic p. define the rth p**-jet functor** $J_{p,r}(X) : \mathsf{CRing} \to \mathsf{Set}$ to be the functor of $W_{p,r}$ valued points of X:

$$J_{p,r}(X)(A) := X(W_{p,r}(A)) \qquad A \in \mathsf{CRing}_R.$$

The natural morphism of ring schemes $\pi_{r,s} : W_{p,r} \to W_{p,s}$ for $r > s$ induce functorial morphisms $J_{p,r}(X) \to J_{p,s}(X)$. The morphisms $\pi_r : W_{p,r} \to \mathcal{O}$ induce functorial morphisms $J_{p,r}(X) \to X$.

Let X/R be a scheme and define Define the p-arc functor $J_p(X) : \mathsf{CRing} \to \mathsf{Set}$ to be the functor which takes $W_{p,\infty}$ valued points of a scheme:

$$J_{p,\infty}(X)(A) := X(W_{p,\infty}(A)) \qquad A \in \mathsf{CRing}_R.$$

Example 2.1. When $X = \operatorname{Spec}(A)$ and A is an R algebra with $R = W_{p,\infty}(k)$ where k is perfect of characteristic p we have that $J_{p,1}(X)$ is representable and

$$J_{p,1}(\operatorname{Spec}(A)) = \operatorname{Spec}(A_{p,1})$$

as schemes over X.

Remark 2.2. Since the constuction $A \mapsto A_{p,1}$ does not localize well one needs to work hard to get that p=jet spaces are representable. In fact, it is an open problem as to whether p-arc spaces are representable.

Example 2.3. Let $R = W_{p,\infty}(k)$ where k is a perfect field of characteritic p. Let $X = \operatorname{Spec}(R[x]/(f))$ (using multi-index notation). There are no sections of the morphism of R-schemes

$$J_{p,1}(X)_0 = J_{p,1}(X_1) \overset{\pi_{p,1}}{\longleftrightarrow} X_1 .$$

This would correspond to a map of rings

$$s^* : R[x, \dot{x}]/(f, p^2, \dot{f}, (\dot{p^2})) = R_0[x, \dot{x}]/(f, \dot{f}) \to R_1[x]/(f) = R[x]/(f, p^2).$$

Theorem 2.4. *Let X/R be an affine scheme which is flat over R (so that multiplication by p is injective).*

1. *The natural morphism $\pi_{p,m,s} : J_{p,m}(X_n) \to J_{p,s}(X_n)$ factors through reduction modulo p^{n-m+1},*

$$J_{p,m}(X)_{n-m} \xrightarrow{(\pi_{p,r+s,s})_{n-r-s}} J_{p,s}(X)_{n-m} .$$

   This is morphism of schemes over R_{n-m}.

2. *Sections of the morphisms*

$$J_{p,1}(X)_m \underset{s}{\overset{\pi_{p,1}}{\longleftrightarrow}} X_n$$

   provide lifts of the Frobenius/p-derivations

$$\delta : \mathcal{O}(X_{n+1}) \to \mathcal{O}(X_n) = \mathcal{O}(X_{n+1})/p.$$

Proof. Let $X = \operatorname{Spec}(R[x]/(f))$ (using multi-index notation). The map $\pi_{m,s}$ gives a map of rings

$$R[x, \dot{x}, \dots, x^{(s)}]/(f, \dot{f}, \dots, f^{(s)}) = \mathcal{O}(J_{p,s}(X)) \to \mathcal{O}(J_{p,m}(X)) = R[x, \dot{x}, \dots, x^{(m)}]/(f, \dot{f}, \dots, f^{(m)}).$$

The first part of the proposition follows from an explicit description of the ideals given previously and the second part follows from the characterization of lifts of the Frobenius on rings of the form $A_n = B/p^{n+1}$. $\square$

Theorem 2.5. *Let X/R be flat where $R = W_{p,\infty}(k)$ where k is perfect and characteristic $k = p$.*

1. *If $i : U \hookrightarrow X$ is an open immersion of affine R-schemes of finite type then*
$$J_{p,r}(i)_n : J_{p,r}(U)_n \hookrightarrow J_{p,r}(X)_n$$
is an open immersion.

2. *If $j : Z \hookrightarrow X$ is a closed immersion of affine R-schemes of finite type then*
$$J_{p,r}(j) : J_{p,r}(Z) \hookrightarrow J_{p,r}(X)$$
is also a closed immersion.

Proof. Let $X = \mathrm{Spec}(A)$ with $A = R[x]/(f)$ (using multi-index notation). It is enough to show that the functor $J_{p,r}()_n$ respects principal open immersions.

$$\begin{aligned}(A_{p,1})_g &= (R[x,\dot{x}]/(f,\dot{f}))_g \\ (A_g)_{p,1} &- R[x,\dot{x},1/g(\dot{1/g})]/(f,\dot{f})] \\ &= R[x,\dot{x},1/g,\frac{-\dot{g}}{g^{2p}}\sum_{j\geq 0}\left(\frac{p\dot{g}}{g^p}\right)^j]/(f,\dot{f})\end{aligned}$$

so we clearly have
$$(A_{p,1})_g \hookrightarrow (A_g)_{p,1}.$$
Reducing modulo p^{n+1} gives
$$\frac{-\dot{g}}{g^{2p}}\sum_{j\geq 0}\left(\frac{-p\dot{g}}{g^p}\right)^j \in ((A_{p,1})_g)_n.$$
This shows
$$((A_{p,1})_g)_n = ((A_g)_{p,1})_n.$$

For the second part
$$\begin{aligned} X &= \mathrm{Spec} R[x]/(f), \qquad J_{p,1}(X) = \mathrm{Spec} R[x,\dot{x}]/(f,\dot{f}) \\ Z &= \mathrm{Spec} R[x]/(f,g) \quad J_{p,1}(X) = \mathrm{Spec} R[x,\dot{x}]/(f,g,\dot{f},\dot{g})\end{aligned}$$
and it is clear that $\dot{f}$ and $\dot{g}$ give extra elements of the ideal. □

The above proposition implies that $J_{p,r}(Y)_n = (\pi_{p,r}^{-1})_n(Y_n)$ if $Y \hookrightarrow X$ is an open or closed immersion of R-schemes when X is flat.

2.2 p-jets as schemes

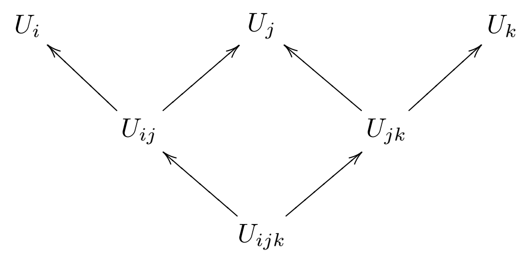
Theorem 2.6 (Buium)**.** *Let $R = W_{p,\infty}(k)$ where k is perfect of characteristic p.*

1. *Let X/R be a flat scheme. The functor $J_{p,r}(X) := X \circ W_{p,r}$ over X, is representable when reduced modulo p for every n,*

2. *Furthermore for every A in CRing_{R_n} we have*
$$J_{p,r}(X)_n(A) = J_{p,r}(X)(A) = X(W_{p,r}(A)) \to X(A) = X_n(A)$$
where the map is $(\pi_{p,r})_A$.

Proof. It is enough to show that $J_{p,r}(X)_n$ is a sheaf on the Zariski site.

Let $(U_i \to X)$ be a Zariski open cover. Let $\widetilde{U}_{ijk}, \widetilde{U}_{kij}, \widetilde{U}_{jki}$ denote U_{ijk} viewed as an open subscheme of U_i, U_j and U_j repsectively. Let $\widetilde{U}_{ij}, \widetilde{U}_{ji}$ be U_{ij} viewed as an open subscheme of U_i and U_j respectively and let α_{ij} be the isomorphism between the two. we have the diagram



We have $\alpha_{ij}\alpha_{jk} = \alpha_{ki}$ where $\alpha_{ij} : J_{p,r}(\widetilde{U}_{ijk}) \to J_{p,r}(\widetilde{U}_{ijk})$.

Since $J_{p,r}(-)_n$ preserves open immersions, we have that $J_{p,r}(-)_n$ gives a diagram satisfying the cocycle condition.

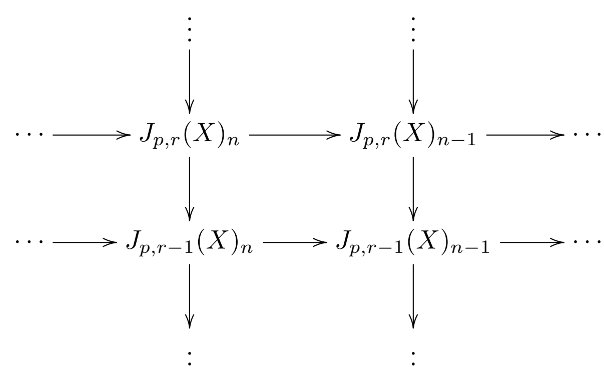
This tells us that morphism patch and that $J_{p,r}(X)_n$ an be viewed as a sheaf in the Zariski topology. $\square$

A more difficult theorem of Borger proves the following:

Theorem 2.7 (Borger)**.** *Let X/R be a scheme and $R = W_{p,\infty}(k)$ with k perfect of characteristic p. The functor $J_{p,r}(X) := X \circ W_{p,r}$ is representable.*

2.3 Various limits of the of p-jet construction

The construction gives a system of maps



One can define the following limits

$$\begin{aligned} \widehat{J}_{p,r}(X) &:= \varinjlim J_{p,r}(X)_n \quad \text{(Buium's } p\text{-formal models)} \\ J_{p,\infty}(X)_n &:= \varprojlim_r J_{p,r}(X)_n \\ \widehat{J_{p,\infty}(X)} &:= \varinjlim_n J_{p,\infty}(X)_n \\ \widehat{J}_{p,\infty}(X) &:= \varprojlim_r \widehat{J}_{p,r}(X) \end{aligned}$$

These exist as Ind-schemes and it is not clear that any of the are representable in the category of schemes.

Remark 2.8. 1. The p-formal schemes $\widehat{J}_{p,r}(X)$ used by Buium behave nicely. In some sense this means that the appropriate place for p-jet spaces would be some variant of the p-adic rigid analytic spaces. No such study of p-jet spaces as rigid analytic objects has been undertaken to our knowledge.

2. We do not understand if $J_{p,\infty}(X)_n$ is representable in the sense. We do not know if $J_{p,\infty}(X)$ exists, hence if the reduction of the limits equals to limit of the reductions. The also implies that we don't know if $\widehat{J}_{p,\infty}(X) = \widehat{J_{p,\infty}(X)}$ since the conjectured scheme we are completing on the right hand side is not known to exist.

3. Buiums's p-jet spaces the p-adic completion of Borger's jet spaces in the following sense: $J_{p,r}(X) \otimes_R R_n \cong J_{p,r}(X)_n$ and $\widehat{J}_{p,r}(X) = \widehat{J_{p,r}(X)}$ when X/R is of finite type.

2.4 examples