Jet Spaces and Diophantine Geometry

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Abstract

Here is how jet spaces got involved in Diophantine geometry. This note didn't really have a place anywhere else.

In [Mor22] it was conjectured that algebraic curves like $y^2 = x^5 + 1$ or $x^n + y^n = 1$ (for $n \geq 3$) have only finitely many rational solutions. More precisely for an algebraic curve C defined over $\mathbb{Q}$ of genus $g > 2$ Mordell conjectured that $\#C(\mathbb{Q}) < \infty$. This problem was generalized by Lang in [Lan60] to the Mordell-Lang problem where he conjectured that a curve in an abelian variety A can only intersect a finitly rank subgroup of $A(\overline{K})$ in finitely many $\overline{K}$-points if K has characteristic zero. This implies the Mordell conjecture since we can always embed a curve into its Jacobian A (an abelian variety) and the group of rational points $A(\mathbb{Q})$ is finitely generated (by the Mordell-Weil Theorem), and the Lang conjecture gives

$$\#C(\mathbb{Q}) = \#(A(\mathbb{Q}) \cap C(\mathbb{Q})) < \infty.$$

A famous variant is the Manin-Mumford problem where $G = A(\overline{K})_{tors}$, the torsion points. When K is a function field the Lang conjecture was first resolved in [Man65], the Manin-Mumford problem was resolved in [Ray83] and Mordell-Lang problem was resolved in [Fal83].

Jet spaces come into the picture when we ask about effectivity — meaning explicit bounds on solutions of Diophantine problems. The idea is roughly that closed subset of jet spaces of a variety correspond to subsets of the original variety cut-out by differential polynomials and that allowing the extra operation of differentiation in the defining equations refines the class of sets we can apply theorems from Algebraic Geometry to. An effective version of the Mordell-Lang problem over a function field K was given by my advisor in [Bui93] where he used "jet space methods". To prove

$$\#(G \cap C(\overline{K})) < N(!)^{6N+6},$$

where $N = \max\{g, r, 4\}$ where g is the genus of the curve C and $r = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$ and $(!)^n$ means iterating the factorial opertation n times.[1] For a fixed prime p my advisor introduced a theory of p-derivations and p-jet in [Bui96] to extend jet space methods to the number field setting; p-derivations are viewed philosophically as "derivations relative to the field with one element" and mathematically are just maps of sets between rings which satisfy a product rule and a sum rule. Using jet space methods he showed that if $X/\mathbb{Q}$ is a curve of genus $g \geq 2$ and $A/\mathbb{Q}$ is its Jacobian then

$$\#(X(\overline{\mathbb{Q}}) \cap A(\overline{\mathbb{Q}})_{tors}) < p^{4g} \cdot 3^g \cdot [p(2g-2) + 6g]g!,$$

where p is the smallest prime of good reduction greater than $2g$. This proof was adapted in [Poo01] to give an algorithm for computing torsion points on curves.

[1] By "over a function field" we mean that C is non-isotrivial.

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