SZEGÖ CURVES, STEEPEST DESCENT ANALYSIS AND THE ZERO BEHAVIOR OF PARTIAL SUMS OF THE EXPONENTIAL FUNCTION

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Abstract. In 1924 Szegö showed that the zeros of normalized partials sums of the exponential function $s_n[e^s](nz)$ accumulate on a curve, now called the Szegö curve. Specifically, he showed that a point is an accumulation point for the zeros of $s_n[e^s](nz)$ as $n \to \infty$ if and only if the point is on the Szegö curve. The present paper derives large n asymptotics for the error term for the partial sums (using the methods of steepest descent) which allows one to discuss how these roots approach the Szegö curve. In the present paper we derive these results for the exponential function using the method of steepest descent. The method used generalizes to several other functions.

1. Introduction to Finding Zeros of Partial Sums of Taylor Series

The nth order Taylor approximant of a function, $s_n[f](z)$, has at most n roots. In the case of the exponential function, the nth partial sum is

$$s_n[e^s](z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}.^1 \tag{1.1}$$

Since the actual function e^z has no roots and the partial sum $s_n[e^s](z)$ has n roots, where do these roots go as $n \to \infty$? We've plotted the zeros of the 10th, 20th, 30th, and 40th degree partial sum for the exponential function in figure 1. As is easily observed from the picture the roots 'fly of to infinity' as $n \to \infty$. The asymptotic behavior of the roots of partial sums for the exponential function was a problem which was first studied by Szegö (1924) [18].[2] Szegö considered the behavior of the roots of the normalized partial sums $s_n[e^s](nz)$. He showed that the roots of

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[1] We denote the nth partial sum by $s_n[f(s)](z)$ or $s_n[f](z)$. The 's' in '$[f(s)]$' is a dummy variable. The operator notation allows one to write expressions like $s_n[(e^s + e^{e^s})/\Gamma(s)](z)$ easier.

[2] Much of the theory that Szegö, Polya and others used in their asymptotic analysis was developed much earlier by Alexander and others. The paper [1] provides a good starting point for a discussion of the history of these methods. Also, a proof that these roots became unbounded as $n \to \infty$ was established first by Eneström [6] and later independently by Kayeya [11] as a corollary of the famous Eneström-Kayeya theorem. Since Eneström and Kayeya, much progress has been made on inequalities for roots of polynomials. The reader is referred to [8], as it is an excellent resource for polynomial inequalities.

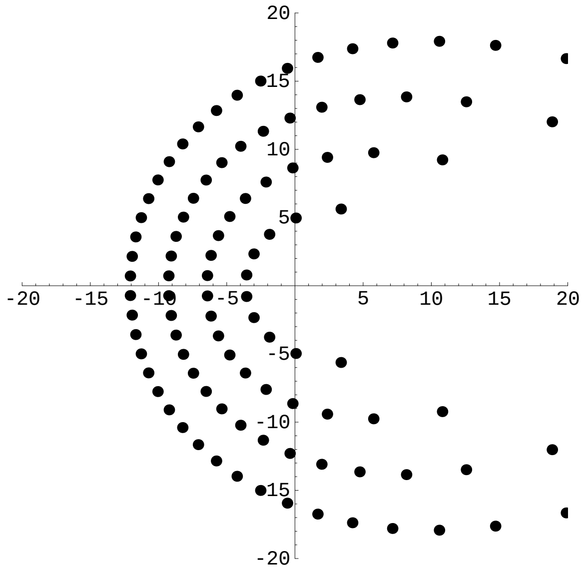

FIGURE 1. Each of the concentric rings correspond to a Taylor polynomial of varying degree; The ring of roots for the higher order polynomials are farther out than the ring of roots for polynomials of lower degree.

$s_n[e^s](nz)$ accumulate on a the so called Szegö curve given by the equation[3]

$$D_\infty = \left\{ \zeta \in \mathbb{C} \mid \left| e^{-\zeta+1}\zeta \right| = 1 \text{ and } |\zeta| \leq 1 \right\}. \tag{1.2}$$

In the 1924 paper, Szegö also considered the roots of the partial sums for the partial sums for sine and cosine functions and showed their accumulation curves as well. (see figure 2).

The question of *rate* of convergence is discussed in an early paper by Buckholtz [3].

Definition 1.1. For two sets A and B the distance function

$$d_h(A,B) := \min \left\{ \sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b| \right\} \tag{1.3}$$

is the *minimal hausdorff distance* and the distance function

$$d_H(A,B) := \max \left\{ \sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b| \right\} \tag{1.4}$$

is the *maximal hausdorff distance*.

If we let

$$R_n[e^s] = R_n = \{ z \in \mathbb{C} \, | s_n[e^s](nz) = 0 \} \tag{1.5}$$

Buckholtz showed

$$d_h(R_n, D_\infty) \leq \frac{2e}{\sqrt{n}}, \tag{1.6}$$

[3]Following the notation of Varga et al

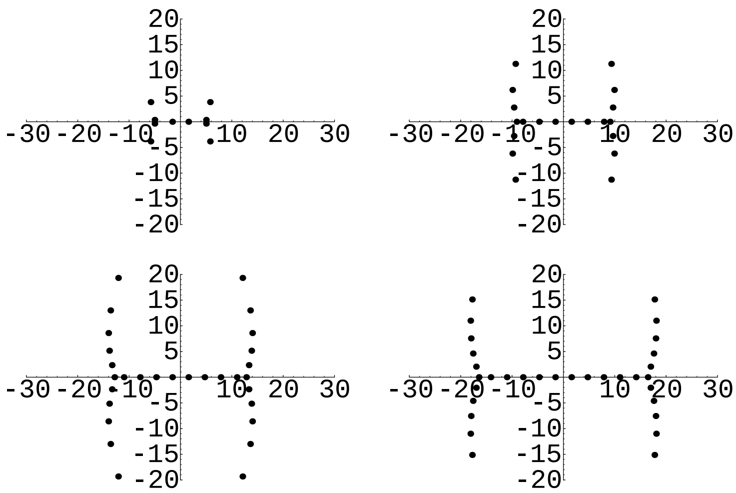

FIGURE 2. These are the roots of the Taylor Approximations for the Cosine function. As we see in the above function, some of the roots tend to the actual roots of the cosine function which occur at at $n\frac{\pi}{2}$ for $n \in \mathbb{Z}$.

as for all n.

This estimate was improved in 1991 a paper by Varga, Carpenter and Waldvogal [20].[4]

Proposition 1.2. *For all $\delta > 0$ there exists some $C \in \mathbb{R}^+$ such that for all n*

$$d_h(R_n \backslash B_\delta(1), D_\infty) \leq C\frac{\log(n)}{n}. \tag{1.7}$$

Or in big 'oh' notation

$$d_h(R_n \backslash B_\delta(1), D_\infty) = O\left(\frac{\log(n)}{n}\right) \tag{1.8}$$

as $n \to \infty$.

This says if one considers the roots away from the point $1 \in D_\infty$ then roots approach the curve like $\log(n)/n$ uniformly.

In the same paper they introduced a family of contours

(1.9)

$$D_n = \left\{ z \in \mathbb{C} \mid \frac{n!e^n\sqrt{2\pi n}}{n^n}\left|\frac{1-z}{z}\right|, |z| < 1, \arg(z) = \arccos(\frac{n-2}{2}), -\pi \leq \arg(z) \leq \pi \right\},$$

which have the property that $D_n \to D_\infty$ as $n \to \infty$. Using these contours and made the following improvement on the location of the roots:

Proposition 1.3. *For all $\delta > 0$*

$$d_h(R_n \backslash B_\delta(1), D_n) = O\left(\frac{1}{n^2}\right) \tag{1.10}$$

as $n \to \infty$.

[4] If the reader is unfamiliar with big 'oh' notation they may jump to section 3 for a definition.

Figure 3 shows these curves.

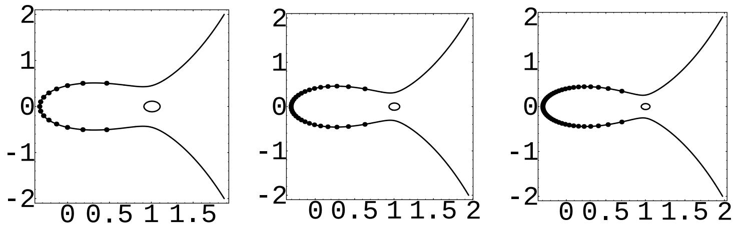

FIGURE 3. Dynamic Szegö curves for R_{10}, R_{15}, and R_{20}.

Remark 1.4. The actual papers of Buckholtz, Saff and Varga et al do not use d_h and use a sup inf distance between a finite set of points P and the contour C.

$$d(P, C) := \max \{ d_\infty(p, C) \, | p \in P \},$$

where

$$d_\infty(p, C) := \min \{ |p - c| \, | c \in C \}.$$

It is easy to show that d and d_h give the same values for distances between a finite set of points and a contour C and we prefer d_h only because it's commutative. Further because the points accumulate on D_∞ this implies converges in d_H as well.

The results on the exponential function then extended in a paper by Kappert[12] where he calculated the "dynamic Szegö curves" for trigonometric functions. A series of papers by Varga et al. [14], [15], [16] give the asymptotics for the zeros and poles of Padé approximations to e^z, as well as inversitgate number theoretic properties for the zeros of the partial sums of the Taylor Series. A good review of Szegö's original method is given in [15].

Since Szegö numerous papers have been published generalizing his results his ideas to a rich family of limiting rational functions including Laurent Series [5], Cesaro Sums [17], Gauss hypergeometric polynomials [10], special orthogonal and combinatorial polynomials: Legendre Polynomials, Laguerre Polynomials [2], Euler Polynomials [21], Chebyshev Polynomials [7], Jacobi Polynomials [4], Hermite Polynomials [9], Fabor polynomials [13], as well as a class of polynomials related to Daubechies Wavelets [19]. In addition studies of particular properties such as root density have been studied [8] [22].

In the present paper we apply the method of steepest descent to obtain these results as it yields to generalization for other functions easier.

2. The Integral Representation of the Taylor Series

We first require the Cauchy Integral Formula.

Theorem 2.1 (Cauchy Integral Formula)**.** *If γ is a simple closed curve, $z \in \gamma^+$, and f is analytic in a region containing γ and its interior then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s - z} ds. \tag{2.1}$$

From the Cauchy integral formula we may derive Taylor's Theorem.

Theorem 2.2 (Taylor's Theorem for Complex Variables)**.** *If A is an open set, $f : A \subset \mathbb{C} \to \mathbb{C}$ analytic, and $z_0 \in A$, Then $\exists R > 0, \forall z \in A$*

$$|z - z_0| < R \implies f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \tag{2.2}$$

The number R is the radius of convergence of the series; R is the supremum of all radii r for which the sequence converges whenever $z \in \{z|\ |z - z_0| < r\}$.

Also, If $\gamma \subset A$ is a closed curve,$\gamma^+ \subset A$ and $z_0 \in \gamma^+$, then the partial sum of equation 2.2 can has the representation

$$s_n[f](z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)} - \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)}\left(\frac{z-z_0}{s-z_0}\right)^{n+1} ds. \tag{2.3}$$

Note that in equation 2.3, the contour only has to contain z_0 and that there is no restriction on z; formula 2.3 is valid for all z! Introducing γ is only a result of an application of the Cauchy integral formula for derivatives in equation 2.2 and has nothing to do with the chosen z.

Equation (2.3) yields useful expression of the error term of Taylor polynomials. For $z \in \gamma^+$ one has

$$f(z) - s_n[f; z_0](z) = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)}\left(\frac{z-z_0}{s-z_0}\right)^{n+1} ds. \tag{2.4}$$

Similarly $z \in \gamma^-$ gives

$$s_n[f; z_0](z) = \frac{-1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)}\left(\frac{z-z_0}{s-z_0}\right)^{n+1} ds. \tag{2.5}$$

We will use these expressions for the error terms later.

3. Asymptotics

In this section we introduce the necessary theory for deriving an asymptotic expansion for the integral

$$\frac{-1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z)}\left(\frac{z-z_0}{s-z_0}\right)^{n+1} ds \tag{3.1}$$

which is valid for large n. To do this we will apply the method of steepest descent, which we will state at the end of this section.[5]

We first must recall some basic notions used in Asymptotic Analysis.

Definition 3.1. **Big 'oh' notation:** A function f is said to be *big 'oh'* of g in the sector $\alpha < \arg(z) < \beta$ where $\alpha, \beta \in \mathbb{R}$ provided $\exists C \in \mathbb{R}^+, \exists M \in \mathbb{R}^+, \forall z \in \mathbb{C}$

$$|z| \geq M \text{ and } \alpha \leq \arg(z) \leq \beta \implies |f(z)| \leq C|g(z)|. \tag{3.2}$$

We write $f(z) = O(g(z))$ as $z \to \infty$.

[5]A proof can be found at http://www.u.arizona.edu/ tdupuy or in a less general form in the internet supplement for *Basic Complex Analysis* by Marsden and Hoffman.

Little 'oh' notation: A function f is said to be *little 'oh'* of g in the sector $\alpha < \arg(z) < \beta$ where $\alpha, \beta \in \mathbb{R}$ provided $\forall \varepsilon > 0, \exists R \in \mathbb{R}^+, \forall z \in \mathbb{C}$

$$|z| \geq R \text{ and } \alpha \leq \arg(z) \leq \beta \implies |f(z)| \leq \varepsilon |g(z)|. \tag{3.3}$$

We write $f(z) = o(g(z))$ as $z \to \infty$.

Note that $f(z) - g(z) = o(1)$ means $\lim_{z\to\infty} |f(z) - g(z)| = 0$.

Definition 3.2. For two functions f and g, we say f is *asymptotic to* g if $f(z) = g(z)(1 + o(1))$ as $z \to \infty$. Stated in more basic terms, $\forall \varepsilon, \exists R \in \mathbb{R}^+$ such that

$$|x| \geq R \implies \frac{|f(x) - g(x)|}{|g(x)|} < \varepsilon. \tag{3.4}$$

We write $f \sim g$ as $z \to \infty$.

Note that $f \sim g$ is an equivalence relation. We have a similar definition for asymptotic series.

Definition 3.3. f *is asymptotic to the series* $S_n(z) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}$, or $S_n(z)$ *is an asymptotic series for* f if $\forall n \in \mathbb{N}$

$$f(z) - S_n(z) = o\left(\frac{1}{z^n}\right). \tag{3.5}$$

We write

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{3.6}$$

as $z \to \infty$.

Note that the above definition requires a function to have a limit as $z \to \infty$, namely $f(z) \to a_0$ as $z \to \infty$. In order for the definition to apply at all it must have a limit! This may seem rather restrictive but was developed exactly for the purpose of obtaining more information about convergence:

If $f(z) \to A$ as $z \to z_0$ we know that

$$\lim_{z_0 \to z} f(z) - A = 0 \text{ and } \lim_{z\to z_0} (z - z_0) = 0 \text{ gives } \lim_{z\to z_0} \frac{(f(z) - A)}{(z - z_0)} = B.$$

Repeating this process of multiplying something which vanishes by something that blows up allows us to get more limits

$$\lim_{z\to z_0} \lim_{z\to z_0} \frac{(f(z) - A)}{(z - z_0)} - B = 0 \text{ and } \lim_{z\to z_0} (z - z_0) = 0 \text{ gives } \lim_{z\to z_0} \frac{\frac{(f(z)-A)}{(z-z_0)} - B}{(z - z_0)} = C.$$

Unraveling the last couple of steps gives

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} A + B(z - z_0) + C(z - z_0)^2.$$

Asymptotic series are unique, you can add them, you can multiply them and it is possible for two different functions to have the same series.

In order to extend the definition we consider expressions of the form

$$g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right).$$

Where $\frac{f(z)}{g(z)} \to a_0$.

Definition 3.4. *f has the asymptotic expansion* $g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right)$ if

$$f(z) = g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n} + o\left(\frac{1}{z^n}\right)\right). \tag{3.7}$$

We write

$$f(z) \sim g(z)\left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right), \tag{3.8}$$

as $z \to \infty$.

Note that this definition corresponds to definition 3.2.

Theorem 3.5 (Steepest Descent)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve. Let h be continuous along γ, g be a function which is continuous and bounded along γ, $\zeta_0 = \gamma(t_0)$ be a critical point of h on γ with g analytic at ζ_0 and $g(\zeta_0) \neq 0$.*

Also let the following conditions hold when $|z| \geq R$ and $\arg(z)$ is fixed:

(1) $\int_\gamma e^{zh(\zeta)} g(\zeta) d\zeta < \infty$.
(2) $\mathrm{Re}(h)$ *has a maximum on γ at ζ_0.*
(3) $\mathrm{Im}(h) = \mathrm{Im}(\zeta_0)$ *in a neighborhood of ζ_0.*

Provided all of the above conditions have been satisfied the function

$$F(z) = \int_\gamma e^{zh(\zeta)} g(\zeta) d\zeta \tag{3.9}$$

has the asymptotic expansion

$$F(z) \sim e^{zh(\zeta_0)} \frac{\sqrt{2\pi}}{\sqrt{z}} \left(a_0 + \frac{a_2}{z} + \frac{a_4 \cdot 1 \cdot 3}{z^2} + \frac{a_6 \cdot 1 \cdot 3 \cdot 5}{z^3}\right) \tag{3.10}$$

as $z \to \infty$ and $\arg(z)$ fixed. (An explanation of these terms and a simplification of this expression is given in the remark below.)

Upon factoring out a_0 of the expansion in 3.10 we get

$$F(z) \sim e^{zh(\zeta_0)} \frac{\sqrt{2\pi}}{\sqrt{z}} \frac{g(\zeta_0)}{\sqrt{-h''(\zeta_0)}} \left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots\right). \tag{3.11}$$

The sign of the square roots are chosen so that $\sqrt{z}\sqrt{-h''(\zeta_0)} \cdot \gamma'(t_0) > 0$.

Remark 3.6. In the derivation of formula 3.10 one defines a function w implicitly by

$$h(\zeta) = h(\zeta_0) + [w(\zeta)]^2.$$

The function w which is shown to be locally one-to-one and analytic. Using the inverse function theorem, the terms $a_0, a_1, a_2 \cdots$ are the coefficients of a Taylor series for a function

$$G(y) := \sqrt{z} \cdot g(\zeta) \cdot w'(\zeta) \tag{3.12}$$

centered at ζ_0 where $y = \sqrt{z}w(\zeta)$. One can see that the Taylor series for G centered at zero corresponds to the expansion of $\sqrt{z} \cdot g(\zeta) \cdot w'(\zeta)$ centered at ζ_0 since $w^{-1}(0) = \zeta_0$.

4. Application of the steepest descent method to the integral representation of the partial sums

The analysis of the partial sums begins with equation 2.4:

$$s_n[e^s](z) = \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-z}ds - \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-z}\left(\frac{z}{s}\right)^{n+1} ds.$$

Denote the error for the partial sum of the Taylor series by $E_n(z)$ (c.f. (2.4)):

$$E_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-z}\left(\frac{z}{s}\right)^{n+1} ds. \tag{4.1}$$

The analysis will proceed as follows:

1. (1) Normalize the partial sums of e^z by some parameter λ_n so that the zeros of the partial sums do not diverge as $n \to \infty$ (Eneström-Kayeya theorem). One considers the normalized partial sums $s_n[e^s](\lambda_n z)$ in order to fix the zeros.
2. (2) Get $E_n(\lambda_n z)$ as given by 4.1 into steepest decent form in order to derive large n asymptotics. This is done by using algebra to get the integral into steepest decent form 3.9.
3. (3) Next, choose a contour for the integral that satisfies the steepest descent conditions found in theorem 3.5.
4. (4) Apply steepest descent to yield the asymptotic expansion.

4.1. Normalization of the partial sums.

In figure 4 we have displayed the roots of the function e^z for the 10th, 20th, 30th, and 40th Taylor polynomials. If we let $M_n = \max\{|z| \mid s_n[e^s](z) = 0\}$, As seen from figure 4, $M_n \to \infty$ as $n \to \infty$. So we want to find $\lambda_n \in \mathbb{R}$, a sequence of numbers, such that all the zeros of $s_n[e^s](\lambda_n z)$ lie don't diverge. The polynomials $s_n[e^s](\lambda_n z)$ are called the *normalized partial sums* of the Taylor series.[6] Again we note that if Z is a zero of nth partial sum, $s_n[e^s](Z) = 0$, then Z/λ_n is the zero in the normalized partial sum.

Thus we have

$$s_n[e^s](\lambda_n z) = \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-\lambda_n z}\, ds - \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-\lambda_n z}\left(\frac{\lambda_n z}{s}\right)^{n+1} ds. \tag{4.2}$$

Let us turn our attention to the error of the approximating series.

$$E_n(\lambda_n z) = \frac{1}{2\pi i}\int_\gamma \frac{e^s}{s-\lambda_n z}\left(\frac{\lambda_n z}{s}\right)^{n+1} ds. \tag{4.3}$$

4.2. Placing the integral representation in steepest descent form.

As stated in the summary, we change the variable of integration to $s = \lambda_n \zeta$:

$$E_n(\lambda_n z) = \frac{z^{n+1}}{2\pi i}\int_\gamma \frac{e^{\lambda_n \zeta}}{\zeta - z}\left(\frac{1}{\zeta}\right)^{n+1} d\zeta.$$

Remark 4.1. Recall that as this point γ is arbitrary. γ will will be chosen later so that it will be a fixed curve in the re-scaled variables.

[6] the business of finding a normalization λ_n is not essential to the problem but rather provides motivation for seeking a normalization in the first place

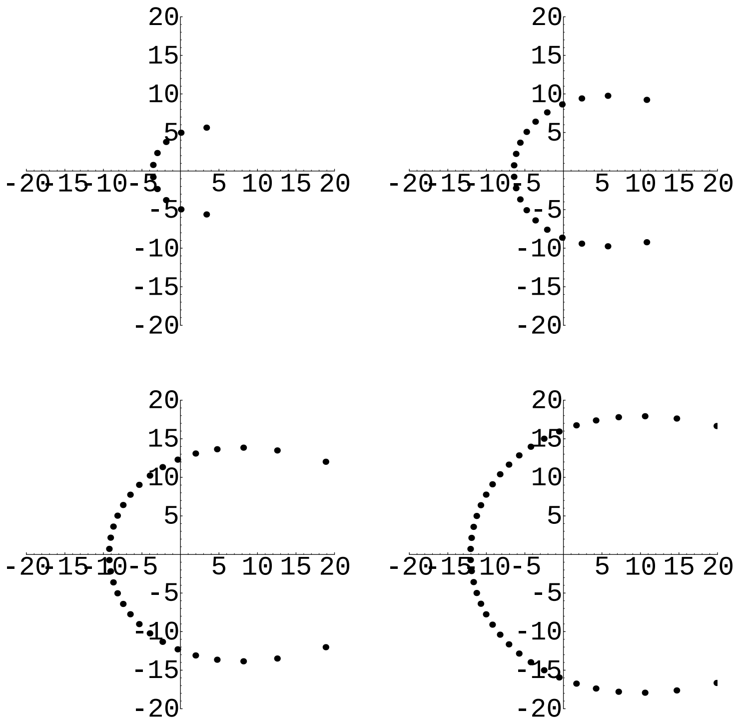

FIGURE 4. These are the roots of the 10th, 20th, 30th, and 40th degree appoximation for e^z

$$E_n(\lambda_n z) = \int_\gamma \frac{e^{\lambda_n \zeta - n\log\zeta}}{\zeta - z}\frac{1}{\zeta}d\zeta. \tag{4.4}$$

Setting $\lambda_n = n$ places the integral in steepest descent form 3.9:

$$E_n(\lambda_n z) = \int_\gamma e^{n(\zeta - \mathrm{Log}\,\zeta)} \cdot \frac{1}{(\zeta - z)\zeta}d\zeta = \int_\gamma e^{nh(\zeta)}g(\zeta)d\zeta. \tag{4.5}$$

where $h(\zeta) = \zeta - \mathrm{Log}(\zeta)$ and $g(\zeta) = g(\zeta, z) = \frac{1}{(\zeta - z)\zeta}$.

4.3. **Contours Which Satisfy the Steepest Descent Conditions.** We now choose a contour that satisfies the conditions in theorem 3.5.

There is one critical point at $\zeta_0 = 1$.

In order for the contour γ to have steepest descent form one must satisfies the conditions given in theorem 3.5. This means γ must pass through the critical point $\zeta_0 = 1$, have $\mathrm{Im}(h(\gamma)) = \mathrm{Im}(h(1)) = 0$ in a region around 1 and achieve a maximum for $\text{Re}(h(\gamma))$ at 1.[7]

$$\text{Im}[\zeta - \text{Log}(\zeta)] = y - \text{Arg}(z) = 0 \tag{4.6}$$

Thus, in order to satisfy the conditions in theorem 3.5, near 1, γ must be given by

$$\gamma_{im} = \{x + iy \in \mathbb{C} \mid y - \text{Arg}(z) = 0\} = \left\{ \frac{\sin(\theta)}{\theta} e^{i\theta} | \theta \in (-\pi, \pi] \right\}. \tag{4.7}$$

Figure 5 shows a plot of this contour.

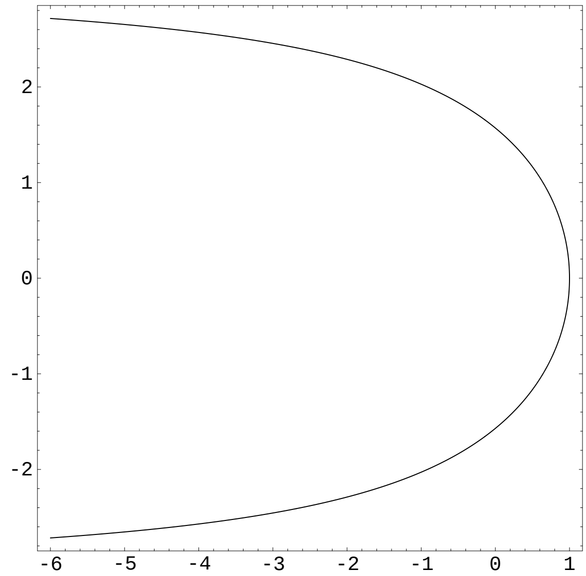

FIGURE 5. Along this contour we find that, $h(\zeta)$ achieves a maximum in the real component. And keeps the imaginary part constant.

Similarly, because $\text{Re}(h)$ must achieve a maximum we consider the contour which holds the real value of $h(z)$ equal to one along it. This is the contour

$$\gamma_{re} = \{x + iy \in \mathbb{C} \mid x - \ln|z| = 1\} = \{x + iy | y^2 = (e^{1-x})^2 - x^2\}. \tag{4.8}$$

A plot of this contour is shown in figure 6. Notice that the values for γ_{re} are greater on the interior of the contour. Figure 7 displays both γ_{re} and γ_{im}.

The steepest descent conditions are satisfied by any closed contour γ which is locally γ_{im} around 1, i.e. there exists some $\delta > 0$ such that $\gamma \cap B_\delta(1) \equiv \gamma_{im} \cap B_\delta(1)$, and does not pass through γ_{re}, i.e. $\gamma \subset \gamma^+$ (where the orientation is for γ_{re} is determined by traversing the curve counter clockwise as usual).

For any such γ, $\text{Re}[h(\gamma)]$ achieves a maximum at $\zeta_0 = 1$ and is locally identical to γ_{im} at $\zeta_0 = 1$.

[7] In a more general context your $h(\zeta)$ may have more than one critical point. If this happens you may break γ up into smaller curves and apply steepest descent to each segment. Since asymptotic expansions are additive, the sum of the resulting expansions will give and expansion for the integral.

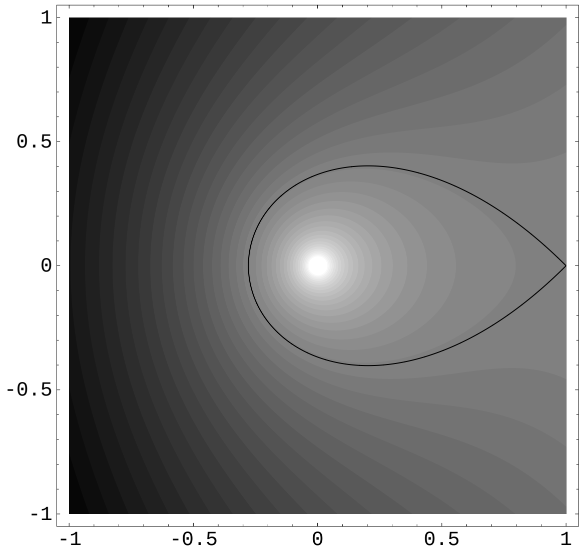

FIGURE 6. Above is the plot for γ_{re}. The lighter regions indicate larger values of $\mathrm{Re}[h]$. From this picture one can see that all z located on the interior of γ_{re}, have larger $\mathrm{Re}[h]$ values than any z' in the exterior of the contour.

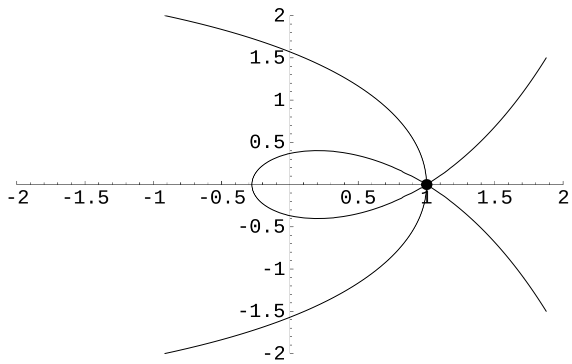

FIGURE 7. A picture of both the contours $Re[h] = 1$ and $Im[h] = 0$ which intersect at the critical point $\zeta_0 = 1$

Proposition 4.2. *Let*

$$\tilde{\gamma}_{re} = \{\zeta \in \gamma_{re} | \operatorname{Re}(\zeta) < 1\}. \tag{4.9}$$

For all simple closed $\mathcal{C}^1$ *curves* γ *such that there exists some* $\delta > 0$ *satisfying* $B_\delta(1) \cap \gamma \equiv B_\delta(1) \cap \gamma_{im}$ *and* $(\gamma \backslash B_\delta(1)) \subset \tilde{\gamma}_{re}^-$ *places the integral*

$$E_n(nz) = \frac{1}{2\pi i} \int_\gamma e^{n(\zeta - \log \zeta)} \cdot \frac{1}{(\zeta - z)\zeta} d\zeta \tag{4.10}$$

in Steepest Descent form (c.f (3.9)).

4.4. **Asymptotics Behavior of the Error Term.**

Corollary 4.3 (Asymptotics for the Error Term). *Provided the conditions for γ in proposition 4.2 are met*

$$E_n[e^s](nz) = \frac{ie^n z^{n+1}}{\sqrt{2\pi n}(1-z)}\left(1 + O\left(\frac{1}{z}\right)\right) \tag{4.11}$$

as $n \to \infty$.

In figure 8 we've plotted the roots of the 40th degree normalized partial sum with the real contour γ_{re}.

We now record the following interesting fact:

Proposition 4.4 (Szegö Curve). *The contour $\tilde{\gamma}_{re}$ (see equation (4.8)) is the the Szegö curve D_∞ (see equation (1.2)).*

4.5. **Rouche's Theorem and Its Application.**

In this section we first show that the roots of the nth normalized partial sum of the exponential function lie in the interior of any valid steepest descent contour for large enough n. Further show that the roots lie just outside the contour γ_{re} (see figure 8). We do this using what we know about the steepest descent contour from proposition 4.2 and the following lemma.

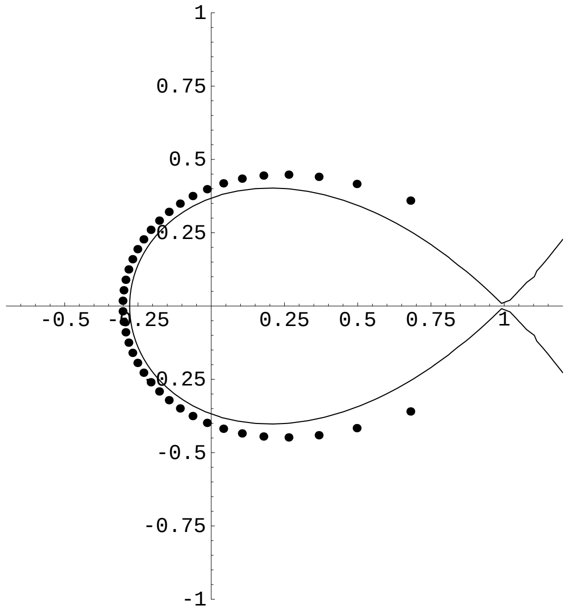

FIGURE 8. Here are the roots of the 40th degree approximant of the exponential function in the normalized plane (The roots are divided by 40). The contour in the picture is γ_{re}, the Szegö curve (Also called D_∞—but proved to be the same curve by proposition proposition 4.4.

Lemma 4.5 (Rouche's Theorem). *Let f and g be analytic functions defined on the interior of a region A. Let γ be a simple closed curve where for all $z \in \gamma$, if z is* *not a root or a pole of f and z is not a root or a pole of g and $|f(z)-g(z)| < |f(z)|$ for all z on γ. Then*

$$Z_g - P_g = Z_f - P_f \tag{4.12}$$

where Z and P are the number of zeros and poles contained γ^+.

Proof. Define

$$h(z) = \frac{g(z)}{f(z)}. \tag{4.13}$$

Since $|h(z)-1| < 1$ for all z on γ, the image of γ under h is contained entirely in a unit disc centered at 1. This means that h is contained part of the complex plane where the real part is greater than zero. This means function $\mathrm{Log}(h(z))$ is analytic can and be defined by

$$\mathrm{Log}(h(z)) = \ln(|h(z)|) + i\,\mathrm{Arg}(h(z)) = \ln(r) + i\theta \tag{4.14}$$

where $h(z) = re^{i\theta} \neq 0$ where we take the principle branch of the argument. By antiderivatives we know

$$0 = \int_\gamma \frac{d}{dz}\,\mathrm{Log}(h(z))dz = \int_\gamma \frac{h'(z)}{h(z)}dz = \int_\gamma \frac{f'(z)}{f(z)}dz - \int_\gamma \frac{g'(z)}{g(z)}dz. \tag{4.15}$$

By the argument principle

$$Z_f - P_f = Z_g - P_g \tag{4.16}$$

where Z and P denote the zeroes and poles of f and g respectively. This proves our result. ☐

Theorem 4.6 (Exterior Bound)**.** *Let γ be any contour which satisfies the conditions in proposition 4.2.*

$\forall$ *closed curves* $\alpha, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}$

$$\gamma \subset \alpha^+ \text{ and } n \geq N \implies R_n[e^{nz}] \subset \alpha^+. \tag{4.17}$$

Proof. Let

$$\frac{ie^n z^{n+1}}{\sqrt{2\pi n}(1-z)} = A(n,z). \tag{4.18}$$

For all $nz = w \in \gamma^-$

$$s_n[e^s](nz) = A(n,z)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{4.19}$$

By equation (4.11) and equation (2.5) the definition of 'big oh' there exists some $M \in \mathbb{R}$ and there exists some $N \in \mathbb{N}$

$$n \geq N \implies |s_n[e^s](nz) + A(n,z)| \leq |A(n,z)|\,M\frac{1}{n} < |A(n,z)|\,. \tag{4.20}$$

because $\forall\varepsilon, \exists N$ such that $n \geq N \implies \frac{M}{n} < \epsilon$ we get

$$|s_n[e^s](nz) + A(n,z)| \leq |A(n,z)|\,\varepsilon < |A(n,z)|\,. \tag{4.21}$$

We can now apply Rouche's theorem to any contour α such that $\alpha^+ \supset \gamma$ (This condition on α preserves (4.21)). Since $s_n[e^s](nz)$ has n roots and $A(n,z)$ has $n+1$ roots and one pole, we have $\{z \in \mathbb{C} | s_n[e^s](nz) = 0\} \subset \alpha^+$ whenever $n \geq N$. ☐

Corollary 4.7 (Convergence to the Szegö Curve)**.** *For all* $\delta > 0$ $\lim_{n\to\infty} d_H(D_\infty \setminus B_\delta, R_n \setminus B_\delta) = 0$.

This is true because *any* steepest descent contour γ must satisfy $\gamma \subset D_\infty^-$ which means $d_H(\gamma, D_\infty) > \varepsilon$ for some $\varepsilon > 0$. But we can just take a sequence of contours which approach D_∞. Take the first contour, which is say, ε away from D_∞, and increase n until the roots are in its interior. Now take the next contour, say $\varepsilon/2$ away, increase the n again until they fall inside. Repeat this until you are within any distance you want of D_∞.

Theorem 4.8 (Interior Bound)**.** *For each* n*, orient the curve*

$$D_n = \left\{ z \in \mathbb{C} \,\middle|\, |e^{1-z}z|^n = \sqrt{2\pi n}\frac{|1-z|}{|z|} \right\} \tag{4.22}$$

in the counterclockwise direction. For all n*,* $R_n[e^{nz}] \subset D_n^-$.

Proof. Take $z \in \gamma^+$, where γ satisfies the steepest descent conditions. The integral formula for Taylor approximants gives

$$s_n(nz) = e^{nz} - \frac{ie^n z^{n+1}}{\sqrt{2\pi n}(1-z)}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{4.23}$$

Consider the set of z such that[8]

$$|e^{nz}| = \left| \frac{ie^n z^{n+1}}{\sqrt{2\pi n}(1-z)} \right|. \tag{4.24}$$

For a fixed n this is the set

$$D_n = \left\{ z \in \mathbb{C} \,\middle|\, |e^{1-z}z|^n = \sqrt{2\pi n}\frac{|1-z|}{|z|} \right\}. \tag{4.25}$$

For all $z \in D_n^+$

$$\left| \frac{ie^n z^{n+1}}{(1-z)} \right| < |e^{nz}|. \tag{4.26}$$

Thus for all $z \in D_n^+$

$$|s_n(nz) - e^{nz}| = \left| \frac{ie^n z^{n+1}}{(1-z)}\left(1 + O\left(\frac{1}{n}\right)\right) \right| < \left| \frac{ie^n z^{n+1}}{(1-z)} \right| < |-e^{nz}|. \tag{4.27}$$

Thus for any curve $\alpha \subset D_n^+$ the above condition holds and we may apply Rouche's theorem which tells us that there are no zeros inside the contour D_n. □

Corollary 4.9 (Convergence to the Family of Curves)**.** *For all* $\delta > 0$ $\lim_{n\to\infty} d_H(D_n \setminus B_\delta, R_n \setminus B_\delta) = 0$.

This result is easy to see since $D_n \to D_\infty$: just let n be large enough to get as close as you want to D_∞. Then increase n again to get the roots within any tolerance of D_∞ (by prop 4.7) to get a bound on how close the roots will be.

[8] Here we are applying the same trick that we did for γ_{re}. We are going to make a contour which gives us an inequality.

Remark 4.10. The contours D_n were obtained by taking a first order approximation to the equation (4.23). If we would have instead obtained formulas using higher order terms in the function "$A(n,z)$" we would have obtained more precise lower[9] bounds on the roots. Similarly, in (4.18) and (4.19), we could have taken a higher order approximation which would have guaranteed a rate of convergence like $O(1/n^k)$ for any k. The payoff is between the uniform bound M and the rate.

Also note that by additivity of asymptotic series it is each to obtain similar results for the functions $e^{z^p}, \cos(z), \sin(z), \cosh(z), \sinh(z)$ and their variants.

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