Some Set Theory

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Abstract

If you aren't going to pay attention to details then you might as well not read this. These notes contain some basic set theory that might helpful in understanding set notation and your homework one and life.

1 Set Notation

A set $S \subset \mathbb{R}$ (A subset of real numbers) can be represented with the notation

$$S = \{x \in \mathbb{R} \mid \mathrm{P}(x)\}. \tag{1}$$

The above sentence is read "S equals the set of all x in the real numbers [denoted by '$x \in \mathbb{R}$'] such that [denoted by '$|$'] the proposition P holds about x ". In the above notation $\mathrm{P}(x)$ is logical some proposition. It can be almost anything. Here are some examples which make the set in equation (1) above make sense: Just replace $\mathrm{P}(x)$ with these propositions.

1. $x < 1$.
2. $x \geq 2$.
3. $x \in \mathbb{N}$. This says x is a natural number. This can also be written as $x \in \{1, 2, 3, 4, \ldots\}$ since $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
4. $|x - 1| < 2$

These propositions give the following sets respectively:

1. $\{x \in \mathbb{R} \mid x < 1\}$. The set of all real numbers less than one.
2. $\{x \in \mathbb{R} \mid x \geq 2\}$. The set of all real numbers greater than or equal to two.
3. $\{x \in \mathbb{R} \mid x \in \mathbb{N}\}$. The set of all real numbers which are also natural numbers. (Which is just the natural numbers obviously. This set can also be written as just $\mathbb{N}$).
4. $\{x \in \mathbb{R} \mid |x - 1| < 2\}$. The set of all real numbers where $|x - 1| < 2$.

Now we can generalize this idea to talk about subsets A of the plane $\mathbb{R}^2$ (when A is a set of points in the cartesian plane $\mathbb{R}^2$ we write $A \subset \mathbb{R}^2$. We say "A is a subset of r-squared" or "A is a set in r-2").

$$A = \{(x, y) \in \mathbb{R}^2 \mid \mathrm{P}(x, y)\}. \tag{2}$$

Note now that the proposition P involves both x and y. These are statements we make about x and y. Here are some examples

1. $x < 1$.
   True for points (x, y) with the x value less than 1. Note that Propositions don't *have* to involve y.

2. $x^2 + y^2 = 1$.
   True for points on the circle.

3. $y = 2x + 5$.
   Points on a line $f(x) = 3x + 5$.

4. $y > 2$ and $x < 2$.
   True points in the plane where the y value is more than 2 and the x value is less than two.

We define the sets similarly in one dimension the same as in two dimensions.

In the fourth example we find something "interesting"—although it main seem extremely obvious bear with me: The proposition $(y > 2)$ is just a proposition about y. Let's call it $\mathrm{Q}(y)$. $\mathrm{Q}(y) = (y > 2)$. Similarly the proposition $(x < 2)$ is just a proposition about x. Let's call it it as $\mathrm{R}(x)$. $\mathrm{R}(x) = (x < 2)$.

We can now write $P(x, y)$ as defined in example 4 as

$$\mathrm{P}(x, y) = (\mathrm{Q}(y) \text{ and } \mathrm{R}(x)),$$

meaning that $\mathrm{P}(x, y)$ is true if and only if both $\mathrm{Q}(y)$ is true and $\mathrm{R}(x)$ is true simultaneously for some point (x, y). We sometimes write "and" as $\wedge$:

$$\mathrm{P}(x, y) = (\mathrm{Q}(y) \wedge \mathrm{R}(x)).$$

Similarly if $\mathrm{A}(x, y)$ is a proposition and $\mathrm{B}(x, y)$ is a proposition then the statement $(\mathrm{A}(x, y) \text{ or } \mathrm{B}(x, y))$ is a proposition. Here we need to make a special point. 'or' does not mean 'either or' in logic. What we mean by 'or' is described in the following cases (you might be familiar with this from highschool)(i'm going to use short-hand notation here):

- IF A is true THEN (A or B) is true.
- IF B is true THEN (A or B) is true.
- IF A is true and B is true THEN (A or B) is true.
- IF both A and B are false THEN (A or B) is false.

So the only way for (A or B) to be false is if both A and B need to be false simultaneously. We write $(\mathrm{A}(x, y) \text{ or } \mathrm{B}(x, y))$ with the symbol '$\vee$':

$$A(x, y) \vee B(x, y)$$

. The symbol '$\vee$' derives from the latin word 'vel' which mean 'or' in the sense I just described.

The following is true:

Theorem 1.1 (Distributive Property of Logic) *If A,B and C are propositions then*

1. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.
2. $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.

you can check these by plugging in values of true and false for A, B and C and check that those equations hold. This is a very useful property.

2 The Point of Set Notation: Union and Intersection

Let $S \subset \mathbb{R}$ be

$$S = \{x \in \mathbb{R} \mid \mathrm{P}(x)\}$$

where P is some proposition (translated: S is a subset of the real numbers where $\mathrm{P}(x)$ holds). Let $T \subset \mathbb{R}$ be

$$T = \{x \in \mathbb{R} \mid \mathrm{Q}(x)\}$$

where Q is some proposition. We can now DEFINE intersection and union in terms of $\wedge$ and $\vee$.

Intersection If S and T are the two sets as described above, then S *intersect* T, written $S \cap T$, is defined to be

$$S \cap T = \{x \in \mathbb{R} \mid \mathrm{P}(x) \wedge \mathrm{Q}(x)\}. \tag{3}$$

Writing this out completely gives

$$\{x \in \mathbb{R} \mid \mathrm{P}(x)\} \cap \{x \in \mathbb{R} \mid \mathrm{Q}(x)\} = \{x \in \mathbb{R} \mid \mathrm{P}(x) \wedge \mathrm{Q}(x)\}. \tag{4}$$

Union If S and T are the two sets as described above, then S *intersect* T, written $S \cup T$, is defined to be

$$S \cup T = \{x \in \mathbb{R} \mid \mathrm{P}(x) \vee \mathrm{Q}(x)\}. \tag{5}$$

Writing this out completely gives

$$\{x \in \mathbb{R} \mid \mathrm{P}(x)\} \cup \{x \in \mathbb{R} \mid \mathrm{Q}(x)\} = \{x \in \mathbb{R} \mid \mathrm{P}(x) \vee \mathrm{Q}(x)\}. \tag{6}$$

This intersection is saying that an element x is in $S \cap T$ (written $x \in (S \cap T)$) if and only if it is in both S and T simultaneously. The union property says a similar thing. x is in $S \cup T$ if you can find it in either one of the sets.

Some examples are in order to prove it the usefulness of this definition.

Example Consider the proposition $\mathrm{P}(x) = (e < x < \pi)$ where $e = 2.718281828459\ldots$ and $\pi = 3.14159265897932462543\ldots$. We can break $P(x)$ into two propsitions. $(e < x)$ and $(x < \pi)$. Meaning

$$\{x \in \mathbb{R} \mid e < x < \pi\} = \{x \in \mathbb{R} \mid e < x\} \cap \{x \in \mathbb{R} \mid x < \pi\}.$$

▲

The next example will be a little more complicated. To prepare you, consider the following fact: if $x \in \mathbb{R}$ then the proposition $((x < \pi) \vee (x \geq \pi))$ is always true. Think about it. A number x is either always less than *pi* or it's more than π. A statement that is always true like this is called a *tautology.*

If P is a proposition that is always true (a tautology), and Q is another proposition, then $Q = P \wedge Q$. That is Q is true whenever Q is true and the statement that is always true is true. Bearing that in mind consider the next example.

Example Consider the statement $(|x - \pi| < x)$. Let us break up the statement $|x - \pi|$ into two cases. We will examine what happens when $x < \pi$ and then we will examine what happens when $x \geq \pi$.

- If $x < \pi$, Then $\pi - x$ is positive which means $|x-\pi| = \pi - x$. So we can write

$$((x < \pi) \wedge (|x-\pi| < x)) = ((x < \pi) \wedge (\pi - x < x))$$

by replacing $|x-\pi|$ with $\pi - x$

- If $x \geq \pi$, Then $|x-\pi| = x - \pi$. So we can write

$$((x \geq \pi) \wedge (|x-\pi| < x)) = ((x \geq \pi) \wedge (x - \pi < x))$$

by replacing $|x-\pi|$ with $x - \pi$.

because $(x < \pi) \vee (x \geq \pi)$ is a tautology

$$(|x-\pi| < x) = (|x-\pi| < x) \wedge ((x < \pi) \vee (x \geq \pi)).$$

This proposition has the form $A \wedge (B \vee C)$ which by the distributive property has $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$, which in our context means

$$\underbrace{(|x-\pi| < x)}_{A} \wedge (\underbrace{(x < \pi)}_{B} \vee \underbrace{(x \geq \pi)}_{C}) = (\underbrace{(|x-\pi| < x)}_{A} \wedge \underbrace{(x < \pi)}_{B}) \vee (\underbrace{(|x-\pi| < x)}_{A} \wedge \underbrace{(x \geq \pi)}_{C}).$$

From our above discussion (the bullets) we can get rid of the absolute values in the right side of the above equation giving

$$((|x-\pi| < x) \wedge (x < \pi)) \vee ((|x-\pi| < x) \wedge (x \geq \pi)) = ((\pi - x < x) \wedge (x < \pi)) \vee ((x - \pi < x) \wedge (x \geq \pi)).$$

so collecting all of our information we have the following equality of propositions

$$(|x-\pi| < x) = ((\pi - x < x) \wedge (x < \pi)) \vee ((x - \pi < x) \wedge (x \geq \pi)). \quad (7)$$

In terms of sets this means

$$\begin{aligned} \{x \in \mathbb{R} \mid |x-\pi| < x\} &= \{x \in \mathbb{R} \mid ((\pi - x < x) \wedge (x < \pi)) \vee ((x - \pi < x) \wedge (x \geq \pi))\} \\ &= \{x \in \mathbb{R} \mid (\pi - x < x) \wedge (x < \pi)\} \cup \{x \in \mathbb{R} \mid (x - \pi < x) \wedge (x \geq \pi)\} \\ &= (\{x \in \mathbb{R} \mid \pi - x < x\} \cap \{x \in \mathbb{R} \mid x < \pi\}) \\ &\quad \cup (\{x \in \mathbb{R} \mid x - \pi < x\} \cap \{x \in \mathbb{R} \mid x \geq \pi\}). \end{aligned}$$

The set on the right hand side is much easier to compute. ▲