# Completing the Square

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**Abstract**

This document shows you how to complete the square.

## 1 Completing the square

We will need the following fact:

**Lemma 1.1** *Let* $A_1, B_1, C_1, A_2, B_2, C_2 \in \mathbb{R}$.

$$A_1x^2 + B_1x + C_1 = A_2x^2 + B_2x + C_2. \tag{1}$$

*If and only if*

$$\begin{cases} A_1 = A_2, \\ B_1 = B_2, \\ C_1 = C_2 \end{cases}. \tag{2}$$

What the above lemma[1] says is that we can only write down a binomial in one way. This is in fact true for all polynomials: two polynomials are equal if and only if they have the same coefficients.[2]

If we want to write a given polynomial $p(x) = Ax^2 + Bx + C$ as $p(x) = c(x-a)^2 + b$, the process of finding $c, a$, and $b$ so this works is called *completing the square*. I will now show you can do this.

**Step 1** Set the two representations equal to each other.

$$Ax^2 + Bx + C = c(x-a)^2 + b. \tag{3}$$

**Step 2** Expand the right side of (3) out and collect terms:

$$c(x-a)^2 + b = c(x^2 - 2ax + a^2) + b = cx^2 - 2cax + ca^2 + b \tag{4}$$

**Step 3** Using

$$Ax^2 + Bx + C = cx^2 - 2cax + ca^2 + b \tag{5}$$

and Theorem 1.1, get the system of equations

$$\begin{cases} A = c, \\ B = -2ca, \\ C = ca^2 + b \end{cases}. \tag{6}$$

[1] A lemma is a proposition/fact/theorem that mathematicians will use to prove something else. You could just as well replace 'lemma' with 'useful fact that I will use to prove something else'.

[2] This is true because the set of polynomials form what is called a *Vector Space*. It has a vector addition and scalar multiplication. You can read about this more on mathworld.wolfram.com or wikipedia.

**Step 4** Solve the system: (you should do this step yourself to check that it's true. Don't trust anyone!)

$$\begin{cases} A = c, \\ B = -2ca, \\ C = ca^2 + b \end{cases} \implies \begin{cases} c = A, \\ a = -B/2A, \\ b = C - A(\frac{-B}{2A})^2 = C - \frac{B^2}{2A} \end{cases} \tag{7}$$

So we are done. This gives us the following theorem:

**Theorem 1.2** *Any polynomial* $p(x) = Ax^2 + Bx + C$ *can be written as*

$$p(x) = c(x - a)^2 + b \tag{8}$$

*where*

$$c = A, \tag{9}$$

$$a = -\frac{B}{2A}, \tag{10}$$

$$b = C - \frac{B^2}{2A}. \tag{11}$$

**Remark** Plugging in the values gives

$$Ax^2 + Bx + C = A\left(x + \frac{B}{2A}\right)^2 + C - \frac{B^2}{2A} \tag{12}$$

which may be familiar.

In practice, instead of memorizing the weird looking equation (12) I suggest carrying out the four step process. Write your polynomial as $b(x - a)^2 - c$, expand it out, then solve for $b, a$ and $c$.