Last Name:
Section:

Math 162 (Loring, Dupuy sections 7 and 8)- Optimization Quiz

- Set up your equations.
- Justify your claims with Math. (Explain why are certain points you find are max/min etc.)
- Explain your steps with English.

1. **(5pts)** Find the cylinder with a volume of 10 in^3 that has a minimal surface area. (Rigorous Analysis. Give me all the parameters.)

   harvest equations The volume of a cylinder is

$$V = \pi r^2 h. \tag{1}$$

The surface area of a cylinder is

$$S = 2\pi r^2 + 2\pi r h. \tag{2}$$

Our constraint on area gives us

$$10 = 2\pi r^2 h. \tag{3}$$

find formula for optimization Because of the constraint we can write h as

$$h = \frac{1}{10\pi r^2}. \tag{4}$$

Putting this back into out equation for surface area give us

$$S = 2\pi r^2 + 2\pi r\left(\frac{1}{10\pi r^2}\right) = 2\pi r^2 + \frac{2}{5r}. \tag{5}$$

find critical points, verify that critical point is a min Now differentiate S with respect to r.

$$\frac{dS}{dr} = 4\pi r - \frac{2}{5r^2}. \tag{6}$$

Setting $\frac{dS}{dr} = 0$ gives

$$4\pi r - \frac{2}{5r^2}. \implies r^3 = \frac{1}{10\pi} \implies r = \left(\frac{1}{10\pi}\right)^{1/3}. \tag{7}$$

(Note: we throw away the non-real cube roots because they are worthless to us.) We know that we have a critical point. To check that it is a min we can look at the second derivative:

$$\frac{d^2S}{dr^2} = 2\pi + \frac{4}{4r^3}. \tag{8}$$

Because

$$\frac{d^2S}{dr^2}\left(\left(\frac{1}{10\pi}\right)^{1/3}\right) > 0 \tag{9}$$

$S(r)$ is concave up at $\left(\frac{1}{10\pi}\right)^{1/3}$ which means we have a local min. This gives

$$S\left(\frac{1}{10\pi}\right)^{1/3} = 2\pi\left(\frac{1}{10\pi}\right)^{2/3} + \frac{2(10\pi)^{1/3}}{5} \tag{10}$$

as our minimal area.

2. **(5pts)**Find the cylinder with surface area equal to 10 in^2 that has maximal volume. (Rigorous Analysis. Give me all the parameters.)

harvest equations The volume of a cylinder is

$$V = \pi r^2 h. \tag{11}$$

The surface area of a cylinder is

$$S = 2\pi r^2 + 2\pi r h. \tag{12}$$

Our constraint on surface area gives us

$$10 = 2\pi r^2 + 2\pi r h. \tag{13}$$

find formula for optimization Because of the constraint we can write h as

$$h = \frac{5}{\pi r} - r \tag{14}$$

Putting this back into our equation for volume gives

$$V = \pi r^2 h = \pi r^2 \left(\frac{5}{\pi r} - r \right) = 5r - \pi r^3 \tag{15}$$

find critical points, verify that critical point is a min Now differentiate S with respect to r. Take the derivative of V with respect to r to get

$$\frac{dV}{dr} = 5 - 3\pi r^2. \tag{16}$$

Setting this equal to zero we get

$$r = \pm\sqrt{\frac{5}{3\pi}}. \tag{17}$$

Throwing away our positive negative root we get $r = \sqrt{\frac{5}{3\pi}}$. To see that this is a min let's apply the second derivative test. The second derivative is

$$\frac{d^2V}{dr^2} = -6\pi r. \tag{18}$$

Because $\frac{d^2V}{dr^2}\left(\sqrt{\frac{5}{3\pi}}\right) < 0$ the function for volume is concave down are $r = \sqrt{\frac{5}{3\pi}}$ which means that we have a maximum there. This means

$$V\left(\sqrt{\frac{5}{3\pi}}\right) = 5\sqrt{\frac{5}{3\pi}} - \pi\left(\sqrt{\frac{5}{3\pi}}\right)^3$$

is our maximal volume.

3. **(3pts)** In problems 1 and 2 what happens if I ask for maximal surface area and minimal volume respectively? (English Sentence) (Hint: think about what could happen when you stretch a cylinder out.)

4. **(5pts)** Given a rectangle with perimeter P what is the shortest possible diagonal it could have. (You should know the answer to this from geometry, but I want you to prove it using calculus. Justify you claims.)(Hint: Instead of minimizing $\sqrt{x^2+y^2}$ just minimize x^2+y^2. This will save you time when taking derivatives.)

harvest equations Our two functions are $D = x^2 + y^2$ and $P = 2x + 2y$. Solving for y gives $y = \frac{P}{2} - x$.

get one equation for D in terms of one variable

$$D = x^2 + y^2 = x^2 + \left(\frac{P}{2} - x\right)^2. \tag{19}$$

find critical points

$$\frac{dD}{dx} = 2x - 2\left(\frac{P}{2} - x\right) = 4x + P. \tag{20}$$

Setting $\dfrac{dD}{dx} = 0$ gives $\frac{P}{4} = x$.

verify that this is a minimum Because

$$\frac{d^2D}{dx^2} = 4 \tag{21}$$

This means that $D(x)$ is concave up at $x = \frac{P}{4}$ which means $x = \frac{P}{4}$ is a minimum.

interpretation The rectangle with the smallest distance across its diagonal is a square.